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COMMON FIXED POINT THEOREMS OF CARISTI TYPE MAPPINGS WITH w-DISTANCE

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Abstract. In this paper, we obtain common fixed point theorems of Caristi type mappings [2] by using the concept of w-distance, which is introduced by Kada, Suzuki and Takahashi [4]. Our results generalize theorems of Bhakta and Basu [1].

1 Introduction and Preliminaries In 1981, Bhakta and Basu [1] proved a common fixed point theorem of Caristi type mappings in a complete metric space. A mapping T from a metric space X to X is said to be orbitally continuous if for every x, $x_0 \in X$, $T^{n_i+1}x$ converges to Tx_0 whenever $T^{n_i}x$ converges to x_0.

Theorem 1. ([1]) Let (X, d) be a complete metric space and let S, T be two orbitally continuous mappings of X into itself. Suppose that there are two functions φ, ψ of X into $[0, \infty)$ satisfying the following condition: For any two points $x, y \in X$,

$$d(Sx, Ty) \leq \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty).$$

Then S and T have a unique common fixed point.

On the other hand, in 1996, Kada, Suzuki and Takahashi [4] introduced the concept of w-distance on a metric space.

Definition 1. ([4]) Let (X, d) be a metric space. Then a function $\rho : X \times X \to [0, \infty)$ is called a w-distance on X if the following are satisfied:

(1) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for any $x, y, z \in X$;

(2) for any $x \in X$, $\rho(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y, z \in X$, $\rho(z, x) \leq \delta$ and $\rho(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

They generalized Caristi's fixed point theorem [2], Ekeland's variational principle [3] and Takahashi's nonconvex minimization theorem [6] by using w-distance ([4], [5]).

In this paper, using the concept of w-distance, we obtain common fixed point theorems on complete metric spaces. Our results generalize theorems of Bhakta and Basu [1].

The following lemma is very important in the proofs of our results.

Lemma 1. ([4]) Let (X, d) be a metric space and let ρ be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to 0, and let $x, y, z \in X$. Then the following hold:

(i) If $\rho(x_n, y) \leq \alpha_n$ and $\rho(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $\rho(x, y) = 0$ and $\rho(x, z) = 0$, then $y = z$;

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(ii) if $\rho(x_n, y_n) \leq \alpha_n$ and $\rho(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;

(iii) if $\rho(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;

(iv) if $\rho(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.

2 Generalized common fixed point theorems of Bhakta and Basu In this section, we generalize results of Bhakta and Basu by using w-distance.

Theorem 2. Let (X, d) be a complete metric space and let ρ be a w-distance on X. Let S, T be two orbitally continuous mappings of X into itself. Suppose that there are two functions φ, ψ of X into $[0, \infty)$ satisfying the following condition: For any two points $x, y \in X$,

$$\max\{\rho(Sx, Ty), \rho(Ty, Sx)\} \leq \varphi(x) - \varphi(Sx) + \psi(y) - \psi(Ty).$$

Then S and T have a unique common fixed point.

Proof. Let x_0 and y_0 be any two points of X. We consider the following sequences

$$x_n = S^n x_0, \ y_n = T^n y_0 \ (n \in \mathbb{N}).$$

Then we have

$$\begin{aligned} \rho(x_i, y_i) &= \rho(Sx_{i-1}, Ty_{i-1}) \\ &\leq \varphi(x_{i-1}) - \varphi(Sx_{i-1}) + \psi(y_{i-1}) - \psi(Ty_{i-1}) \\ &= \varphi(x_{i-1}) - \varphi(x_i) + \psi(y_{i-1}) - \psi(y_i) \end{aligned}$$

for all $i \in \{1, 2, \ldots, n\}$. So,

$$\begin{aligned} \sum_{i=1}^{n} \rho(x_i, y_i) &\leq \sum_{i=1}^{n} \{\varphi(x_{i-1}) - \varphi(x_i) + \psi(y_{i-1}) - \psi(y_i)\} \\ &= \varphi(x_0) - \varphi(x_n) + \psi(y_0) - \psi(y_n) \\ &\leq \varphi(x_0) + \psi(y_0). \end{aligned}$$

