EXISTENCE THEORY FOR IMPULSIVE PARTIAL HYPERBOLIC FUNCTIONAL DIFFERENTIAL EQUATIONS INVOLVING THE CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this paper we investigate the existence and uniqueness of solutions of a class of partial impulsive hyperbolic differential equations with fixed time impulses involving the Caputo fractional derivative. Our main tool is a fixed point theorem.

1 Introduction This paper concerns the existence results to fractional order initial value problems (IVP for short), for the system

$$(1)\qquad ({}^cD_0^r u)(x,y)=f(x,y,u(x,y)),\ \text{if } (x,y)\in J,\ x\neq x_k,\ k=1,\dots,m,$$

$$(2)\qquad u(x_k^+,y)=u(x_k^-,y)+I_k(u(x_k^-,y)),\quad \text{if } y\in[0,b],\ k=1,\dots,m,$$

$$(3)\qquad u(x,0)=\varphi(x),\ u(0,y)=\psi(y),\quad \text{if } x\in[0,a]\ \text{ and } y\in[0,b],$$

where $J=[0,a]\times[0,b],\ a,b>0,\ \ {}^cD_0^r$ is the fractional Caputo derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1],\ 0=x_0<x_1<\cdots<x_m<x_{m+1}=a,\ f:J\times\mathbb{R}^n\to\mathbb{R}^n$ and $I_k:\mathbb{R}^n\to\mathbb{R}^n,\ k=0,1,\dots,m$ are given functions, $\varphi:[0,a]\to\mathbb{R}^n,\ \psi:[0,b]\to\mathbb{R}^n$ are absolutely continuous functions with $\varphi(0)=\psi(0)$.

Next we consider the following nonlocal initial value problem

$$(4)\qquad ({}^cD_0^r u)(x,y)=f(x,y,u(x,y)),\ \text{if } (x,y)\in J,\ x\neq x_k,\ k=1,\dots,m,$$

$$(5)\qquad u(x_k^+,y)=u(x_k^-,y)+I_k(u(x_k^-,y)),\quad \text{if } y\in[0,b],\ k=1,\dots,m,$$

$$(6)\qquad u(x,0)+Q(u)=\varphi(x),\ u(0,y)+K(u)=\psi(y),\quad \text{if } x\in[0,a]\ \text{ and } y\in[0,b],$$

where $f,\ \varphi,\ \psi,\ I_k;\ k=1,...m$, are as in problem (1)-(3) and $Q,K:PC(J,\mathbb{R}^n)\to\mathbb{R}^n$ are continuous functions. $PC(J,\mathbb{R}^n)$ is a Banach space to be specified later (see Section 3).

The problem of existence of solutions of Cauchy-type problems for ordinary differential equations of fractional order in spaces of integrable functions was studied in [18], a similar problem in spaces of continuous functions was studied in [31]. We can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [14, 17, 25]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see

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the monographs of Kilbas *et al.* [20], Lakshmikantham *et al.* [22], Miller and Ross [23], Podlubny [27], Samko *et al.* [29], the papers of Agarwal *et al.* [3, 4], Abbas and Benchohra [1, 2], Belarbi *et al.* [8], Benchohra *et al.* [9, 10, 12], Diethelm [14, 15], Kilbas and Marzan [19], NGuérékata [24], Shi and Zhang [30], Vityuk and Golushkov [32], Zhang [33], Zhou *et al.* [34], and the references therein.

The theory of impulsive differential equations have become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory in recent years, especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs of Benchohra *et al.* [11], Lakshmikantham *et al* [21], and Samoilenko and Perestyuk [28], and the references therein.

Very recently, some extensions to impulsive fractional order differential equations have been obtained by Agarwal *et al.* [5], Ahmad and Sivasundaram [6], Ait Dads *et al.* [7], Benchohra and Slimani [13].

