AROUND THE NAGATA'S METRICS

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Dedicated to the memory of Professor Jun-iti Nagata

Professor Jun-iti Nagata is a distinguished topologist who is one of the leaders of general topology in the second half of 20th century. He mostly devoted in the theories of metrization, uniform spaces, generalized metric spaces, rings of continuous functions, and dimension. In particular, his work about metrization and dimension theory are remarkable. In his research, some "special metrics", which induce particular topological properties, are key notions in the fields. We so-called such metrics as Nagata's metrics. Nagata's metrics connect between the metrization theory and dimension theory. We review Nagata's contribution on special metrics, and how the Nagata's metrics influenced further developments of general topology.

All topological spaces we will consider in this paper are metrizable spaces. Our terminology mostly follows [24] and [25], and we refer the reader to [14] for a survey of special metrics.

1 Nagata's metric and dimension For a metrizable space X, let $\dim X$ denote the covering dimension of X. We say that a metrizable space X *admits a metric* ρ if the metric ρ induces the original topology of X. A particular property of a metrizable space X is, of course, X admits a metric. We say generally that an admitting metric is *special* if it characterizes a (topological) property of a metrizable space. Typical examples of special metrics are totally bounded, and complete metrics. They characterize the separability and the topological completeness, respectively. The non-Archimedean metric is also a special metric. A metric ρ on a space X is called a *non-Archimedean metric* if ρ is a metric on X which satisfies the *strong triangle inequality*: $\rho(x,y) \leq \max\{\rho(x,z),\rho(y,z)\}$ for each $x,y,z \in X$. J. de Groot [10] proved that a metrizable space X is strongly zero-dimensional (i.e., $\dim X = 0$) if and only if X admits a non-Archimedean metric. Then de Groot extended the theorem to higher dimensions.

Theorem 1 ([11]) *Let n be a non-negative integer and X a separable metrizable space. Then $\dim X \leq n$ if and only if X admits a totally bounded metric ρ such that for every $n+3$ points $x, y_1, y_2, \ldots, y_{n+2}$ of X, there are i,j,k satisfying $i \neq j$ and $\rho(y_i,y_j) \leq \rho(x,y_k)$.*

Corollary 1 ([11]) *Let n be a non-negative integer and X a compact metrizable space. Then $\dim X \leq n$ if and only if X admits a metric ρ such that for every $n+3$ points $x, y_1, y_2, \ldots, y_{n+2}$ of X, there are i,j,k satisfying $i \neq j$ and $\rho(y_i,y_j) \leq \rho(x,y_k)$.*

On the other hand, Nagata obtained another characterization of the covering dimension in terms of a special metric in metrizable spaces.

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Theorem 2 ([18]) *Let n be a non-negative integer and X a metrizable space. Then $\dim X \le n$ if and only if X admits a metric ρ such that for every $n+3$ points $x, y_1, y_2, \dots, y_{n+2} \in X$ and every $\varepsilon > 0$ with $\rho(S_{\frac{\varepsilon}{2}}(x), y_i) < \varepsilon$ for each $i = 1, 2, \dots, n+2$, there are $i \ne j$ such that $\rho(y_i, y_j) < \varepsilon$, where $S_\varepsilon(x) = \{y : \rho(x, y) < \varepsilon\}$ is the spherical neighborhood of x.*

J. de Groot asked whether the corollary above holds for every metrizable space. Although the question still remains open, Nagata partially answered the question as follows.