Again,

$$\begin{aligned} \rho(y_i, x_{i+1}) &= \rho(Ty_{i-1}, Sx_i) \\ &\leq \varphi(x_i) - \varphi(Sx_i) + \psi(y_{i-1}) - \psi(Ty_{i-1}) \\ &= \varphi(x_i) - \varphi(x_{i+1}) + \psi(y_{i-1}) - \psi(y_i) \end{aligned}$$

for all $i \in \{1, 2, \ldots, n\}$. So,

$$\begin{aligned} \sum_{i=1}^{n} \rho(y_i, x_{i+1}) &\leq \sum_{i=1}^{n} \{\varphi(x_i) - \varphi(x_{i+1}) + \psi(y_{i-1}) - \psi(y_i)\} \\ &= \varphi(x_1) - \varphi(x_{n+1}) + \psi(y_0) - \psi(y_n) \\ &\leq \varphi(x_1) + \psi(y_0). \end{aligned}$$

Since, $\rho(x_i, x_{i+1}) \leq \rho(x_i, y_i) + \rho(y_i, x_{i+1})$, we have

$$\begin{aligned} \sum_{i=1}^{n} \rho(x_i, x_{i+1}) &\leq \sum_{i=1}^{n} \{\rho(x_i, y_i) + \rho(y_i, x_{i+1})\} \\ &\leq \varphi(x_0) + \varphi(x_1) + 2\psi(y_0). \end{aligned}$$

This gives that series $\sum_{i=1}^{\infty} \rho(x_i, x_{i+1})$ is convergent. Let n and m be any two positive integers with $m > n$. Then

$$\rho(x_n, x_m) \leq \sum_{i=n}^{m-1} \rho(x_i, x_{i+1}) \to 0 \text{ as } n \to \infty.$$

By Lemma 1(iii), $\{x_n\}$ is a Cauchy sequence.

Similarly, we have

$$\begin{aligned} \rho(x_{i+1}, y_{i+1}) &= \rho(Sx_i, Ty_i) \\ &\leq \varphi(x_i) - \varphi(Sx_i) + \psi(y_i) - \psi(Ty_i) \\ &= \varphi(x_i) - \varphi(x_{i+1}) + \psi(y_i) - \psi(y_{i+1}) \end{aligned}$$

for all $i \in \{1, 2, \ldots, n\}$. So,

$$\begin{aligned} \sum_{i=1}^{n} \rho(x_{i+1}, y_{i+1}) &\leq \sum_{i=1}^{n} \{\varphi(x_i) - \varphi(x_{i+1}) + \psi(y_i) - \psi(y_{i+1})\} \\ &= \varphi(x_1) - \varphi(x_{n+1}) + \psi(y_1) - \psi(y_{n+1}) \\ &\leq \varphi(x_1) + \psi(y_1). \end{aligned}$$

Since, $\rho(y_i, y_{i+1}) \leq \rho(y_i, x_{i+1}) + \rho(x_{i+1}, y_{i+1})$, we have

$$\begin{aligned} \sum_{i=1}^{n} \rho(y_i, y_{i+1}) &\leq \sum_{i=1}^{n} \{\rho(y_i, x_{i+1}) + d(x_{i+1}, y_{i+1})\} \\ &\leq 2\varphi(x_1) + \psi(y_0) + \psi(y_1). \end{aligned}$$

This gives that series $\sum_{i=1}^{\infty} \rho(y_i, y_{i+1})$ is convergent. In the same way, $\{y_n\}$ is a Cauchy sequence.

Since X is complete, each of them is convergent, that is,

$$x_n \to \bar{x}, \; y_n \to \bar{y}$$

as $n \to \infty$ for some $\bar{x}$, $\bar{y} \in X$. Since S and T are orbitally continuous, $Sx_n \to S\bar{x}$, $Ty_n \to T\bar{y}$, that is,

$$x_{n+1} \to S\bar{x}, \; y_{n+1} \to T\bar{y}$$

as $n \to \infty$. This gives that $S\bar{x} = \bar{x}$ and $T\bar{y} = \bar{y}$.