In this paper, we shall present existence and uniqueness results for our problems. Our results initiate the study of hyperbolic fractional differential equations subject to impulsive effect. We present two results for the problem (1)-(3), the first one is based on Banach's contraction principle (Theorem 3.4) and the second one on the nonlinear alternative of Leray-Schauder type (Theorem 3.5). As an extension to nonlocal problems, we present two similar results for the problem (4)-(6). Finally we present an illustrative example.

2 Preliminaries In this section, we introduce notations and definitions which are used throughout this paper. By $L^1(J,\mathbb{R}^n)$ we denote the space of Lebesgue-integrable functions $f : J \to \mathbb{R}^n$ with the norm

$$\|f\|_1 = \int_0^a \int_0^b \|f(x,y)\|dydx,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.
Let $a_1 \in [0,a]$, $z^+ = (a_1,0) \in J$, $J_z = [a_1,a] \times [0,b]$, $r_1, r_2 > 0$ and $r = (r_1,r_2)$. For $f \in L^1(J_z,\mathbb{R}^n)$, the expression

$$(I^r_{z^+}f)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)dtds,$$

where $\Gamma(.)$ is the Euler gamma function, is called the left-sided mixed Riemann-Liouville integral of order r.

Denote by $D^2_{xy} := \frac{\partial^2}{\partial x \partial y}$ the mixed second order partial derivative.

Definition 2.1 ([32]). *For $f \in L^1(J_z,\mathbb{R}^n)$ where $D^2_{xy}f$ is Lebesque integrable on $[x_k,x_{k+1}] \times [0,b]$, $k = 0,\dots,m$, the Caputo fractional-order derivative of order r is defined by the expression $({}^cD^r_{z^+}f)(x,y) = (I^{1-r}_{z^+}D^2_{xy}f)(x,y)$.*

3 Main Results In what follows set

$$J_k := (x_k, x_{k+1}] \times [0,b].$$

To define the solutions of problems (1)-(3), we shall consider the space

$$\begin{aligned} PC(J,\mathbb{R}^n) = \big\{ & u : J \to \mathbb{R}^n : u \in C(J_k,\mathbb{R}^n);\ k = 0,1,\dots,m, \text{ and there} \\ & \text{exist } u(x_k^-,y) \text{ and } u(x_k^+,y);\ k = 1,\dots,m, \\ & \text{with } u(x_k^-,y) = u(x_k,y) \text{ for each } y \in [0,b] \big\}. \end{aligned}$$

This set is a Banach space with the norm

$$\|u\|_{PC} = \sup_{(x,y)\in J} \|u(x,y)\|.$$

Set

$$J' := J\backslash\{(x_1, y), \ldots, (x_m, y),\ y \in [0, b]\}.$$

Definition 3.1 A function $u \in PC(J, \mathbb{R}^n)$ whose r-derivative exists on J' is said to be a solution of (1)-(3) if u satisfies $({}^cD_0^r u)(x, y) = f(x, y, u(x, y))$ on J' and conditions $(2), (3)$ are satisfied.

Let $h \in C([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n)$, $z_k = (x_k, 0)$, and

$$\mu_k(x, y) = u(x, 0) + u(x_k^+, y) - u(x_k^+, 0), \quad k = 0, \ldots, m.$$

For the existence of solutions for the problem $(1) - (3)$, we need the following lemma:

Lemma 3.2 *A function $u \in AC([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n);\ k = 0, \ldots, m$ is a solution of the differential equation*

$$({}^cD^r_{z_k^+} u)(x, y) = h(x, y);\ (x, y) \in [x_k, x_{k+1}] \times [0, b],$$

if and only if $u(x, y)$ satisfies

$$u(x, y) = \mu_k(x, y) + (I^r_{z_k^+} h)(x, y);\ (x, y) \in [x_k, x_{k+1}] \times [0, b]. \tag{7}$$

Proof: Let $u(x, y)$ be a solution of

$$({}^cD^r_{z_k^+} u)(x, y) = h(x, y);\ (x, y) \in [x_k, x_{k+1}] \times [0, b].$$