Theorem 3 ([21]) *Let n be a non-negative integer and X a metrizable space. Then $\dim X \le n$ if and only if X admits a metric ρ such that for every $n+3$ points $x, y_1, y_2, \dots, y_{n+2}$ of X, there are $i \ne j$ satisfying $\rho(y_i, y_j) \le \rho(x, y_j)$.*

Nagata's proofs of Theorems 2 and 3 are based on the theory of normal sequences of open coverings. Nagata said in his reminiscences ([26]) that he read the monograph of Tukey [29] when he was a university student. I guess the monograph might be influenced to his idea of the proof of theorems above. His construction of the special metric of Theorem 2 is too long, and he asked the question in his book [22]: Find a simpler proof of Theorem 2. Then simpler proofs of Theorem 2 were found by S. Buzási [7] and P. Assouad [1] independently, after twenty years since Nagata originally proved the theorem. We notice that the technique due to P. Assouad is applied in the geometric group theory, and it makes a new development in topology (see the next section). Theorem 3 was obtained by P. A. Ostrand independently in [28].

Since the condition in Theorem 1 is weaker than that of Theorem 3, it follows that every metrizable space X with $\dim X \le n$ admits a metric ρ satisfying the condition of Theorem 1. We notice that if a metrizable space X admits a metric ρ satisfying the condition of Theorem 1, then $\mu \dim(X, \rho) \le n$, where $\mu \dim(X, \rho)$ is the metric dimension of X: For a metric space (X, ρ), the *metric dimension* $\mu \dim(X, \rho) \le n$ if there is a sequence $\{\mathcal{U}_k : k = 1, 2, \dots\}$ of open coverings of X such that $\operatorname{ord} \mathcal{U}_k \le n+1$ for each $k = 1, 2, \dots$ and $\lim_{k\to\infty} \operatorname{mesh} \mathcal{U}_k = 0$. It is clear that $\mu \dim(X, \rho) \le \dim X$ holds for every metrizable space X and an admitting metric ρ on X. We also notice that $\dim X \le 2\mu \dim(X, \rho)$ holds for every metric space (X, ρ) [15]. The reader refers [17] for the details of the metric dimension and other metric dependent dimensions which are also related to Nagata's metrics.

For a metric space (X, ρ) Nagata [27] called the smallest number $n+2$ which satisfies the condition in Theorem 3 is the *crowding number* of a metric space (X, ρ). We notice that the crowding number of the one-dimensional Euclidean space $\mathbb{R}$ is 3 and that of the two-dimensional Euclidean space $\mathbb{R}^2$ is 7. ([27]). Nagata asked the following questions.

Question 1 ([27]) Find the crowding number of the n-dimensional Euclidean space $\mathbb{R}^n$ for $n \ge 3$.

Question 2 ([27]) Find the metric ρ of the two-dimensional Euclidean space $\mathbb{R}^2$ such that the crowding number of $(\mathbb{R}^2, \rho)$ is 4.

Nagata furthered a study on special metrics in dimension theory.

Theorem 4 ([20]) *Let n be a non-negative integer and X a metrizable space. Then $\dim X \le n$ if and only if X admits a metric ρ such that for every point $x \in X$ and every $\varepsilon > 0$ $\dim \operatorname{Bd} S_\varepsilon(x) \le n-1$ and $\{S_\varepsilon(x) : x \in X\}$ is closure-preserving.*

Theorem 5 ([20]) *Let n be a non-negative integer and X a metrizable space. Then $\dim X \leq n$ if and only if X admits a metric ρ such that for every closed subspace F of X and every natural number i $\dim \mathrm{Bd}\, S_{\frac{1}{i}}(F) \leq n-1$, where $S_{\frac{1}{i}}(F) = \{y : \rho(y, F) < \frac{1}{i}\}$.*

Nagata also considered on infinite dimensional spaces (cf. [19] and [24]). A metrizable space X is called a *countable dimensional* (*strongly countable dimensional*) space if X is a countable union of zero-dimensional (resp. closed finite dimensional) subsets. Nagata extended Corollary 1 and Theorem 3 to strongly countable dimensional spaces.

Theorem 6 ([23]) *For a metrizable space X, the following are equivalent.*

(1) X is a strongly countable dimensional space.

(2) X admits a metric ρ such that for every point $x \in X$ there is a natural number $n(x)$ such that for every point $x \in X$ and every $n(x)+2$ points $y_1, y_2, \ldots, y_{n(x)+2}$ of X, there are $i \neq j$ satisfying $\rho(y_i, y_j) \leq \rho(x, y_j)$.