Now,

$$\begin{aligned} \rho(\bar{x}, \bar{x}) &\leq \rho(\bar{x}, \bar{y}) + \rho(\bar{y}, \bar{x}) \\ &= \rho(S\bar{x}, T\bar{y}) + \rho(T\bar{y}, S\bar{x}) \\ &\leq 2[\varphi(\bar{x}) - \varphi(S\bar{x}) + \psi(\bar{y}) - \psi(T\bar{y})] = 0 \end{aligned}$$

and

$$\begin{aligned} \rho(\bar{x}, \bar{y}) &= \rho(S\bar{x}, T\bar{y}) \\ &\leq \varphi(\bar{x}) - \varphi(S\bar{x}) + \psi(\bar{y}) - \psi(T\bar{y}) = 0. \end{aligned}$$

Hence, by Lemma 1(i), $\bar{x} = \bar{y}$.

Let $\bar{z} \in X$ satisfying $S\bar{z} = \bar{z}$. Then

$$\begin{aligned}\rho(\bar{z}, \bar{z}) &\leq \rho(\bar{z}, \bar{x}) + \rho(\bar{x}, \bar{z}) \\ &= \rho(S\bar{z}, T\bar{x}) + \rho(T\bar{x}, S\bar{z}) \\ &\leq 2[\varphi(\bar{z}) - \varphi(S\bar{z}) + \psi(\bar{x}) - \psi(T\bar{x})] = 0\end{aligned}$$

and

$$\begin{aligned}\rho(\bar{z}, \bar{x}) &= \rho(S\bar{z}, T\bar{x}) \\ &\leq \varphi(\bar{z}) - \varphi(S\bar{z}) + \psi(\bar{x}) - \psi(T\bar{x}) = 0.\end{aligned}$$

So, $\bar{z} = \bar{x}$. Thus $\bar{x}$ is the only fixed point of S. Similarly, we can show that $\bar{x}$ is the only fixed point of T. This proves the theorem. □

Corollary 1. Let (X, d) be a complete metric space and let ρ be a w-distance on X. Let $\mathcal{F} = \{S_\alpha \mid \alpha \in \Lambda\}$ be a family of orbitally continuous mappings of X into itself. Suppose that for each mapping $S \in \mathcal{F}$, there is a function φ_S of X into $[0, \infty)$ satisfying the following condition: For any two mappings $S, T \in \mathcal{F}$ and for any $x, y \in X$,

$$\max\{\rho(Sx, Ty), \rho(Ty, Sx)\} \leq \varphi_S(x) - \varphi_S(Sx) + \varphi_T(y) - \varphi_T(Ty).$$

Then the family $\mathcal{F}$ have a unique common fixed point.

Proof. Let S and T be two mappings of $\mathcal{F}$. Then S, T and φ_S, φ_T satisfy the conditions of Theorem 2. Hence S and T have a unique common fixed point x_0. Let S' be any other mapping of $\mathcal{F}$. Again by using Theorem 2, S and S' have a unique common fixed point x_1. As x_0 is the unique fixed point of S, $x_0 = x_1$. Hence x_0 is a unique common fixed point of S, T and S'. As S' is an arbitrary mapping of $\mathcal{F}$, it follows that x_0 is a unique common fixed point of the mappings of $\mathcal{F}$. □

Next we consider common fixed point theorems for set-valued maps. A set-valued mapping S from a metric space X into 2^X is said to be upper semicontinuous if for every $x \in X$ and every open set V with $Sx \subset V$, there exists a neighborhood U of x such that $Sz \in V$ for all $z \in U$. See [7].