Then, taking into account the definition of the derivative $({}^cD^r_{z_k^+} u)(x, y)$, we have

$$I^{1-r}_{z_k^+}(D^2_{xy} u)(x, y) = h(x, y).$$

Hence, we obtain

$$I^r_{z_k^+}(I^{1-r}_{z_k^+} D^2_{xy} u)(x, y) = (I^r_{z_k^+} h)(x, y),$$

then

$$I^1_{z_k^+} D^2_{xy} u(x, y) = (I^r_{z_k^+} h)(x, y).$$

Since

$$I^1_{z_k^+}(D^2_{xy} u)(x, y) = u(x, y) - u(x, 0) - u(x_k^+, y) + u(x_k^+, 0),$$

we have

$$u(x, y) = \mu_k(x, y) + (I^r_{z_k^+} h)(x, y).$$

Now let $u(x, y)$ satisfies (7). It is clear that $u(x, y)$ satisfy

$$({}^cD^r_{z_k^+} u)(x, y) = h(x, y),\ \text{on } [x_k, x_{k+1}] \times [0, b].$$

In all what follows set

$$\mu_0(x, y) = \mu(x, y),\ (x, y) \in J.$$

Lemma 3.3 *Let $0 < r_1, r_2 \leq 1$ and let $h : J \to \mathbb{R}^n$ be continuous. A function u is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases} \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds; \\ \ \textit{if } (x,y) \in [0,x_1] \times [0,b], \\ \mu(x,y) + \sum_{i=1}^k (I_i(u(x_i^-,y)) - I_i(u(x_i^-,0))) \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds \\ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds; \\ \ \textit{if } (x,y) \in (x_k, x_{k+1}] \times [0,b], \ k = 1, \ldots, m, \end{cases} \tag{8}$$

if and only if u is a solution of the fractional IVP

$$^cD^r_{z_k^+} u(x,y) = h(x,y), \quad (x,y) \in J', \quad k = 1, \ldots, m, \tag{9}$$

$$u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)), \ y \in [0,b], \quad k = 1, \ldots, m. \tag{10}$$

Proof. Assume u satisfies (9)-(10). If $(x,y) \in [0,x_1] \times [0,b]$ then

$$^cD^r_0 u(x,y) = h(x,y).$$

Lemma 3.2 implies

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds.$$

If $(x,y) \in (x_1, x_2] \times [0,b]$ then Lemma 3.2 implies

$$\begin{aligned} u(x,y) &= \mu_1(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds \\ &= u(x,0) + u(x_1^+, y) - u(x_1^+, 0) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds \\ &= u(x,0) + u(x_1^-, y) - u(x_1^-, 0) + I_1(u(x_1^-, y)) - I_1(u(x_1^-, 0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds \\ &= u(x,0) + u(x_1, y) - u(x_1, 0) + I_1(u(x_1^-, y)) - I_1(u(x_1^-, 0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds \\ &= \mu(x,y) + I_1(u(x_1^-, y)) - I_1(u(x_1^-, 0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_0^y (x_1-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds. \end{aligned}$$

If $(x,y) \in (x_2, x_3] \times [0,b]$ then from Lemma 3.2 we get

$$\begin{aligned}
u(x,y) &= \mu_2(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_2}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds \\
&= u(x,0) + u(x_2^+, y) - u(x_2^+, 0) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_2}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds \\
&= u(x,0) + u(x_2^-, y) - u(x_2^-, 0) + I_2(u(x_2^-, y)) - I_2(u(x_2^-, 0)) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_2}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds \\
&= u(x,0) + u(x_2, y) - u(x_2, 0) + I_2(u(x_2^-, y)) - I_2(u(x_2^-, 0)) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_2}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds \\
&= \mu(x,y) + I_2(u(x_2^-, y)) - I_2(u(x_2^-, 0)) + I_1(u(x_1^-, y)) - I_1(u(x_1^-, 0)) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{0}^{x_1} \int_0^y (x_1-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_0^y (x_2-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_2}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s,t)dtds.
\end{aligned}$$

If $(x,y) \in (x_k, x_{k+1}] \times [0,b]$ then again from Lemma 3.2 we get (8).

