ON ONE NAGATA'S IDEA

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In memory of Professor J.Nagata

Abstract. The article concerns the influence of one Nagata's idea with the dimension theory of general topological spaces. The addition to the article contains proofs of very general sufficient conditions for the Ind-product theorem (for Tychonoff products) and for the Ind-subset theorem (which were published without proofs earlier).

The aim of this article to show how one notion due to Nagata stimulated obtaining many important assertions of dimension theory (the product theorem, the subspace theorem, the inverse limit theorem are among them) in as general as possible situations.

Below, "space" is a topological space and "map" is a continuous mapping between spaces.

The product theorem in dimension theory states that for a topological product $X \times Y$, $X \cup Y \neq \emptyset$,

$$\dim\ X \times Y \leq \dim\ X + \dim\ Y \tag{1}$$

or

$$\mathrm{Ind}\ X \times Y \leq \mathrm{Ind}\ X + \mathrm{Ind}\ Y. \tag{2}$$

Note that inequality (2) with ind instead of Ind (let it be inequality (3)) is also considered but rather seldom. Note also that (2) concerns normal spaces only (because there is no satisfied definition of the dimension Ind even for Tychonoff spaces).

In 1952 Katětov [6] and in 1954 Morita [15] established that (1) (and (2) because $\dim\ X = \mathrm{Ind}\ X$ for any metrizable space X) is true for any metrizable spaces X and Y. After this the following problem arose:
When do inequalities (1) *and* (2) *hold for non-metrizable spaces?*

In 1960 Nagami [17] proved that (2) is true if X is metrizable and Y is a perfectly normal paracompactum; in 1963 Kimura [8] proved (2) for metrizable X and totally normal countably compact $X \times Y$; in 1966 Katuta [7] proved that (2) holds if $X \times Y$ is totally normal and either $X \times Y$ is strongly paracompact or X is a paracompactum and Y is a locally compact paracompactum.

In 1967 Nagata [18] gave the following definitions (recall that a subset $A \times B$ of a topological product $X \times Y$ is called a *closed* (respectively, (*functionally*) *open*) *rectangle* of $X \times Y$ if A is a closed (respectively, (functionally) open) subset of X and B is a closed (respectively, (functionally) open) subset of Y).

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Nagata's definition. A pair $\{\mathcal{F},\mathcal{U}\}$, consisting of the family $\mathcal{F}$ of closed sets F_α in a space X and the family $\mathcal{U}$ of open sets U_α in a space X, $\alpha \in A$, is called a *tissue* of X if $F_\alpha \subset U_\alpha$ for every α. The tissue $\{\mathcal{F},\mathcal{U}\}$ of X separates a pair $\{K, L\}$ of disjoint closed sets of X if there is $B \subset A$ for which

$\cup\{F_\alpha : \alpha \in B\} = X$ and $\{U_\alpha : \alpha \in B\}$ is a refinement of $\{X \setminus K, X \setminus L\}$.

A topological product $X \times Y$ is called an *F-product* if for any pair $\{K, L\}$ of disjoint closed sets of $X \times Y$ there is a tissue $\{\mathcal{F},\mathcal{U}\}$ of $X \times Y$ such that

i) $\mathcal{F}$ consists of closed rectangles;

ii) $\mathcal{U}$ consists of open rectangles and is σ-locally finite;

iii) $\{\mathcal{F},\mathcal{U}\}$ separates $\{K, L\}$.

At first time, the definition seems too artificial, too complicated. But the main idea of the definition seems very interesting, because the use of rectangles in the products allows to reduce the consideration of dimensional properties of topological products to the consideration of dimensional properties of their factors.

It was clarified later that very many topological products are F-products and yet more of topological products are rectangular and piece-wise rectangular (see below).