Lemma 2. Let (X, d) be a metric space and let S be an upper semicontinuous mapping of X into 2^X. For any $x \in X$, Sx is nonempty and closed. Let $x_0 \in X$ and $\{x_n\}$ is a sequence in X. Then,

$$\begin{cases} x_{n+1} \in Sx_n \ (n \in \mathbb{N}) \\ x_n \to x_0 \end{cases} \Rightarrow x_0 \in Sx_0.$$

Proof. Assume that $x_{n+1} \in Sx_n$ and $x_n \to x_0$. Suppose that $x_0 \notin Sx_0$. Since X is a metric space and Sx_0 is a closed set, there exists two open sets G_1 and G_2 such that

$$x_0 \in G_1,\ Sx_0 \subseteq G_2 \text{ and } G_1 \cap G_2 = \emptyset.$$

¿From upper semicontinuity of S, there exists a neighborhood U_{x_0} of x_0 such that

$$Sx \subseteq G_2 \ \text{ for any } x \in U_{x_0}.$$

Since $x_n \to x_0$, $x_n \in U_{x_0}$ for large enough n, therefore $x_{n+1} \in Sx_n \subseteq G_2$, and we have $x_0 \in \mathrm{cl}G_2$. This is a contradiction. □

Theorem 3. Let (X, d) be a complete metric space and let ρ be a w-distance on X. Let S, T be two upper semicontinuous mappings of X into 2^X. For any $x \in X$, Sx and Tx are nonempty and closed. Suppose that there are two functions φ, ψ of X into $[0, \infty)$ satisfying the following condition: For any two points $x, y \in X$ and for any $u \in Sx$, $v \in Ty$,

$$\max\{\rho(u, v), \rho(v, u)\} \le \varphi(x) - \varphi(u) + \psi(y) - \psi(v).$$

Then S and T have a unique common fixed point.

Proof. Let x_0 and y_0 be any two points of X and $\{x_n\}$ and $\{y_n\}$ be sequences satisfying

$$x_n \in Sx_{n-1},\ y_n \in Ty_{n-1}\ (n \in \mathbb{N}).$$

Then we have

$$\rho(x_i, y_i) \le \varphi(x_{i-1}) - \varphi(x_i) + \psi(y_{i-1}) - \psi(y_i),$$
$$\rho(y_i, x_{i+1}) \le \varphi(x_i) - \varphi(x_{i+1}) + \psi(y_{i-1}) - \psi(y_i)$$

and

$$\rho(x_{i+1}, y_{i+1}) \le \varphi(x_i) - \varphi(x_{i+1}) + \psi(y_i) - \psi(y_{i+1})$$

for all $i \in \mathbb{N}$. In similar way to proof of Theorem 2, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is complete, each of them is convergent, that is,

$$x_n \to \bar{x},\ y_n \to \bar{y}$$

as $n \to \infty$ for some $\bar{x}$, $\bar{y} \in X$. By Lemma 2, we have that $\bar{x} \in S\bar{x}$ and $\bar{y} \in T\bar{y}$.

Now,

$$\begin{aligned} \rho(\bar{x}, \bar{x}) &\le \rho(\bar{x}, \bar{y}) + \rho(\bar{y}, \bar{x}) \\ &\le 2[\varphi(\bar{x}) - \varphi(\bar{x}) + \psi(\bar{y}) - \psi(\bar{y})] = 0 \end{aligned}$$

and

$$\rho(\bar{x}, \bar{y}) \le \varphi(\bar{x}) - \varphi(\bar{x}) + \psi(\bar{y}) - \psi(\bar{y}) = 0.$$

Hence, by Lemma 1(i), $\bar{x} = \bar{y}$. We obtain $\bar{x} \in S\bar{x} \cap T\bar{x}$.

