ISOMORPHIC THEOREM BETWEEN ALGEBRA GENERATED BY IDEMPOTENTS AND ALGEBRA OF THEIR SYMBOLS

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Abstract. In this paper we constructed separately the algebras generated by the idempotents operators, and by nilpotents operators, and prove the isomorphic theorems between original algebras and algebras of symbols. In particular, Theorems 2.1, 2.2 prove that an operator is generalized invertible if and only if so its symbol is.

1 Introduction and preliminary In present paper, desire to have the criterion of the invertibility of linear operators belonged to the algebras generated by the idempotent operators or by the nilpotent operators is motivated by the relative studies as follows: the Gohberg-Krupnik-Sarason symbol calculus for algebras of Toeplitz, Hakel, Cauchy, and Carleman operators on L^p-spaces, the isomorphism between algebras (or C^*-algebras) of Cauchy singular integral operators with Carleman shift and algebras of symbols (or C^*-algebras in certain sense) and the Nöetherian theory of Cauchy singular integral operators with Carleman shift, generalized invertibility of Wiener-Hopf operators and the cross factorization of those operators. Many works dealing with the relationship between the invertibility of elements in an algebra and that of so-called symbols in other algebra have been appeared (see [6, 9, 13, 14, 15, 16, 17, 28, 29] and references therein). Namely, the aim of this paper is devoted to the algebras (denoted by $\widehat{X}$ through the paper) generated by the system of the complete orthogonal projectors, and by the system of the nilpotent operators defined in the linear space. What we are interested in is criterion for generalized invertibility of operators in $\widehat{X}$. As the criterion should be well verifiable, we aim the following precision: whether or not it is possible to associate with every operators $M \in \widehat{X}$ a certain-matrix function, called the symbol of M, having the characterization that M is generalized invertible if and only if so its symbol is. This is impossible in general cases as indicated in [5]. Fortunately, in the frame of this work, the answer is positive.

The paper contains two sections and organized as follows. The followings of this section are the well-known concepts concerning the generalized invertibility of linear operators. In Section 2, we make separately two algebraic structures for the sets, by means of the idempotents and nilpotent operators, so that these sets become the subalgebras (denoted by the same notation as $\widehat{X}$) of the algebra $L_0(X)$. Hereafter, for given element $K \in \widehat{X}$ we determine its symbol σ_K, and prove that $\widehat{X}$ is isomorphic to the algebra of all symbols of elements in $\widehat{X}$ (through the paper this algebra is denoted by $[\![\sigma]\!]$). Theorems 2.1, 2.3 show that the element $K \in \widehat{X}$ is generalized invertible, generalized right invertible if and only if so its symbol σ_K is, respectively. Furthermore, Theorems 2.2, 2.4 provide the necessary sufficient conditions for the matrice being the symbol of an element in $\widehat{X}$.

Let X be a linear space over the scalar field $\mathcal{F}$ ($\mathcal{F} = \mathbb{R}$, or $\mathcal{F} = \mathbb{C}$). Denoted by $L_0(X)$ the set of all operators whose domain is entire X. Obviously, $L_0(X)$ is the algebra with unit being the identity operator I. Let $\widehat{X}$ be a subalgebra of $L_0(X)$.

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We recall some definitions.

Definition 1.1 ([23, 27]). *An operator $D \in \widehat{X}$ is said to be right invertible (abbreviated RI) in $\widehat{X}$ if there is an $R \in \widehat{X}$ such that $DR = I$.*

The set of all RI-operators in $\widehat{X}$ is denoted by $\widehat{R}(\widehat{X})$.

Definition 1.2 ([23, 27]). *An operator $V \in \widehat{X}$ is said to be generalized invertible (abbreviated GI) in $\widehat{X}$ if there is a $W \in \widehat{X}$ such that $VWV = V$. Then W is called a generalized inverse of V.*

The set of all GI-operators in $\widehat{X}$ is denoted by $\widehat{W}(\widehat{X})$.