Nagata himself showed in [18] that *a topological product $X \times Y$ is an F-product if*

1. *X and Y are paracompact M-spaces, or*
2. *X is metrizable and Y is a normal P-space, or*
3. *X is a locally compact paracompactum and Y is a paracompactum.*

Nagata's theorem ([18])**.** (2) *is true if $X \times Y$ is an F-product and the space $X \times Y$ is totally normal.*

Independently, Lifanov [11]–[14] also discovered a very important role of rectangles in the products for obtaining inequality (2) in some classes of compacta. (For example, he showed that (2) is true for totally normal compacta X and Y and for compacta X, Y with $\mathrm{Ind}\ X = \mathrm{Ind}\ Y = 1$).

Very interesting Nagata's idea of F-product, Lifanov's papers and Morita's and Filippiv's results concerning inequality (1) (for example, see [10] about them) stimulated me (in particular, for considering the covering dimension in the class of arbitrary (not necessary normal) spaces) to modify the notion of the F-product in the following way.

Definition 1 ([20, 21] for finite topological products)**.** A topological product $X \times Y$ is called *rectangular* if for any finite functionally open cover λ of $X \times Y$ there exists its σ-locally finite refinement consisting of functionally open rectangles of $X \times Y$.

It is not difficult to prove that *if the space $X \times Y$ is normal then $X \times Y$ is rectangular iff it is an F-product.*

Remark 1. Many cases of rectangularity (and piecewise rectangularity (see below)) and non-rectangularity of topological products are considered in [5],[22] and [10].

The following two assertions showed that Nagata's intuition did not deceive him when he decided to introduce and consider the notion of F-product.

Theorem 1 ([19, 20, 21, 25])**.** *If the product $X \times Y$ is rectangular, the space $X \times Y$ is normal* (thus $X \times Y$ is an F-product) *and the finite sum theorem for Ind holds in the factors X and Y* (for example, X and Y are totally normal or $\mathrm{Ind}\ X = \mathrm{Ind}\ Y = 1$) *then* (2) *is true.*

Theorem 2 ([21, 22])**.** *If the product $X \times Y$ is rectangular then* (1) *is true.*

Considering of rectangular products led to the following their generalization.

Definition 2 ([23, 24]). A topological product $X \times Y$ is called *piecewise rectangular* if for any finite functionally open cover λ of $X \times Y$, there exists its σ-locally finite refinement μ with the following property:

For any $U \in \mu$ there is a functionally open rectangle $V(U)$ of the product $X \times Y$ such that U is closed-open in $V(U)$.

The class of piecewise rectangular products is interesting by three reasons. In the first place, this class is essentially wider than the class of rectangular products. For example, the topological square of the Sorgenfrey line is piecewise rectangular but not rectangular. In the second place, the following is true.

Theorem 3 ([23, 24, 25]). *Theorems 1 and 2 remain true if to consider piecewise rectangular products instead of rectangular ones in them.*

Finally, the piecewise rectangularity allows to characterize 0-dimensional products in the following way.

Theorem 4 ([23, 24]). *If* $\dim\ X = \dim\ Y = 0$, *then* $\dim\ X \times Y = 0$ *iff the product* $X \times Y$ *is piecewise rectangular.*

Problem 1. To find criteria (or as general as possible sufficient conditions) for $\dim\ X \times Y \leq m + n$ if $\dim\ X = m$ and $\dim\ Y = n$.

Theorems 1-3 may be generalized in the following way.

Definition 3 ([9, 10]). A subset A of a product $X \times Y$ is called *piecewise rectangularly posed* (respectively, *rectangularly posed*) if for any finite functionally open cover λ of A, there exists its σ-locally finite refinement μ with the following property:

For any $U \in \mu$ there is a functionally open rectangle $V(U)$ of the product $X \times Y$ such that U is closed-open in $A \cap V(U)$ (respectively, such that $U = A \cap V(U)$).

Theorem 5 ([9, 10]). *If a subset A of a product $X \times Y$ is piecewise rectangularly posed, then* $\dim\ A \leq \dim\ X + \dim\ Y$.

In connection with Theorem 5, note that there exist products $X \times Y$ such that $\dim\ X \times Y > \dim\ X + \dim\ Y$.

Note yet that we consider only topological products of two factors. But Definitions 1-3 may be given for any finite topological products and for Tychonoff products and Theorems 1-5 are true for all these products (see, for example, [10]).