Let $\bar{z} \in X$ satisfying $\bar{z} \in S\bar{z}$. Then

$$\begin{aligned} \rho(\bar{z}, \bar{z}) &\le \rho(\bar{z}, \bar{x}) + \rho(\bar{x}, \bar{z}) \\ &\le 2[\varphi(\bar{z}) - \varphi(\bar{z}) + \psi(\bar{x}) - \psi(\bar{x})] = 0 \end{aligned}$$

and

$$\rho(\bar{z}, \bar{x}) \le \varphi(\bar{z}) - \varphi(\bar{z}) + \psi(\bar{x}) - \psi(\bar{x}) = 0.$$

So, $\bar{z} = \bar{x}$. Thus $\bar{x}$ is the only fixed point of S. Similarly, we can show that $\bar{x}$ is the only fixed point of T. This proves the theorem. □

Corollary 2. Let (X, d) be a complete metric space and let ρ be a w-distance on X. Let $\mathcal{F} = \{S_\alpha \mid \alpha \in \Lambda\}$ be a family of upper semicontinuous mappings of X into 2^X. For any $x \in X$ and $S \in \mathcal{F}$, Sx is nonempty and closed. Suppose that there is a family $\{\varphi_S \mid S \in \mathcal{F}\}$ of functions of X into $[0, \infty)$ satisfying the following condition: For any two mappings $S, T \in \mathcal{F}$ and for any $x, y \in X$ and $u \in Sx$, $v \in Ty$,

$$\max\{\rho(u, v), \rho(v, u)\} \le \varphi_S(x) - \varphi_S(u) + \varphi_T(y) - \varphi_T(v).$$

Then the family $\mathcal{F}$ have a unique common fixed point.

3 More common fixed point theorems Let (X, d) be a metric space and let ρ be w-distance on X. In this section, we consider the following condition

$$\max\{\rho(x,y),\rho(y,x)\}+\rho(x,Sx)+\rho(y,Ty)\leq\varphi(x)-\varphi(Sx)+\psi(y)-\psi(Ty)$$

instead of

$$\max\{\rho(Sx,Ty),\rho(Ty,Sx)\}\leq\varphi(x)-\varphi(Sx)+\psi(y)-\psi(Ty)$$

for orbitally continuous mappings S and T of X into itself. The inequality

$$\max\{\rho(Sx,Ty),\rho(Ty,Sx)\}\leq\max\{\rho(x,y),\rho(y,x)\}+\rho(x,Sx)+\rho(y,Ty)$$

does not hold in general. For example, let $X = \{a, b, c\}$, $d(x, y) = 1$ if $x \neq y$, $d(x, y) = 0$ if $x = y$, $\rho(a, b) = 3$, $\rho(x, y) = d(x, y)$ whenever $(x, y) \neq (b, a)$, $Sa = Sb = a$, $Sc = b$ and $Ta = Tc = a$, $Tb = c$. If $(x, y) = (c, a)$,

$$\max\{\rho(Sx,Ty),\rho(Ty,Sx)\}>\max\{\rho(x,y),\rho(y,x)\}+\rho(x,Sx)+\rho(y,Ty).$$

Theorem 4. Let (X, d) be a complete metric space and let ρ be a w-distance on X. Let S, T be two orbitally continuous mappings of X into itself. Suppose that there are two functions φ, ψ of X into $[0, \infty)$ satisfying the following condition: For any two points $x, y \in X$,

$$\max\{\rho(x,y),\rho(y,x)\}+\rho(x,Sx)+\rho(y,Ty)\leq\varphi(x)-\varphi(Sx)+\psi(y)-\psi(Ty).$$

Then S and T have a unique common fixed point.

Proof. Let x_0 and y_0 be any two points of X. We consider the following sequences

$$x_n = S^n x_0,\ y_n = T^n y_0\ (n \in \mathbb{N}).$$

Then we have

$$\begin{aligned}\rho(x_{i-1},x_i) &\leq \max\{\rho(x_{i-1},y_{i-1}),\rho(y_{i-1},x_{i-1})\}+\rho(x_{i-1},x_i)+\rho(y_{i-1},y_i)\\ &\leq \varphi(x_{i-1})-\varphi(x_i)+\psi(y_{i-1})-\psi(y_i)\end{aligned}$$

for all $i \in \{1, 2, \ldots, n\}$. So,

$$\begin{aligned}\sum_{i=1}^{n}\rho(x_{i-1},x_i) &\leq \sum_{i=1}^{n}\{\varphi(x_{i-1})-\varphi(x_i)+\psi(y_{i-1})-\psi(y_i)\}\\ &= \varphi(x_0)-\varphi(x_n)+\psi(y_0)-\psi(y_n)\\ &\leq \varphi(x_0)+\psi(y_0).\end{aligned}$$

This gives that series $\sum_{i=1}^{\infty}\rho(x_{i-1}, x_i)$ is convergent, and $\{x_n\}$ is a Cauchy sequence in similar way to proof of Theorem 2. Also we have $\{y_n\}$ is a Cauchy sequence.