Definition 1.3 ([18, 20]). *An operator $V \in \widehat{X}$ is said to be generalized right invertible of degree r in $\widehat{X}$ (abbreviated GRI-r) if there exists a $W \in \widehat{X}$ such that $VWV = V$, $V^{r+1}W = V^r$, where $V^0 := I$. We call W a generalized right inverse of degree r of V (abbreviated GRI-r of V).*

The set of all GRI-r operators is denoted by $\widehat{R}_r(\widehat{X})$. It is clear that if $r = 0$, then the set of all GRI-0 operators is identical with the set of RI operators, i.e $\widehat{R}_0(\widehat{X}) \equiv \widehat{R}(\widehat{X})$. For any $V \in \widehat{R}_r(\widehat{X})$ let $\mathcal{R}_V^r$ denote the set of all GRI-r operators of V. Algebra $\widehat{X}$ can be classified by means of the degrees of invertibility of elements in which $\widehat{R}(\widehat{X})$ is just the first class, i.e.

$$\widehat{R}(\widehat{X}) \equiv \widehat{R}_0(\widehat{X}) \subseteq \widehat{R}_1(\widehat{X}) \subseteq \widehat{R}_2(\widehat{X}) \subseteq \dots \subseteq \widehat{R}_n(\widehat{X}) \subseteq \dots \subseteq \widehat{W}(\widehat{X}) \subseteq \widehat{X} \subseteq L_0(X).$$

It is easy to check that if W is a generalized inverse of V, then so $W_1 := WVW$ is, and the following identity yeilds: $W_1VW_1 = W_1$. The term generalized inverse, actually, is sometimes used as a synonym for pseudoinverse, or Moore-Penrose inverse, which was independently described in the works of Moore [21] and Penrose [22]. The class of generalized invertible operators is a natural generalization of the class of invertible, and one-sided invertible operators that attracts attention of many authors (see [1, 2, 3, 4, 7, 8, 10, 11, 12, 23, 24, 25, 26]). In our view, the class of generalized invertible operators deserves the interest.

What folowed in this paper is focused on the second class $\widehat{R}_1(\widehat{X})$. It is worth saying that the class $\widehat{R}_1(\widehat{X})$ really contains not only some well-known operators in analysis such as projectors, integral-differential operators but also a class of algebraic operators (see [18, 19, 20, 30]).

2 Main results

2.1 Algebra generated by the orthogonal projectors Let $l(X)$ denote the subalgebra in $L_0(X)$, and let $\{P_i\}_{i=\overline{1,n}}$ be the complete system of orthogonal projectors in $L_0(X)$, i.e.

$$\begin{cases} P_iP_j = \delta_{ij}P_j, \quad i,j = 1,\dots,n \\ \sum\limits_{i=1}^{n} P_i = I, \end{cases} \tag{2.1}$$

where δ_{ij} is the Kronecker symbol. Assume that the system $\{P_i\}_{i=\overline{1,n}}$ is linear independent on algebra $l(X)$, i.e. if $\sum\limits_{k=1}^{n} A_kP_k = 0$ $(A_k \in l(X),\ k = 1,\dots,n)$, then $A_k = 0$ for every

$k = 1, \dots, n$. Obviously, if $A \in l(X)$ and if $AP_k = 0$, then $A = 0$. We now set

$$\widehat{X} = \left\{ B = \sum_{i=1}^{n} B_i P_i : B_i \in l(X) \right\}. \tag{2.2}$$

Loosely speaking, $\widehat{X}$ is generated by $\{P_i\}_{i=\overline{1,n}}$ with respect to $l(X)$. Assume that for any $M \in l(X)$ and for each of P_i $(i = 1, \dots, n)$, there exists a unique system $\{M_k\}_{k=\overline{1,n}} \in l(X)$ such that $P_i M = \sum\limits_{k=1}^{n} M_k P_k$. Then, for $M = \sum\limits_{j=1}^{n} M_j P_j \in l(X)$, we have

$$\begin{aligned} M &= \sum_{i=1}^{n} P_i \sum_{j=1}^{n} M_j P_j = \sum_{i=1}^{n} \sum_{j=1}^{n} P_i M_j P_j \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} \sum_{k=1}^{n} M_{ijk} P_k P_j = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} M_{ijj} \Big) P_j, \end{aligned} \tag{2.3}$$

where $\{M_{ijk}\}_{k=\overline{1,n}} \in l(X)$. In the sequel, we write $M_{ij} := M_{ijj}$ $(i, j = 1, \dots, n)$ as there is no danger of confusion, and set $\sigma_M = [M_{ij}]_{i,j=\overline{1,n}}$. Clearly, σ_M is the square-matrix of order n. We call σ_M the symbol of M. Let $[\![\sigma]\!]$ denote the set of all symbols of operators in $\widehat{X}$.