Further considering of rectangular and piecewise rectangular products showed a connection between the product and the subset theorems. Indeed, it is well known that a) (1) is true for any compacta X and Y [4] and b) $\dim \beta X = \dim X$ for any Tychonoff space X and $\operatorname{Ind} \beta X = \operatorname{Ind} X$ for any normal Hausdorff space X. Hence $\dim \beta X \times \beta Y \leq \dim \beta X + \dim \beta Y = \dim X + \dim Y$ and $X \times Y \subset \beta X \times \beta Y$. Thus we can use sufficient conditions for $\dim X \times Y \leq \dim \beta X \times \beta Y$ to obtain (1). Such a condition is contained in the following definition.

Definition 4 ([23, 24]). A subset A of a space X is called *d-posed* (respectively, *d-right*) if for any finite functionally open cover λ of A, there exists its σ-locally finite (and functionally open) refinement μ with the following property:

For any $U \in \mu$ there is a functionally open set $V(U)$ in X such that $U = A \cap V(U)$ (respectively, U is closed-open in $A \cap V(U)$).

Remark 2. Evidently, Definition 3 is close to Definitions 1 and 2.

Remark 3. More detailed information on d-posedness and d-rightness may by found in [10]. In [3] Filippov defined a property of d-posedness of a subset of a space that is equivalent to d-posedness from Definition 3 in the class of normal spaces.

Theorem 6 ([23, 24])**.** *If a subset A of a space X is d-right (in particular, d-posed), then*

$$\dim A \leq \dim X. \tag{4}$$

Theorem 7 ([23, 24])**.** *Let* $\dim X = 0$ *and* $A \subset X$*. Then* (4) *is true iff A is a d-right subset of X.*

Remark 4. Possibly, at this moment d-rightness is the most general sufficient condition for the subset theorem for the covering dimension dim outside of the class of hereditarily normal spaces (but perhaps more general sufficient conditions for this theorem are contained in [1] and [26]).

Theorem 8 ([23, 24])**.** *A topological product of Tychonoff spaces X and Y is rectangular (respectively, piecewise rectangular) iff $X \times Y$ is a d-posed (respectively, d-right) subset of $\beta X \times \beta Y$.*

Remark 5. Theorems 6 and 8 imply Theorem 2 and the part of Theorem 3 concerning the covering dimension *dim* for Tychonoff spaces X and Y.

Pass to the dimension Ind.

Theorem 9 ([23])**.** *Let a space X and its subspace A be normal. If A is a d-right subset of X and the finite sum theorem holds either in X or in A, then*

$$\operatorname{Ind} A \leq \operatorname{Ind} X. \tag{5}$$

The d-rightness plays very important role in the proof of the following assertion.

Theorem 10 ([10])**.** *Any paracompactum (respectively, non-zero-dimensional paracompactum) Y is the image of a paracompactum $X = X(Y)$ of dimension* $\dim X = 1$ *under perfect open map with path-connected (respectively, zero-dimensional) fibers.*

The following is an analog of Theorems 6 and 9.

Theorem 11 ([10])**.** *If B is a non-empty subparacompactum of a paracompactum Y and Y is the image of a paracompactum X with* $\dim X = 0$ *under a closed* $\leq (n+1)$*-to-*1 *map, then B also is the image of a paracompactum A with* $\dim A = 0$ *under a closed* $\leq (n+1)$*-to-*1 *map,* $n = 0, 1, 2, \dots$

The d-rightness is connected with the following class of maps.

Definition 5 ([24, 10])**.** A map $f : A \to X$ is called *d-right* if for any finite functionally open cover λ of A, there exists a σ-locally finite (functionally open) refinement μ with the following property:

For any $U \in \mu$ there is a functionally open set $V(U)$ in X such that U is closed-open in $f^{-1}V(U)$.

Remark 6. In [24] d-right maps are called strongly decomposing.

The following is a generalization of Theorem 6.

Theorem 12 ([10])**.** *If a map $f : A \to X$ is d-right, then we have* (4).