Conversely, assume that u satisfies the impulsive fractional integral equation (8). If $(x,y) \in [0, x_1] \times [0,b]$ and using the fact that ${}^cD^r_{z_k^+}$ is the left inverse of $I^r_{z_k^+}$ we get

$${}^cD^r_0 u(x,y) = h(x,y), \quad \text{for each } (x,y) \in [0,x_1] \times [0,b].$$

If $(x,y) \in [x_k, x_{k+1}) \times [0,b]$, $k = 1, \ldots, m$ and using the fact that ${}^cD^r_{z_k^+} C = 0$, where C is a constant, we get

$${}^cD^r_{z_k^+} u(x,y) = h(x,y), \text{for each } (x,y) \in [x_k, x_{k+1}) \times [0,b].$$

Also, we can easily show that

$$u(x_k^+, y) = u(x_k^-, y) + I_k(u(x_k^-, y)), \quad y \in [0,b], k = 1, \ldots, m.$$

Our first result is based on Banach fixed point theorem.

Theorem 3.4 *Assume that*

(H1) There exists a constant $l > 0$ such that

$$\|f(x,y,u) - f(x,y,\overline{u})\| \le l\|u - \overline{u}\|, \ \textit{for each } (x,y) \in J, \textit{and each } u, \overline{u} \in \mathbb{R}^n.$$

(H2) There exists a constant $l^* > 0$ such that

$$\|I_k(u) - I_k(\overline{u})\| \le l^*\|u - \overline{u}\|, \ \textit{for each } u, \overline{u} \in \mathbb{R}^n, \ k = 1, \ldots, m.$$

If

$$2ml^* + \frac{2la^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1, \tag{11}$$

then (1)-(3) *has a unique solution on* J.

Proof. We transform the problem (1)-(2) into a fixed point problem. Consider the operator $F : PC(J,\mathbb{R}^n) \to PC(J,\mathbb{R}^n)$ defined by

$$\begin{aligned} F(u)(x,y) &= \mu(x,y) + \sum_{0<x_k<x} (I_k(u(x_k^-,y)) - I_k(u(x_k^-,0))) \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{0<x_k<x} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t))dtds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t))dtds. \end{aligned}$$

Clearly, the fixed points of the operator F are solution of the problem (1)-(3). We shall use the Banach contraction principle to prove that F has a fixed point. We shall show that F *is a contraction.* Let $u, v \in PC(J,\mathbb{R}^n)$. Then, for each $(x,y) \in J$, we have

$$\begin{aligned} &\|F(u)(x,y) - F(v)(x,y)\| \\ &\leq \sum_{k=1}^{m} (\|I_k(u(x_k^-,y)) - I_k(v(x_k^-,y))\| + \|I_k(u(x_k^-,0)) - I_k(v(x_k^-,0))\|) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} \|f(s,t,u(s,t)) - f(s,t,v(s,t))\| dtds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|f(s,t,u(s,t)) - f(s,t,v(s,t))\| dtds \\ &\leq \sum_{k=1}^{m} l^* (\|u(x_k^-,y) - v(x_k^-,y)\| + \|u(x_k^-,0) - v(x_k^-,0)\|) \\ &+ \frac{l}{\Gamma(r_1)\Gamma(r_2)} \sum_{k=1}^{m} \int_{x_{k-1}}^{x_k} \int_0^y (x_k - s)^{r_1-1}(y-t)^{r_2-1} \|u(s,t) - v(s,t)\| dtds \\ &+ \frac{l}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|u(s,t) - v(s,t)\| dtds \\ &\leq [2ml^* + \frac{la^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \frac{la^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}] \|u-v\|_\infty \\ &\leq \big[2ml^* + \frac{2la^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\big] \|u-v\|_\infty. \end{aligned}$$

By the condition (11), we conclude that F is a contraction. As a consequence of Banach fixed point theorem, we deduce that F has a fixed point which is a solution of the problem $(1) - (3)$.