Theorem 2.1. *$M \in \widehat{X}$ is GI, GRI-1 in $\widehat{X}$ if and only if so σ_M is in $[\![\sigma]\!]$, respectively.*

To prove Theorem 2.1 we need the following lemmas.

Lemma 2.1. *$\widehat{X}$ is the subalgebra of $L_0(X)$.*

Proof. Obviously, $\widehat{X}$ is the linear space. Let $A = \sum\limits_{i=1}^{n} A_i P_i$, $B = \sum\limits_{i=1}^{n} B_i P_i$, where $A_i, B_i \in l(X)$. We have

$$\begin{aligned} AB &= \sum_{i=1}^{n} \sum_{j=1}^{n} A_i P_i B_j P_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} A_i B_{ijk} P_k P_j \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} A_i B_{ijj} P_j = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} A_i B_{ijj} \Big) P_j = \sum_{j=1}^{n} C_j P_j, \end{aligned} \tag{2.4}$$

where $C_j = \sum\limits_{i=1}^{n} A_i B_{ijj} \in l(X)$, $j = 1, \dots n$. This implies $AB \in \widehat{X}$. Thus, $\widehat{X}$ is an algebra. The lemma is proved. $\square$

Remark 2.1. *$l(X)$ is the subalgebra of $\widehat{X}$ as $A = \sum\limits_{i=1}^{n} AP_i$ for $A \in l(X)$.*

Lemma 2.2. *$[\![\sigma]\!]$ has an algebraic structure.*

Proof. Obviously, $[\![\sigma]\!]$ is a linear space. Let σ_A, σ_B denote the symbols of $A = \sum\limits_{i=1}^{n} A_i P_i$, $B = \sum\limits_{i=1}^{n} B_i P_i \in \widehat{X}$ in $\widehat{X}$ respectively. We then have $\sigma_A = [A_{ikk}]_{i,k=\overline{1,n}}$, $\sigma_B = [B_{ikk}]_{i,k=\overline{1,n}}$, where

$A_{ikk}, B_{ikk},\ \ i,k = 1,2,\dots,n$ are determined as in (2.3). We shall prove that $\sigma_A\sigma_B = \sigma_{AB} \in [\![\sigma]\!]$. Indeed, suppose that $\sigma_A\sigma_B = [C_{ik}]_{i,k=\overline{1,n}}$. Due to conditions (2.3), we have

$$C_{ik} = \sum_{j=1}^{n} A_{ijj}B_{jkk},\ \ i,k = 1,2,\dots,n. \tag{2.5}$$

On the other hand,

$$\begin{aligned} AB &= \sum_{j=1}^{n}\sum_{k=1}^{n} A_jP_jB_kP_k = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} P_iA_jP_jB_kP_k \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} A_{ijl}P_lP_jB_kP_k = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} A_{ijj}P_jB_kP_k \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} A_{ijj}B_{jkl}P_lP_k = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} A_{ijj}B_{jkk}P_k \\ &= \sum_{k=1}^{n}\sum_{i=1}^{n}\Big(\sum_{j=1}^{n} A_{ijj}B_{jkk}\Big)P_k = \sum_{k=1}^{n}\Big(\sum_{i=1}^{n} C^*_{ik}\Big)P_k, \end{aligned}$$

where

$$C^*_{ik} = \sum_{j=1}^{n} A_{ijj}B_{jkk}. \tag{2.6}$$

This implies $C_{ik} = C^*_{ik},\ \ i,k = 1,2,\dots,n$. Hence, $[C_{ik}]_{i,k=\overline{1,n}} = [A][B]$ is the symbol of $AB \in l(X)$. The lemma is proved. □

Corollary 2.1. *Algebra $\widehat{X}$ is isomorphic to $[\![\sigma]\!]$.*

Proof. Consider the map

$$\begin{aligned} \delta : \widehat{X} &\longrightarrow [\![\sigma]\!] \\ A &\longrightarrow \sigma_A. \end{aligned}$$

By the definition of $[\![\sigma]\!]$, δ is the linear map from $\widehat{X}$ onto $[\![\sigma]\!]$. Obviously, if $\delta(A) = 0$, i.e. $\sigma_A = 0$, then $A = 0$. Hence, δ is the linear isomorphism beetwen $\widehat{X}$ and $[\![\sigma]\!]$. Thanks to the proof of Lemma 2.2, $\delta(AB) = \sigma_A\sigma_B = \delta(A)\delta(B)$. Thus, δ is the algebraic isomorphism. The corollary is proved. □