Remark 7. It is not difficult to prove that $\dim f^{-1}x \leq 0$ in Theorem 12 if the space A is normal and X is a T_1-space. Hence for d-right maps, Hurewicz's formula $\dim A \leq \dim X + \dim\ f$ is true.

Theorem 12 may be generalized in the following way.

Recall that a map of a space A to an inverse system $S = \{X_\alpha, p_{\beta\alpha}; \alpha \in \mathcal{A}\}$ is a system φ of maps $\varphi_\alpha : A \to X_\alpha$, $\alpha \in \mathcal{A}$, such that $f_\alpha = p_{\beta\alpha} \circ f_\beta$ if $\alpha \leq \beta$.

Definition 6 ([10] **).** A map $\varphi = \{\varphi_\alpha : \alpha \in \mathcal{A}\}$ of a space A to an inverse system $S = \{X_\alpha, p_{\beta\alpha}; \alpha \in \mathcal{A}\}$ is called *d-right* if for any finite functionally open cover λ of A there exists a σ-locally finite (functionally open) refinement μ with the following property:

For any $U \in \mu$ there are $\alpha(U) \in \mathcal{A}$ and a functionally open set $V(U)$ in $X_{\alpha(U)}$ such that U is closed-open in $\varphi^{-1}_{\alpha(U)} V(U)$.

Theorem 13 ([10])**.** *If a space A admits a d-right map to an inverse system $S = \{X_\alpha, p_{\beta\alpha}; \alpha \in \mathcal{A}\}$, then*

$$\dim A \leq \sup\{\dim X_\alpha : \alpha \in \mathcal{A}\}. \tag{6}$$

Remark 8. Evidently, Theorem 6 is a very special case of Theorem 13 (when $|\mathcal{A}| = 1$).

Corollary 1 (The inverse limit theorem) ([24, 10]) **.** *Let A be the limit of an inverse system $S = \{X_\alpha, p_{\beta\alpha}; \alpha \in \mathcal{A}\}$ and let the system φ of all projections $p_\alpha : A \to X_\alpha$ be d-right. Then we have* (6).

Let now S be an inverse sequence $\{X_i, p_{ji}; i \in \mathbb{N}\}$ with limit X and projections $p_i : X \to X_i$. Then, by Corollary 1,

$$\dim X \leq \sup\{\dim X_i : i \in \mathbb{N}\} \tag{7}$$

if the system of all projections $\{p_i\}$ is d-right.

However, for inverse sequences, more simple sufficient condition for d-rightness of φ may be found.

Corollary 2 ([10]) **.** *Let $A \subset X$ and for any functionally open set U in A there exist (functionally open in A) sets $U_i \subset A$, $i \in \mathbb{N}$, with the following property:*

For any i there is a functionally open set V_i in X_i such that U_i is closed-open in $A \cap p_i^{-1} V_i$ and $U = \cup\{U_i : i \in \mathbb{N}\}$.

Then A is d-right in X and $\dim A \leq \sup\{\dim X_i : i \in \mathbb{N}\}$.

The inverse sequence limit theorem. *If $A = X$ in Corollary 2, then we have* (7).

Formally, at this moment, this corollary generalize all previous sufficient conditions for (7) (see [10], 1.6).

My short review shows that Nagata's idea has turned out to be very fruitful. It may be considered as one from origins of dimension theory of general topological spaces.

Addition

The addition contains proofs of a product theorem for Tychonoff products and a subset theorem for the dimension Ind. They were published in [23] without proof.

Below, let Π be the Tychonoff product of spaces $X_i \neq \emptyset$, $i \in I$, and I_f denote the set of all non-empty finite subsets of I. Set $m = \sup\{\operatorname{Ind} X_{i_1} + \ldots + \operatorname{Ind} X_{i_k} : \{i_1, \ldots, i_k\} \in I_f\}$.

A set of the form $\{x = (x_i)_{i \in I} \in \Pi : x_i \in O_i \text{ for } i \in a \in I_f\}$, where O_i is functionally open in X_i, will be called a *functionally open rectangle* of the product Π. A closed-open subset of a functionally open rectangle is called a *functionally open rectangular piece* of Π. A cover of the product Π by functionally open rectangular pieces (rectangles) is called *functionally open piecewise rectangular* (*rectangular*).