In the following theorem we give an existence result for the problem (1)-(3) by applying the nonlinear alternative of Leray-Schauder type.

Theorem 3.5 *Let* $f(\cdot,\cdot,u) \in PC(J,\mathbb{R}^n)$ *for each* $u \in \mathbb{R}^n$. *Assume that the following conditions hold:*

(H3) There exists $\phi_f \in C(J, \mathbb{R}_+)$ *and* $\psi_* : [0, \infty) \to (0, \infty)$ *continuous and nondecreasing such that*

$$\|f(x, y, u)\| \leq \phi_f(x, y)\psi_*(\|u\|) \quad \textit{for all } (x, y) \in J,\ u \in \mathbb{R}^n.$$

(H4) There exists $\psi^* : [0, \infty) \to (0, \infty)$ *continuous and nondecreasing such that*

$$\|I_k(u)\| \leq \psi^*(\|u\|) \quad \textit{for all } u \in \mathbb{R}^n.$$

(H5) There exists an number $\overline{M} > 0$ *such that*

$$\frac{\overline{M}}{\|\mu\|_\infty + 2m\psi^*(\overline{M}) + \frac{2a^{r_1}b^{r_2}\phi_f^0\psi_*(\overline{M})}{\Gamma(r_1+1)\Gamma(r_2+1)}} > 1,$$

where $\phi_f^0 = \sup\{\phi_f(x, y) : \ (x, y) \in J\}$.

Then (1)-(3) *has at least one solution on* J.

Proof: Consider the operator F defined in Theorem 3.4.

Step 1: F *is continuous.*

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(J, \mathbb{R}^n)$. There exists $\eta > 0$ such that $\|u_n\| \leq \eta$. Then for each $(x, y) \in J$, we have

$$\begin{aligned}
&\|F(u_n)(x, y) - F(u)(x, y)\| \\
&\leq \sum_{k=1}^{m}(\|I_k(u_n(x_k^-, y)) - I_k(u(x_k^-, y))\| + \|I_k(u_n(x_k^-, 0)) - I_k(u(x_k^-, 0))\|) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x_k - s)^{r_1-1}(y - t)^{r_2-1}\|f(s, t, u_n(s, t)) - f(s, t, u(s, t))\|dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x - s)^{r_1-1}(y - t)^{r_2-1}\|f(s, t, u_n(s, t)) - f(s, t, u(s, t))\|dtds.
\end{aligned}$$

Since f and I_k, $k = 1, \ldots, m$ are continuous functions, we have

$$\|F(u_n) - F(u)\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Step 2: F *maps bounded sets into bounded sets in* $PC(J, \mathbb{R}^n)$.

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant ℓ such that for each $u \in B_{\eta^*} = \{u \in PC(J, \mathbb{R}^n) : \|u\|_\infty \leq \eta^*\}$, we have $\|F(u)\|_\infty \leq \ell$. (H4) and (H5) implies that for each $(x, y) \in J$,

$$\begin{aligned}
\|F(u)(x, y)\| &\leq \|\mu(x, y)\| + \sum_{k=1}^{m}(\|I_k(u(x_k^-, y))\| + \|I_k(u(x_k^-, 0))\|) \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^y (x_k - s)^{r_1-1}(y - t)^{r_2-1}\|f(s, t, u(s, t))\|dtds \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y (x - s)^{r_1-1}(y - t)^{r_2-1}\|f(s, t, u(s, t))\|dtds \\
&\leq \|\mu\|_\infty + 2m\psi^*(\eta^*) + \frac{2a^{r_1}b^{r_2}\phi_f^0\psi_*(\eta^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} := \ell.
\end{aligned}$$