Let $[A_{ij}]_{i,j=\overline{1,n}}$ be a matrix of order n whose elements belong to $l(X)$. Put $A^*_j = \sum_{i=1}^{n} A_{ij},\ j = 1,\dots,n$. Suppose that

$$P_iA^*_j = \sum_{k=1}^{n} A^*_{ijk}P_k,\ i,j = 1,\dots,n. \tag{2.7}$$

Theorem 2.2 below gives the necessary and sufficient condition for the square-matrix being symbol of an operator in $\widehat{X}$.

Theorem 2.2. *Matrix $[A_{ij}]_{i,j=\overline{1,n}}$ is the symbol of an operator in $\widehat{X}$ if and only if $A^*_{ijj} = A_{ij}$ for every $i,j = 1,\dots,n$, where A^*_{ijj} are defined as in (2.7).*

Proof. Necessity. Suppose that $[A_{ij}]_{i,j=\overline{1,n}}$ is the symbol of $A = \sum_{j=1}^{n} K_j P_j$. We have

$$\begin{aligned} A &= \sum_{i=1}^{n} P_i \sum_{j=1}^{n} K_j P_j = \sum_{i=1}^{n}\sum_{j=1}^{n} P_i K_j P_j \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} K_{ijk} P_k P_j = \sum_{j=1}^{n}\Big(\sum_{i=1}^{n} K_{ijj}\Big) P_j. \end{aligned}$$

It follows that $\sigma_A = [K_{ijj}]_{i,j=\overline{1,n}} = [A_{ij}]_{i,j=\overline{1,n}}$. Hence, $A_{ij} = K_{ijj}$ $(i,j = 1,\dots,n)$. Furthermore,

$$\begin{aligned} A &= \sum_{j=1}^{n}\Big(\sum_{i=1}^{n} K_{ijj}\Big) P_j = \sum_{j=1}^{n}\Big(\sum_{i=1}^{n} A_{ij}\Big) P_j = \sum_{j=1}^{n} A_j^* P_j \\ &= \sum_{i=1}^{n} P_i \sum_{j=1}^{n} A_j^* P_j = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} A_{ijk}^* P_k P_j = \sum_{j=1}^{n}\Big(\sum_{i=1}^{n} A_{ijj}^*\Big) P_j. \end{aligned}$$

It implies that $\sigma_A = [A_{ijj}^*]_{i,j=\overline{1,n}} = [A_{ij}]_{i,j=\overline{1,n}}$. Thus, $A_{ij} = A_{ijj}^*$, $(i,j = 1,\dots,n)$.

Sufficiency. Suppose that $A_{ijj}^* = A_{ij}, i,j = 1,\dots,n$. Set $A = \sum_{j=1}^{n} A_j^* P_j$. We have

$$\begin{aligned} A &= \sum_{i=1}^{n}\sum_{j=1}^{n} P_i A_j^* P_j \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} A_{ijk}^* P_k P_j = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ijj}^* P_j \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} P_j = \sum_{j=1}^{n}\Big(\sum_{i=1}^{n} A_{ij}\Big) P_j. \end{aligned}$$

Thus, $[A_{ij}]_{i,j=\overline{1,n}}$ is the symbol of A. The theorem is proved. □

PROOF OF THEOREM 2.1.

Necessity. Suppose that $MWM = M$. By Corollary 2.1,

$$\sigma_M = \delta(M) = \delta(MWM) = \delta(M)\delta(W)\delta(M) = \sigma_M \sigma_W \sigma_M.$$

Moreover, if $M^2 W = M$, then

$$\sigma_M = \delta(M) = \delta(M^2 W) = \delta(M)\delta(M)\delta(W) = \sigma_M^2 \sigma_M.$$

Sufficiency. Suppose that σ_M is the symbol of M, and σ_M is GI in $[\![\sigma]\!]$. There exists $\sigma_W := [W_{kl}]_{k,l=\overline{1,n}} \in [\![\sigma]\!]$ such that $\sigma_M \sigma_W \sigma_M = \sigma_M$. By Corollary 2.1, we have

$$M = \delta^{-1}(\sigma_M) = \delta^{-1}(\sigma_M \sigma_W \sigma_M) = \delta^{-1}(\sigma_M)\delta^{-1}(\sigma_W)\delta^{-1}(\sigma_M) = MWM.$$