Definition 7. The Tychonoff product Π is called *piecewise rectangular* [23, 24] (*rectangular* [3]) if each of its finite functionally open cover has a σ-locally finite functionally open piecewise rectangular (rectangular) refinement.

Theorem 14 ([23])**.** *If the Tychonoff product Π is piecewise rectangular, the space Π is normal and the finite sum theorem for Ind holds in every factor X_i then $Ind\,\Pi \le m$.*

The proof will be given later. We need some auxiliary assertions.

If $I = I_1 \cup I_2$, $I_1 \cap I_2 = \emptyset$, $I_1 \neq \emptyset \neq I_2$, $\Pi_k = \prod\{X_i : i \in I_k\}$, $k = 1, 2$, then we shall identify Π with $\Pi_1 \times \Pi_2$ identifying $x = (x_i)_{i\in I} \in \Pi$ with $((x_i)_{i\in I_1}, (x_i)_{i\in I_2}) \in \Pi_1 \times \Pi_2$. Note that every functionally open rectangle $\{x = (x_i)_{i\in I} \in \Pi : x_i \in O_i \text{ for } i \in a \in I_f\}$ of Π, where O_i is functionally open in X_i, is a functionally open rectangle $U_1 \times U_2$ of $\Pi_1 \times \Pi_2$, where $U_k = \{x = (x_i)_{i\in I_k} \in \Pi_k : x_i \in O_i \text{ for } i \in a \cap I_k\}$, $k = 1, 2$. It follows from this the following assertion.

Lemma 1. *If the product Π is piecewise rectangular (rectangular) and $I = I_1 \cup I_2$, $I_1 \cap I_2 = \emptyset$, $I_1 \neq \emptyset \neq I_2$, $\Pi_k = \prod\{X_i : i \in I_k\}$, $k = 1, 2$, then the product $\Pi_1 \times \Pi_2$ is also piecewise rectangular (rectangular).*

Lemma 2. *If the product Π is piecewise rectangular (rectangular) and $I_1 \subset I$, then the subproduct $\Pi_1 = \prod\{X_i : i \in I_1\}$ is also piecewise rectangular (rectangular).*

Proof. Fix a point $x^0 = (x_i^0)_{i\in I} \in \Pi$ and identify Π_1 with a subspace of Π identifying every $x_1 = (x_i)_{i\in I} \in \Pi_1$ with $x \in \Pi$ with coordinates x_i for $i \in I_1$ and x_i^0 for $I \setminus I_1$. Take a finite functionally open cover Ω of Π_1. If pr is the projection of Π onto Π_1, then $\Omega' = pr^{-1}\Omega$ is a finite functionally open cover of Π. Hence we can take a σ-locally finite piecewise rectangular (rectangular) refinement ω of Ω'. Then $\Pi_1 \wedge \omega = \{\Pi_1 \cap O : O \in \omega\}$ is a σ-locally finite piecewise rectangular (rectangular) refinement of Ω. □

Now we consider the dimensional invariant Id. Recall some necessary definitions.

Let X be a space. A system σ of closed subsets of X is called *additive* if $\cup\sigma' \in \sigma$ for any finite $\sigma' \subset \sigma$. And σ is called *monotone* if every closed subset of any $F \in \sigma$ is an element of σ.

For systems λ and μ of closed subsets of X, we say that λ *breaks* μ if for every $F \in \mu$ and any closed subsets A and B of F, that are functionally separated in X, there exists a partition $C \in \lambda$ in F between A and B.

A system λ of subsets of X is called *finite relatively to a system* ω of subsets of X if for any $O \in \omega$ the system $\{F \in \lambda : F \cap O \neq \emptyset\}$ is finite. A system λ of subsets of X is called *uniformly locally finite* (ULF, for short) if λ is finite relatively to a functionally open locally finite (FOLF, for short) cover of X.