Since X is complete, each of them is convergent, that is,

$$x_n \to \bar{x},\ y_n \to \bar{y}$$

as $n \to \infty$ for some $\bar{x}$, $\bar{y} \in X$. Since S and T are orbitally continuous, $Sx_n \to S\bar{x}$, $Ty_n \to T\bar{y}$, that is,

$$x_{n+1} \to S\bar{x},\ y_{n+1} \to T\bar{y}$$

as $n \to \infty$. This gives that $S\bar{x} = \bar{x}$ and $T\bar{y} = \bar{y}$.

Now,

$$\begin{aligned} \max\{\rho(\bar{x},\bar{y}),\rho(\bar{y},\bar{x})\} &\le \max\{\rho(\bar{x},\bar{y}),\rho(\bar{y},\bar{x})\} + \rho(\bar{x},S\bar{x}) + \rho(\bar{y},T\bar{y}) \\ &\le \varphi(\bar{x}) - \varphi(S\bar{x}) + \psi(\bar{y}) - \psi(T\bar{y}) = 0, \end{aligned}$$

then $\rho(\bar{x},\bar{y}) = \rho(\bar{y},\bar{x}) = 0$. And also

$$\rho(\bar{x},\bar{x}) \le \rho(\bar{x},\bar{y}) + \rho(\bar{y},\bar{x}) = 0,$$

hence $\rho(\bar{x},\bar{y}) = \rho(\bar{x},\bar{x}) = 0$. By Lemma 1(i), $\bar{x} = \bar{y}$.

Let $\bar{z} \in X$ satisfying $S\bar{z} = \bar{z}$. Then

$$\begin{aligned} \max\{\rho(\bar{z},\bar{x}),\rho(\bar{x},\bar{z})\} &\le \max\{\rho(\bar{z},\bar{x}),\rho(\bar{x},\bar{z})\} + \rho(\bar{z},S\bar{z}) + \rho(\bar{x},T\bar{x}) \\ &\le \varphi(\bar{z}) - \varphi(S\bar{z}) + \psi(\bar{x}) - \psi(T\bar{x}) = 0. \end{aligned}$$

In the same way, we have $\bar{z} = \bar{x}$. Thus $\bar{x}$ is the only fixed point of S. Similarly, we can show that $\bar{x}$ is the only fixed point of T. This proves the theorem. □

Corollary 3. Let (X, d) be a complete metric space and let ρ be a w-distance on X. Let $\mathcal{F} = \{S_\alpha \mid \alpha \in \Lambda\}$ be a family of orbitally continuous mappings of X into itself. Suppose that there is a family $\{\varphi_S \mid S \in \mathcal{F}\}$ of functions of X into $[0,\infty)$ satisfying the following condition: For any two mappings $S, T \in \mathcal{F}$ and for any $x, y \in X$,

$$\max\{\rho(x,y),\rho(y,x)\} + \rho(x,Sx) + \rho(y,Ty) \le \varphi_S(x) - \varphi_S(Sx) + \varphi_T(y) - \varphi_T(Ty).$$

Then the family $\mathcal{F}$ have a unique common fixed point.

The proof is similar to Corollary 1, and omitted.

Theorem 5. Let (X, d) be a complete metric space and let ρ be a w-distance on X. Let S, T be two upper semicontinuous mappings of X into 2^X. For any $x \in X$, Sx and Tx are nonempty and closed. Suppose that there are two functions φ, ψ of X into $[0,\infty)$ satisfying the following condition: For any two points $x, y \in X$ and for any $u \in Sx$, $v \in Ty$,

$$\max\{\rho(x,y),\rho(y,x)\} + \rho(x,u) + \rho(y,v) \le \varphi(x) - \varphi(u) + \psi(y) - \psi(v).$$

Then S and T have a unique common fixed point.