Moreover, if $\sigma_M^2 \sigma_W = \sigma_M$, then

$$M = \delta^{-1}(\sigma_M) = \delta^{-1}(\sigma_M^2 \sigma_W) = \delta^{-1}(\sigma_M)\delta^{-1}(\sigma_M)\delta^{-1}(\sigma_W) = M^2 W.$$

Theorem 2.1 is proved. □

2.2 Algebra generated by the nilpotent operators Let $\{Q_k^i\}_{k=\overline{1,n}}^{i=\overline{0,r_i-1}}$ be the N $(= r_1 + r_2 + r_3 + \cdots + r_n)$ operators in $L_0(X)$ satisfying the following condition

$$(2.8)\qquad \begin{cases} Q_i Q_j = \delta_{ij} Q_j^2, \\ Q_i^{r_i} = 0, \quad (\text{if } r_i > 1), \\ Q_i^0 Q_j^0 = \delta_{ij} Q_j^0, \\ \sum\limits_{j=1}^{n} Q_j^0 = I \end{cases}$$

(note that the Q_j^0 denotes the element in $L_0(X)$, not the power of zero-order of $Q_j, j = 1, \ldots, n$, see [27, p. 65]). Assume that the system $\{Q_k^i\}_{k=\overline{1,n}}^{i=\overline{0,r_i-1}}$ are independent on $l(X)$. We now set

$$(2.9)\qquad \widehat{X} = \left\{ A = \sum_{k=1}^{n} \sum_{i=0}^{r_k-1} A_{ki} Q_k^i : \; A_{ki} \in l(X) \right\}.$$

Roughly speaking, $\widehat{X}$ is generated by $\{Q_k^i\}_{k=\overline{1,n}}^{i=\overline{0,r_k-1}}$ with respect to $l(X)$. Suppose that for any $T \in l(X)$ and for $k = 1, \ldots, n, \; j = 0, 1, \ldots, r_k - 1$ there exist a unique system of operators $\{T_{ki}^{(m,\mu)}\}_{m=\overline{1,n}}^{\mu=\overline{0,r_m-1}} \in l(X)$ such that

$$(2.10)\qquad Q_k^i T = \sum_{m=1}^{n} \sum_{\mu=0}^{r_m-1} T_{ki}^{(m,\mu)} Q_m^\mu.$$

By this assumption, if $T = \sum\limits_{k=1}^{n} \sum\limits_{j=0}^{r_k-1} T_{kj} Q_k^j \in l(X)$, then there exist $T_{kilj}^{(m,\mu)} \in \widehat{X}$ such that $Q_k^i T_{lj} = \sum\limits_{m=1}^{n} \sum\limits_{\mu=0}^{r_m-1} T_{kilj}^{(m,\mu)} Q_m^\mu$. Put $\widehat{T}_{i,\nu}^{(k,l)} = \sum\limits_{j=0}^{\nu} T_{kilj}^{(l,\nu-j)}, \quad k,l = 1, \ldots, n; \; i = 0, 1, \ldots, r_k - 1; \; \nu = 0, 1, \ldots, r_l - 1$. Write

$$\widehat{T}^{(k,l)} = \left[\widehat{T}_{i,\nu}^{(k,l)} \right]_{i=\overline{0,r_k-1}}^{\nu=\overline{0,r_l-1}}, \; k,l = 1, \ldots n.$$

It is clear that $\widehat{T}^{(k,l)}$ is the matrix of order $r_k \times r_l$ whose elements belong to $\widehat{X}$. We denote

$$\sigma_T := \left[\widehat{T}^{(k,l)} \right]_{k,l=\overline{1,n}} = \left[T_{i,\nu}^{(k,l)} \right]_{\substack{k=1,\ldots,n \\ i=\overline{0,r_k-1}}}^{\substack{l=1,\ldots,n \\ \nu=\overline{0,r_l-1}}}.$$

Obviously, σ_T is the square-matrix of order N whose elements belong to $\widehat{X}$. We call σ_T the symbol of T. Let $[\![\sigma]\!]$ denote the set of symbols of all operators in $\widehat{X}$.

Theorem 2.3. *$T \in \widehat{X}$ is GI, GRI-1 in $\widehat{X}$ if and only if so is σ_T in $[\![\sigma]\!]$, respectively.*

We need the following lemmas.