(3) X admits a metric ρ such that for every point $x \in X$ there is a natural number $n(x)$ such that for every point $x \in X$ and every $n(x)+2$ points $y_1, y_2, \ldots, y_{n(x)+2}$ of X, there are i, j, k satisfying $i \neq j$ and $\rho(y_i, y_j) \leq \rho(x, y_k)$.

We notice that Theorem 2 was extended to strongly countable dimensional spaces by Y. Hattori [13]: A metrizable space X is strongly countable dimensional if and only if X admits a metric ρ such that for every point $x \in X$ there is a natural number $n(x)$ such that for every $\varepsilon > 0$ and every $n(x)+2$ points $y_1, y_2, \ldots, y_{n(x)+2}$ of X with $\rho(S_{\frac{\varepsilon}{2}}(x), y_i) < \varepsilon$ for each $i = 1, 2, \ldots, n(x)+2$, there are $i \neq j$ such that $\rho(y_i, y_j) < \varepsilon$.

Remark 1 Nagata [19] obtained some interesting characterizations of countable dimensional spaces which related to metrization theory.

In [23], Nagata considered the following conditions on a metric space (X, ρ) which are generalizations of the conditions in Theorems 1 and 3:

$(1)_\omega$ For every point $x \in X$ and every sequence $y_1, y_2, \ldots$ in X there are $i \neq j$ such that $\rho(y_i, y_j) \leq \rho(x, y_j)$.

$(2)_\omega$ For every point $x \in X$ and every sequence $y_1, y_2, \ldots$ in X there are i, j, k such that $i \neq i$ and $\rho(y_i, y_j) \leq \rho(x, y_k)$.

Theorem 7 ([23]) *Every metrizable space X admits a metric ρ satisfying the condition $(2)_\omega$.*

Question 3 ([23]) Does every metrizable space X admit a metric ρ satisfying the condition $(1)_\omega$?

The question still remains open. However, a partial answer is given by Y. Hattori [13]: Every metrizable space X admits a metric ρ such that for every point $x \in X$ and every sequence $y_1, y_2, \ldots$ in X with $\{i : \rho(x, y_i) \geq \delta\}$ is infinite for some $\delta > 0$ there are $i \neq j$ such that $\rho(y_i, y_j) \leq \rho(x, y_j)$.

2 Other aspects on special metrics Theorem 7 and Question 3 lead us the following.

Theorem 8 ([13]) *Every metrizable space X admits a metric ρ such that for every $\varepsilon > 0$, every point $x \in X$ and every sequence $y_1, y_2, \ldots$ in X with $\rho(S_{\frac{\varepsilon}{2}}(x), y_i) < \varepsilon$ for each $i = 1, 2, \ldots$, there are $i \neq j$ such that $\rho(y_i, y_j) < \varepsilon$.*

It is easy to see that the condition in Theorem 8 implies that for every $\varepsilon > 0$ $\{S_\varepsilon(x) : x \in X\}$ has a locally finite subcover. Hence every metrizable space has a σ-locally finite base consisting of open balls. This seems to be interesting by comparing with Nagata-Smirnov Metrization Theorem. It is noticed that Z. Balogh and G. Gruenhage [2] proved that the hedgehog with uncountably many spines does not admit a metric ρ such that $\{S_\varepsilon(x) : x \in X\}$ is locally finite for each $\varepsilon > 0$. (Y. Ziqiu and H. Junnila [30] also proved the same result for the hedgehog of the weight $(2^c)^+$.) Further, they proved that the metric satisfying the condition above characterizes the strong metrizability. A regular space X is called a *strongly metrizable space* if X has a base which is a countable union of star finite open coverings.