Proof. Let x_0 and y_0 be any two points of X and $\{x_n\}$ and $\{y_n\}$ be sequences satisfying

$$x_n \in Sx_{n-1},\ y_n \in Ty_{n-1}\ (n \in \mathbb{N}).$$

Then we have

$$\rho(x_{i-1},x_i) \le \varphi(x_{i-1}) - \varphi(x_i) + \psi(y_{i-1}) - \psi(y_i)$$

and

$$\rho(y_{i-1},y_i) \le \varphi(x_{i-1}) - \varphi(x_i) + \psi(y_{i-1}) - \psi(y_i)$$

for all $i \in \mathbb{N}$. In similar way to proof of Theorem 4, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Since X is complete, each of them is convergent, that is,

$$x_n \to \bar{x},\ y_n \to \bar{y}$$

as $n \to \infty$ for some $\bar{x}$, $\bar{y} \in X$. By Lemma 2, we have that $\bar{x} \in S\bar{x}$ and $\bar{y} \in T\bar{y}$.

Now,

$$\begin{aligned}\max\{\rho(\bar{x},\bar{y}),\rho(\bar{y},\bar{x})\} &\leq \max\{\rho(\bar{x},\bar{y}),\rho(\bar{y},\bar{x})\}+\rho(\bar{x},\bar{x})+\rho(\bar{y},\bar{y})\\ &\leq \varphi(\bar{x})-\varphi(\bar{x})+\psi(\bar{y})-\psi(\bar{y})=0,\end{aligned}$$

then $\rho(\bar{x},\bar{y})=\rho(\bar{y},\bar{x})=0$. And also

$$\rho(\bar{x},\bar{x})\leq\rho(\bar{x},\bar{y})+\rho(\bar{y},\bar{x})=0,$$

hence $\rho(\bar{x},\bar{y})=\rho(\bar{x},\bar{x})=0$. By Lemma 1(i), $\bar{x}=\bar{y}$. We obtain $\bar{x}\in S\bar{x}\cap T\bar{x}$.

Let $\bar{z}\in X$ satisfying $\bar{z}\in S\bar{z}$. Then

$$\begin{aligned}\max\{\rho(\bar{z},\bar{x}),\rho(\bar{x},\bar{z})\} &\leq \max\{\rho(\bar{z},\bar{x}),\rho(\bar{x},\bar{z})\}+\rho(\bar{z},\bar{z})+\rho(\bar{x},\bar{x})\\ &\leq \varphi(\bar{z})-\varphi(\bar{z})+\psi(\bar{x})-\psi(\bar{x})=0.\end{aligned}$$

In the same way, we have $\bar{z}=\bar{x}$. Thus $\bar{x}$ is the only fixed point of S. Similarly, we can show that $\bar{x}$ is the only fixed point of T. This proves the theorem. □

Corollary 4. Let (X,d) be a complete metric space and let ρ be a w-distance on X. Let $\mathcal{F}=\{S_\alpha\mid\alpha\in\Lambda\}$ be a family of upper semicontinuous mappings of X into 2^X. For any $x\in X$ and $S\in\mathcal{F}$, Sx is nonempty and closed. Suppose that there is a family $\{\varphi_S\mid S\in\mathcal{F}\}$ of functions of X into $[0,\infty)$ satisfying the following condition: For any two mappings $S,T\in\mathcal{F}$ and for any $x,y\in X$ and $u\in Sx$, $v\in Ty$,

$$\max\{\rho(x,y),\rho(y,x)\}+\rho(x,u)+\rho(y,v)\leq\varphi_S(x)-\varphi_S(u)+\varphi_T(y)-\varphi_T(v).$$

Then the mappings of $\mathcal{F}$ have a unique common fixed point.

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