Step 3: *F maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R}^n)$.*

Let $(\tau_1,y_1),(\tau_2,y_2)\in[0,a]\times[0,b]$, $\tau_1<\tau_2$ and $y_1<y_2$, B_{η^*} be a bounded set of $PC(J,\mathbb{R}^n)$ as in Step 2, and let $u\in B_{\eta^*}$. Then for each $(x,y)\in J$, we have

$$
\begin{aligned}
&\|F(u)(\tau_2,y_2)-F(u)(\tau_1,y_1)\|\\
&\leq \|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)\|+\sum_{k=1}^{m}(\|I_k(u(x_k^-,y_1))-I_k(u(x_k^-,y_2))\|)\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^{y_1}(x_k-s)^{r_1-1}[(y_2-t)^{r_2-1}-(y_1-t)^{r_2-1}]\\
&\times f(s,t,u(s,t))dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_{y_1}^{y_2}(x_k-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,u(s,t))\|dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}-(\tau_1-s)^{r_1-1}(y_1-t)^{r_2-1}]\\
&\times f(s,t,u(s,t))dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,u(s,t))dtds\|\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,u(s,t))dtds\|\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_0^{y_1}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}\|f(s,t,u(s,t))dtds\|\\
&\leq \|\mu(\tau_1,y_1)-\mu(\tau_2,y_2)\|+\sum_{k=1}^{m}(\|I_k(u(x_k^-,y_1))-I_k(u(x_k^-,y_2))\|)\\
&+\frac{\phi_f^0\psi_*(\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_0^{y_1}(x_k-s)^{r_1-1}[(y_2-t)^{r_2-1}-(y_1-t)^{r_2-1}]dtds\\
&+\frac{\phi_f^0\psi_*(\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\sum_{k=1}^{m}\int_{x_{k-1}}^{x_k}\int_{y_1}^{y_2}(x_k-s)^{r_1-1}(y_2-t)^{r_2-1}dtds\\
&+\frac{\phi_f^0\psi_*(\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}-(\tau_1-s)^{r_1-1}(y_1-t)^{r_2-1}]dtds\\
&+\frac{\phi_f^0\psi_*(\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds\\
&+\frac{\phi_f^0\psi_*(\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds\\
&+\frac{\phi_f^0\psi_*(\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_0^{y_1}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds.
\end{aligned}
$$

As $\tau_1\longrightarrow\tau_2$ and $y_1\longrightarrow y_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $F:PC(J,\mathbb{R}^n)\to PC(J,\mathbb{R}^n)$ is completely continuous.

Step 4: *A priori bound.*

For $\lambda \in [0,1]$, let u be such that for each $(x,y) \in J$ we have $u(x,y) = \lambda(Fu)(x,y)$. For each $(x,y) \in J$, then from (H3) and (H4) we have

$$\frac{\|u\|_\infty}{\|\mu\|_\infty + 2m\psi^*(\|u\|) + \frac{2a^{r_1}b^{r_2}\phi_f^0\psi_*(\|u\|)}{\Gamma(r_1+1)\Gamma(r_2+1)}} \leq 1.$$

By condition (H5), there exists $\overline{M}$ such that $\|u\|_\infty \neq \overline{M}$. Let

$$U = \{u \in PC(J, \mathbb{R}^n) : \|u\|_\infty < \overline{M}\}.$$

The operator $F : \overline{U} \to PC(J, \mathbb{R}^n)$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda F(u)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [16], we deduce that F has a fixed point u in $\overline{U}$ which is a solution of the problem (1)-(3).

Now we present two existence results for the nonlocal problem (4)-(6). Their proofs are similar to those for problem (1)-(3).