Theorem 9 ([2], [30]) *For a metrizable space X the following are equivalent.*
(1) X is strongly metrizable.
(2) X admits a metric ρ such that for every $\varepsilon > 0$ $\{S_\varepsilon(x) : x \in X\}$ is locally finite.
(3) X admits a metric ρ such that for every $\varepsilon > 0$ $\{S_\varepsilon(x) : x \in X\}$ is star finite.

Slight modification of the condition in Theorem 8 also implies the strong metrizability.

Theorem 10 ([13]) *A metrizable space X is strongly metrizable if and only if X admits a metric ρ such that for every $\varepsilon > 0$, every point $x \in X$ and every sequence $y_1, y_2, \dots$ in X with $\rho(S_\varepsilon(x), y_i) < \varepsilon$ for each $i = 1, 2, \dots$, there are $i \neq j$ such that $\rho(y_i, y_j) < \varepsilon$.*

Recently, the condition of Theorem 2 makes a new development in topology. P. Assouad [1] said that the condition in Theorem 2 is the *n-dimensional Nagata property*, and defined the *Nagata dimension* N-dim(X, ρ) of a metric space (X, ρ) as follows: N-dim$(X, \rho) \leq n$ if there exists a constant C such that for each $r > 0$ there is an open covering $\mathcal{U}_r$ of X such that mesh$\,\mathcal{U}_r \leq Cr$ and $|\{U \in \mathcal{U}_r : U \cap S_r(x) \neq \emptyset\}| \leq n + 1$ for each $x \in X$, where $S_r(x)$ is the r-spherical neighborhood of x. He proved in [1] that

(1) for a metrizable space X $\dim X \leq n$ if and only if X admits a metric ρ such that N-dim$(X, \rho) \leq n$, and

(2) for a metric space (X, ρ) N-dim$(X, \rho) \leq n$ if and only if there are $p > 0$ and a metric δ on X such that (X, δ) satisfies the n-dimensional Nagata property and δ^p is Lipschitz equivalent to ρ.

Now, the Nagata dimension is called the *Assouad-Nagata dimension* and denoted by $\dim_{AN}$, and the Assouad-Nagata dimension is applied and extended to the coarse topology and the asymptotic geometry [16], [7], [6]. Then the asymptotic version of the Assouad-Nagata diemsnion, say *asymptotic Assouad-Nagata dimension*, is studying by several authors (cf. [3], [9], [8]). It is remarkable that the Nataga's metric has important applications after more than a half century since he found it.

3 Memory of Professor Nagata I knew the death of Professor Nagata by a phone call from Professor Okuyama on December, 2007 after a month since he has passed away. I was very surprised and came a big sadness.

Professor Nagata came back to Japan in 1982 as a professor at Osaka Kyoiku University after a long stay in abroad, US and the Netherlands. When Professor Nagata came to Osaka, I was a research assistant at Osaka Kyoiku University and studying in dimension theory. (The first paper I read is the paper of Nagata with Bruijning concerning a characterization of the covering dimension by a generalization of the theorem of Pontrjagin-Schnirelmann [4].) At the first seminar at Osaka Kyoiku University since Nagata came there, he presented a lecture about several topics in general topology, which included some topics in dimension theory, generalized metric spaces etc. The lecture gave me a direction of my research. The theory of infinite dimensional spaces is one of the topics at his lecture, and since then I am studying in this field. He showed us the background of the theory and posed several open questions that made help me very much.

Since Professor Nagata came back to Osaka in 1982, he organized the seminar at Osaka Kyoiku University on every Tuesday afternoon. More than fifteen people usually attended the seminar. At the breaks of the seminar, we talked on several things not only mathematics. Sometimes Professor Nagata told us about fishing and travels with his pleasure. I also remember his smile. His face usually looks to be serious, but we could relax with his smile when he sometimes had a short laugh.

I was working with him at Osaka Kyoiku University in five years, and then I left Osaka. Professor Nagata used to be encouraging me even I left Osaka. I never forget everything of Professor Nagata, especially his kindness, his personality, his distinguished mathematical work and his guidances for my study.

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