Lemma 2.3. *$\widehat{X}$ is an subalgebra of $L_0(X)$.*

Proof. Let $A=\sum\limits_{k=1}^{n}\sum\limits_{i=0}^{r_k-1}A_{ki}Q_k^i$, $B=\sum\limits_{l=1}^{n}\sum\limits_{j=0}^{r_l-1}B_{lj}Q_l^j$. We have

$$AB=\sum_{k=1}^{n}\sum_{i=0}^{r_k-1}\sum_{l=1}^{n}\sum_{j=0}^{r_l-1}A_{ki}Q_k^iB_{lj}Q_l^j=\sum_{k=1}^{n}\sum_{i=0}^{r_k-1}\sum_{l=1}^{n}\sum_{j=0}^{r_l-1}\sum_{m=1}^{n}\sum_{\mu=0}^{r_m-1}A_{ki}B_{kilj}^{(m,\mu)}Q_m^\mu Q_l^j$$
$$=\sum_{l=1}^{n}\sum_{j=0}^{r_l-1}\Big(\sum_{k=1}^{n}\sum_{i=0}^{r_k-1}\sum_{\mu=0}^{r_l-1}A_{ki}B_{kilj}^{(l,\mu)}\Big)Q_l^j.$$

Hence, $AB\in\widehat{X}$. The lemma is proved. □

Lemma 2.4. *Let $T=\sum\limits_{k=1}^{n}\sum\limits_{j=0}^{r_k-1}T_{kj}Q_k^j$, and let $\sigma_T=\Big[T_{i,\nu}^{(k,l)}\Big]_{k=\overline{1,2,\dots,n}\atop i=\overline{0,r_k-1}}^{l=\overline{1,2,\dots,n}\atop \nu=\overline{0,r_l-1}}$ be the symbol of T. Then T can be represented of the form*

$$T=\sum_{k=1}^{n}\sum_{\mu=0}^{r_k-1}\Big(\sum_{j=1}^{n}\widehat{T}_{0,\mu}^{(j,k)}\Big)Q_k^\mu.$$

Proof. We have

$$T=\sum_{k=1}^{n}\sum_{i=0}^{r_k-1}T_{ki}Q_k^i=\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=0}^{r_k-1}Q_j^0T_{ki}Q_k^i=\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=0}^{r_k-1}\sum_{l=1}^{n}\sum_{s=0}^{r_l-1}T_{j0ki}^{(l,s)}Q_l^sQ_k^i$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=0}^{r_k-1}\sum_{s=0}^{r_k-1}T_{j0ki}^{(k,s)}Q_k^{s+i}=\sum_{k=1}^{n}\sum_{\mu=0}^{r_k-1}\sum_{j=1}^{n}\Big(\sum_{i=0}^{\mu}T_{j0ki}^{(k,\mu-i)}\Big)Q_k^\mu$$
$$=\sum_{k=1}^{n}\sum_{\mu=0}^{r_k-1}\sum_{j=1}^{n}\widehat{T}_{0,\mu}^{(j,k)}Q_k^\mu.$$

The lemma is proved. □

Let $M=\Big[M_{i,\nu}^{(k,l)}\Big]_{k=\overline{1,2,\dots,n}\atop i=\overline{0,r_k-1}}^{l=\overline{1,2,\dots,n}\atop \nu=\overline{0,r_l-1}}$ be the square-matrix of order N whose elements belong to $\widehat{X}$. Put $M_{lj}^*=\sum\limits_{i=1}^{n}M_{0,j}^{(i,l)}$. Suppose that

$$Q_k^iM_{lj}^*=\sum_{m=1}^{n}\sum_{\mu=0}^{r_m-1}(M^*)_{kilj}^{(m,\mu)}Q_m^\mu,\quad \widehat{(M^*)}_{i,\nu}^{(k,l)}=\sum_{j=0}^{\nu}(M^*)_{kilj}^{(l,\nu-j)}.$$

Theorem 2.4. *M is the symbol of $T\in\widehat{X}$ if and only if $\widehat{(M^*)}_{i,\nu}^{(k,l)}=M_{i,\nu}^{(k,l)}$, $k,l=1,\dots,n;$ $i=0,1,\dots,r_k-1;$ $\nu=0,1,\dots,r_l-1$.*