Definition 3.6 *A function $u \in PC(J, \mathbb{R}^n)$ whose r-derivative exists on J' is said to be a solution of (4)-(6) if u satisfies $({}^cD_0^r u)(x,y) = f(x,y,u(x,y))$ on J' and conditions* (5), (6) *are satisfied.*

Theorem 3.7 *Assume $(H_1), (H_2)$ and the following conditions*

(H_1') *There exists $\tilde{l} > 0$ such that*

$$\|Q(u) - Q(v)\| \leq \tilde{l}\|u - v\|, \text{ for any } u, v \in PC(J, \mathbb{R}^n).$$

(H_1'') *There exists $\tilde{l}^* > 0$ such that*

$$\|K(u) - K(v)\| \leq \tilde{l}^*\|u - v\|, \text{ for any } u, v \in PC(J, \mathbb{R}^n)$$

hold. If

$$\tilde{l} + \tilde{l}^* + 2ml^* + \frac{2la^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1,$$

then there exists a unique solution for IVP (4)-(6) on J.

Theorem 3.8 *Let $f(\cdot,\cdot,u) \in PC(J, \mathbb{R}^n)$ for each $u \in \mathbb{R}^n$. Assume $(H_3), (H_4)$ and the following conditions*

(H_3') *There exists $\tilde{d} > 0$ such that*

$$\|Q(u)\| \leq \tilde{d}(1 + \|u\|), \text{ for any } u \in PC(J, \mathbb{R}^n).$$

(H_3'') *There exists $d^* > 0$ such that*

$$\|K(u)\| \leq d^*(1 + \|u\|), \text{ for any } u \in PC(J, \mathbb{R}^n).$$

(H_3''') *There exists an number $\overline{M}_* > 0$ such that*

$$\frac{\overline{M}_*}{(\tilde{d} + d^*)(1 + \overline{M}_*) + \|\mu\|_\infty + 2m\psi^*(\overline{M}_*) + \frac{2a^{r_1}b^{r_2}\phi_f^0\psi_*(\overline{M}_*)}{\Gamma(r_1+1)\Gamma(r_2+1)}} > 1,$$

hold, then there exists at least one solution for IVP (4)-(6) on J.

4 An Example As an application of our results we consider the following impulsive partial hyperbolic differential equations of the form

(12)
$$({}^cD_0^r u)(x,y) = \frac{1}{(10e^{x+y+2})(1+|u(x,y)|)}, \quad \text{if } (x,y) \in J = [0,1]\times[0,1],\ x \neq x_k,\ k = 1,\dots,m,$$

(13)
$$u(x_k^+,y) = u(x_k^-,y) + \frac{1}{(6e^{x+y+4})(1+|u(x_k^-,y)|)}, \quad \text{if } y \in [0,1],\ k = 1,\dots,m,$$

(14)
$$u(x,0) = x,\ u(0,y) = y^2,\ \text{if } \ x \in [0,1] \text{ and } y \in [0,1].$$

Set
$$f(x,y,u) = \frac{1}{(10e^{x+y+2})(1+|u|)},\ (x,y) \in [0,1]\times[0,1],$$
$$I_k(u(x_k^-,y)) = \frac{1}{(6e^{x+y+4})(1+|u(x_k^-,y)|)},\ y \in [0,1].$$

For each $u,\ \overline{u} \in \mathbb{R}$ and $(x,y) \in [0,1]\times[0,1]$ we have
$$|f(x,y,u) - f(x,y,\overline{u})| \leq \frac{1}{10e^2}|u-\overline{u}|,$$
and
$$|I_k(u) - I_k(\overline{u})| \leq \frac{1}{6e^4}|u-\overline{u}|.$$

Hence condition (H_1) and (H_2) are satisfied with $l = \frac{1}{10e^2}$ and $l^* = \frac{1}{6e^4}$. We shall show that condition (11) holds with $a = b = 1$. Indeed, if we assume, for instance, that the number of impulses $m = 3$, than we have
$$2ml^* + \frac{2la^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} = \frac{1}{e^4} + \frac{1}{5e^2\Gamma(r_1+1)\Gamma(r_2+1)} < 1,$$

which is satisfied for each $(r_1,r_2) \in (0,1]\times(0,1]$. Consequently Theorem 3.4 implies that problem (12)-(14) has a unique solution defined on $[0,1]\times[0,1]$.

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