Definition 8 ([25])**.** A system λ' of subsets of X *uniformly generates* a system λ if for every $L \in \lambda$ there is a ULF system μ_L consisting of closed subsets of some members of λ' such that $L = \cup\mu_L$.

Definition 9 ([19, 21])**.** Let $\mathrm{Id}\,X = -1$ iff $X = \emptyset$. Put $\mathrm{Id}\,X \le n$, $n = 0, 1, 2, \dots,$ if there are systems of closed subsets σ_i, $i = -1, 0, 1, \dots, k \le n$, in X satisfying the following conditions:

(a) $\sigma_{-1} = \{\emptyset\}$, $X \in \sigma_k$, $\sigma_i \subset \sigma_{i+1}$, $-1 \le i \le k-1$;
(b) σ_i breaks σ_{i+1}, $i < k$;
(c) σ_i is additive for any i;
(d) σ_i is monotone for any i.

Remark 9. The word "determines" is used instead of the word "separates" on line 7 of Definition 1 from [19] in Russian and on line 6 of Definition 1 from the translation of [19] in English. (These words have a similar look in Russian.)

Remark 10. Sometimes we shall use the following condition (a′) instead of (a) (not assume that $k \leq n$):

(a′) $\sigma_{-1} = \{\emptyset\}$, $\sigma_i \subset \sigma_{i+1}$, $-1 \leq i \leq k-1$.

It is not difficult to prove the following.

Lemma 3. *Let X be a normal space. Then we have the following.*

1. *If systems of closed subsets σ_i, $i = -1, 0, 1, \dots, k$, in X have properties* (a′) *and* (b) *from Definition* 9, *then* $\operatorname{Ind} F \leq i$ *for any $F \in \sigma_i$, $i = -1, 0, 1, \dots, k$.*

2. *The systems* $\sigma_{(i,\mathrm{Ind})} = \{F : F \text{ is closed in } X, \operatorname{Ind} F \leq i\}$, $i = -1, \cdots, k$ (respectively, and $k = \operatorname{Ind} X$), *have properties* (a′) (respectively, (a)),(b),(d) *from Definition* 9. (Note that $\sigma_{(i,\mathrm{Ind})}$ has property (c) if the finite sum theorem for Ind is true in X.)

Proposition 1 ([19, 23, 25])**.** *Let X be a normal space. Then 1)* $\operatorname{Ind} X \leq \operatorname{Id} X$ *and 2)* $\operatorname{Ind} X = \operatorname{Id} X$ *if the finite sum theorem for Ind is satisfied in X.*

Proposition 2. *If systems σ_i of closed sets in a space X, $i = -1, 0, 1, \dots, k$, have properties (a') (respectively, (a)),$(b) - (d)$ from Definition 9 and for all i, σ'_i is uniformly generated by σ_i. Then the systems σ'_i also have properties (a') (respectively, (a)),$(b) - (d)$ from Definition 9 and, additionally,*

$(c)_{ULF}$ for any i, the system σ''_i uniformly generated by σ'_i coincides with σ'_i.

Proof. The systems σ'_i are monotone, additive and (a') (respectively, (a)) is true for them. By Lemma 2 from [26] ($\equiv$ Lemma 2 from [27]) σ'_i breaks σ'_{i+1}. Finally, $(c)_{ULF}$ follows from Lemma 3 of [26]. $\square$

Corollary 3 (The uniformly locally finite sum theorem for Ind) ([27])**.** *Let the finite sum theorem for Ind be true in a normal space X. If F is the union of an ULF system of closed sets F_α in X with* $\operatorname{Ind} F_\alpha \leq r$, *$\alpha \in \mathcal{A}$, then* $\operatorname{Ind} F \leq r$, *$r = -1, 0, \dots$*

Proof. By Lemma 3, the systems $\sigma_{(i,\mathrm{Ind})}$, $i \leq r$, have properties (a′),(b)-(d). By Proposition 2 (using notation of it), the systems $\sigma'_{(i,\mathrm{Ind})}$ also have properties (a′),(b)-(d). By Lemma 3 (because $F \in \sigma_{(r,\mathrm{Ind})}$), $\operatorname{Ind} F \leq r$. $\square$