Proof. Necessity. Suppose that M is the symbol of $T\in X$. By Lemma 2.4,

$$T=\sum_{l=1}^{n}\sum_{j=0}^{r_l-1}\Big(\sum_{i=0}^{n}M_{0,j}^{(i,l)}\Big)Q_l^j=\sum_{l=1}^{n}\sum_{j=0}^{r_l-1}M_{lj}^*Q_l^j.$$

Then $\sigma_T = \left[\widehat{(M^*)}^{(k,l)}_{i,\nu}\right]^{l=1,2,\dots,n}_{\substack{\nu=\overline{0,r_l-1}\\ k=1,2,\dots,n\\ i=\overline{0,r_k-1}}}$. Hence, $\widehat{(M^*)}^{(k,l)}_{i,\nu} = M^{(k,l)}_{i,\nu},\ k,l=1,\dots,n;\ i=0,1,\dots,r_k-1;\ \nu=0,1,\dots,r_l-1.$

Sufficiency. Consider $T = \sum\limits_{l=1}^{n}\sum\limits_{j=0}^{r_l-1} M^*_{lj}Q^j_l$. We then have

$$\sigma_T = \left[\widehat{(M^*)}^{(k,l)}_{i,\nu}\right]^{\substack{l=1,2,\dots,n\\ \nu=\overline{0,r_l-1}}}_{\substack{k=1,2\dots,n\\ i=\overline{0,r_k-1}}}.$$

As $\widehat{(M^*)}^{(k,l)}_{i,\nu} = M^{(k,l)}_{i,\nu},\ k,l=1,\dots,n;\ i=0,1,\dots,r_k-1;\ \nu=0,1,\dots,r_l-1$, M is the symbol of T. The theorem is proved. □

Lemma 2.5. *$[\![\sigma]\!]$ has an algebraic structure.*

Proof. It is sufficient to prove that $\sigma_A\sigma_B = \sigma_{AB}$ for any $A,B\in\widehat{X}$. Let us first prove the following identities for the basic elements Q^0_p, Q_p and an arbitray element $A\in l(X)$. Namely, we shall prove two identities:

$$\sigma_A\sigma_{Q_p} = \sigma_{AQ_p},\ p=1,\dots,n. \tag{2.11}$$

$$\sigma_A\sigma_{Q^0_p} = \sigma_{AQ^0_p},\ p=1,\dots,n. \tag{2.12}$$

Proof of (2.11). We determine the symbols of three elements AQ_p, A, Q_p. We have

$$AQ_p = \sum_{l=1}^{n}\sum_{j=0}^{r_l-1}\delta_{lp}\delta_{j1}AQ^j_l,\quad Q^i_k(\delta_{lp}\delta_{j1}A) = \sum_{m=1}^{n}\sum_{\mu=0}^{r_m-1}\delta_{lp}\delta_{j1}A^{(m,\mu)}_{ki}Q^\mu_m,$$

$$\widehat{(AQ_p)}^{(k,l)}_{i,\nu} = \sum_{j=0}^{\nu}\delta_{lp}\delta_{j1}A^{(l,\nu-j)}_{ki} = \delta_{lp}A^{(l,\nu-1)}_{ki}.$$

where we admit $A^{(l,0-1)}_{ki}=0$. Hence, $\sigma_{AQ_p} = \left[\delta_{lp}A^{(l,\nu-1)}_{ki}\right]^{\substack{l=1,2,\dots,n\\ \nu=\overline{0,r_l-1}}}_{\substack{k=1,2,\dots,n\\ i=\overline{0,r_k-1}}}$. Similarly,

$$A = \sum_{l=1}^{n}\sum_{j=0}^{r_l-1}\delta_{j0}AQ^j_l,\quad Q^i_k(\delta_{j0}A) = \sum_{m=1}^{n}\sum_{\mu=0}^{r_m-1}\delta_{j0}A^{(m,\mu)}_{ki}Q^\mu_m,$$

$$\widehat{A}^{(k,l)}_{i,\nu} = \sum_{j=0}^{\nu}\delta_{j0}A^{(l,\nu-1)}_{ki} = A^{(l,\nu)}_{ki}.$$

We then have $\sigma_A = \left[A^{(l,\nu)}_{ki}\right]^{\substack{l=1,2,\dots,n\\ \nu=\overline{0,r_l-1}}}_{\substack{k=1,2,\dots,n\\ i=\overline{0,r_k-1}}}$. Finally,

$$Q_p = \sum_{l=1}^{n}\sum_{j=0}^{r_l-1}\delta_{lp}\delta_{j1}Q^j_l,\qquad Q^i_k(\delta_{lp}\delta_{j1}) = \sum_{m=1}^{n}\sum_{\mu=0}^{r_l-1}\delta_{lp}\delta_{j1}\delta_{mk}\delta_{\mu i}Q^\mu_m$$

$$\widehat{(Q_p)}^{(k,l)}_{i,\nu} = \sum_{j=0}^{\nu}\delta_{lp}\delta_{j1}\delta_{lk}\delta_{(\nu-j)i} = \delta_{lp}\delta_{lk}\sum_{j=0}^{\nu}\delta_{j1}\delta_{(\nu-j)i} = \delta_{lp}\delta_{lk}\delta_{(\nu-1)i},$$

where we admit $\delta_{(0-1)i} \equiv \delta_{-1i} = 0$. Thus, $\sigma_{Q_p} = \Big[\delta_{lp}\delta_{lk}\delta_{(\nu-1)i}\Big]_{k=\overline{1,\dots,n}\atop i=\overline{0,r_k-1}}^{l=\overline{1,\dots,n}\atop \nu=\overline{0,r_l-1}}$. By the role of multiplication of two matrices, and by

$$\sum_{m=1}^{n}\sum_{\mu=0}^{r_m-1} A_{ki}^{(m,\mu)}\delta_{lp}\delta_{lm}\delta_{(\nu-1)\mu} = \delta_{lp}A_{ki}^{(l,\nu-1)},$$

we have $\sigma_A\sigma_{Q_p} = \sigma_{AQ_p}$.

Proof of (2.12). Replacing δ_{j1} with δ_{j0} in above identities, we get

$$\sigma_{AQ_p^0} = \Big[\delta_{lp}A_{ki}^{(l,\nu)}\Big]_{k=\overline{1,2,\dots,n}\atop i=\overline{0,r_k-1}}^{l=\overline{1,2,\dots,n}\atop \nu=\overline{0,r_l-1}}, \quad \sigma_{Q_p^0} = \Big[\delta_{lp}\delta_{lk}\delta_{\nu i}\Big]_{k=\overline{1,\dots,n}\atop i=\overline{0,r_k-1}}^{l=\overline{1,\dots,n}\atop \nu=\overline{0,r_l-1}}.$$

Since $\sum\limits_{m=1}^{n}\sum\limits_{\mu=0}^{r_m-1} A_{ki}^{(m,\mu)}\delta_{lp}\delta_{lm}\delta_{\nu\mu} = \delta_{lp}A_{ki}^{(l,\nu)}$, we get $\sigma_{AQ_p^0} = \sigma_A\sigma_{Q_p^0}$.

The identity $\sigma_A\sigma_B = \sigma_{AB}$ for every $A, B \in \widehat{X}$ now is an immediate consequence of (2.9), (2.11), (2.12). The lemma is proved. □

Corollary 2.2. *Algebra $\widehat{X}$ is isomorphic to the algebra $[\![\sigma]\!]$.*

Proof. By the natural multiplication of matrices and Lemma 2.5, we have $\sigma_A\sigma_B = \sigma_{AB} \in [\![\sigma]\!]$. Consider the map

$$\begin{aligned} \delta : \widehat{X} &\longrightarrow [\![\sigma]\!] \\ A &\longrightarrow \sigma_A, \end{aligned}$$

where σ_A is the symbol of $A \in \widehat{X}$.

Obviously, δ is the linear map from $\widehat{X}$ onto $[\![\sigma]\!]$, and $\delta(A) = 0$ if and only if $A = 0$. Moreover, it is clearly that $\delta(AB) = \delta(A)\delta(B)$ for every $A, B \in \widehat{X}$. □

PROOF OF THEOREM 2.3

The proof follows immediately from Lemma 2.5. Indeed, since the map

$$\begin{aligned} \delta : \widehat{X} &\longrightarrow [\![\sigma]\!] \\ A &\longrightarrow \sigma_A \end{aligned}$$

is the morphism, we have

1. $TMT = T$ if and only if $\sigma_T\sigma_M\sigma_T = \sigma_T$.
2. $T^2M = T$ if and only if $\sigma_T^2\sigma_M = \sigma_T$.

Theorem 2.3 is proved. □

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