Proposition 3. *For any space X, the equalities* $\dim X = 0$ *and* $\operatorname{Id} X = 0$ *are equivalent.*

Proof. Let $\operatorname{Id} X = 0$. Then there are closed families σ_{-1} and σ_0 in X such that $\sigma_{-1} = \{\emptyset\}$, $\sigma_0 \ni X$ and σ_{-1} breaks σ_0. Take a finite functionally open cover $\Omega = \{O_i : i = 1, \dots, s\}$ of X. Then there exist a separable metrizable space Y, its finite functionally open cover $\lambda = \{L_1, \dots, L_s\}$ and a map f of X onto Y such that $f^{-1}L_i = O_i$. Take a finite closed cover $\{F_i : i = 1, \dots, s\}$ of Y such that $F_i \subset L_i$. Then $\{G_i = f^{-1}F_i : i = 1, \dots, s\}$ is a closed cover of X such that G_i and $X \setminus O_i$ are functionally separated. Since σ_{-1} breaks σ_0, there is an empty partition between G_i and $X \setminus O_i$ and so for any i, there is a closed-open set U_i in X such that $G_i \subset U_i \subset O_i$. Hence $\{U_i : i = 1, \dots, s\}$ is a closed-open refinement of Ω and $U_1, U_2 \setminus U_1, \dots, U_s \setminus (U_1 \cup \dots U_{s-1})$ is an open disjoint refinement of Ω. Thus *dim* $X = 0$.

Now let $\dim X = 0$. Take closed families $\sigma_{-1} = \{\emptyset\}$ and σ_0 consisting of all closed subsets of X. Evidently, these families are monotone and additive. Prove that σ_{-1} breaks

σ_0. Let subsets A and B of $F \in \sigma_0$ be closed and functionally separated in X. Take a map $f : X \to [0,1]$ such that $A \subset f^{-1}0$ and $B \subset f^{-1}1$. Then $\Omega = \{O_0 = f^{-1}(0,1], O_1 = O_0 = f^{-1}[0,1)\}$ is a finite functionally open cover of X. Hence there exists a finite open disjoint refinement ω of Ω. Without loss of generality, we can suppose that $\omega = \{U_0, U_1\}$ with $U_i \subset O_i$, $i = 0, 1$. Hence there is an empty partition between A and B in X and in F. Thus $\operatorname{Id} X = 0$. □

Now we can prove an assertion that implies Theorem 14.

Theorem 15 ([23]). *If the Tychonoff product* Π *is piecewise rectangular, then*

$$\operatorname{Id}\Pi \le \sup\{\operatorname{Id} X_{i_1} + \ldots + \operatorname{Id} X_{i_k} : \{i_1, \ldots, i_k\} \in I_f\} \qquad (8).$$

Proof. We put $l = \sup\{\operatorname{Id} X_{i_1} + \ldots + \operatorname{Id} X_{i_k} : \{i_1, \ldots, i_k\} \in I_f\}$. First suppose that I is finite. It is proved in [25] that the theorem is true for $|I| = 2$. Suppose that it is true for all I with $|I| < n$, $n > 2$, and let $|I| = n$. Then $I = I_1 \cup I_2$, where $|I_1| = n - 1$ (and $|I_2| = 1$). Then by Lemmas 1 and 2 and the inductive hypothesis, (for $\Pi_1 = X_1 \times \ldots X_{n-1}$ and $\Pi_2 = X_n$) $\operatorname{Id}\Pi = \operatorname{Id}\Pi_1 + \operatorname{Id} X_n \le (\operatorname{Id} X_1 + \ldots + \operatorname{Id} X_{n-1}) + \operatorname{Id} X_n = l$.

Now let $|I| \ge \aleph_0$ and $I_1 = \{i \in I : \operatorname{Id} X_i > 0\}$. If I_1 is infinite, then the theorem is true. Let I_1 be finite and $I_2 = I \setminus I_1$. For $\Pi_k = \prod\{X_i : i \in I_k\}$, $k = 1, 2$, by Lemma 3 and since I_1 is finite, $\operatorname{Id}\Pi \le \operatorname{Id}\Pi_1 + \operatorname{Id}\Pi_2 \le \sum\{\operatorname{Id} X_i : i \in I_1\} + \operatorname{Id}\Pi_2$.

Prove that $\operatorname{Id}\Pi_2 = 0$. By Proposition 3, $\dim X_i = \operatorname{Id} X_i = 0$ for any $i \in I_2$. It follows from [24] that $\dim \Pi_2 = 0$. Hence $\operatorname{Id}\Pi_2 = 0$ and so $\operatorname{Id}\Pi \le \sum\{\operatorname{Id} X_i : i \in I_1\} + 0 \le l$. □

Proof of Theorem 14. By Proposition 1 and Theorem 15, $\operatorname{Ind}\Pi \le \operatorname{Id}\Pi \le \sup\{\operatorname{Id} X_{i_1} + \ldots + \operatorname{Id} X_{i_k} : \{i_1, \ldots, i_k\} \in I_f\} = \sup\{\operatorname{Ind} X_{i_1} + \ldots + \operatorname{Ind} X_{i_k} : \{i_1, \ldots, i_k\} \in I_f\}$. □

Pass to a subset theorem for Ind (see Theorem 9).

Lemma 4. *Let a subset* A *of a space* X *be d-right,* λ^* *and* μ^* *be monotone families of closed subsets in* X *and* λ^* *break* μ^*. *If* $\lambda^\wedge = A \wedge \lambda^* = \{A \cap L : L \in \lambda^*\}$, $\mu^\wedge = A \wedge \mu^* = \{A \cap M : M \in \mu^*\}$ *and* λ *is uniformly generated by* $\lambda^\wedge$, *then* λ *breaks* $\mu^\wedge$.

Proof. First note that the monotonicity of λ^* implies the monotonicity of $\lambda^\wedge$.

Now take $F \in \mu^\wedge$ and closed subsets C and D of F functionally separated in A. Then there exist functionally open sets U and V in A such that $C \subset U$, $D \subset V$, $A = U \cup V$ and $C \cap V = \emptyset = D \cap U$.

Since A is d-right in X, there exists a σ-locally finite refinement Ω of the cover $\{U, V\}$ of A with the following property: For any $O \in \Omega$ there is a functionally open set O^* in X such that O is closed-open in $O^\wedge = A \cap O^*$. For every $O \in \Omega$, take a map $f_O : X \to [0,1]$ with $O^* = f_O^{-1}(0,1]$. For $j = 2, 3, \ldots$ and $O \in \Omega$, let $O_j = O \cap f_O^{-1}(1/j, 1]$ and $O_j^c = O \cap f_O^{-1}[1/j, 1]$. Then

$$(*) \quad O_j \subset O_j^c \subset O_{j+1}.$$

Since O is closed-open in $O^\wedge$, all O_j are functionally open and all O_j^c are functionally closed in A.

Let Ω be the union of locally finite families Ω_l, $l \in \mathbb{N}$. Put $\Omega_{lj} = \{O_j : O \in \Omega_l\}$ and $\Omega_{lj}^c = \{O_j^c : O \in \Omega_l\}$. Then the families Ω_{lj} are FOLF in A and the families Ω_{lj}^c are functionally closed in A. By [16] (see Lemma 6 from [25] ($\equiv$ Lemma 5 from [27] (on the first line of this lemma must be O_α instead of Q_α))) and by $(*)$,

$$(**)\ \text{ all families } \Omega_{lj} \text{ are ULF in } A.$$

Besides,

$$(***)\ \cup\Omega_l = \cup\{\cup\Omega_{lj} : j = 2, 3, \dots\}.$$

Let $F^* \in \mu^*$ be such that $A \cap F^* = F$. For any $O \in \Omega$ and $j = 2, 3, \dots$, the sets $(F^*_{Oj})' = F^* \cap f_O^{-1}[1/j, 1]$ and $(F^*_{Oj})'' = F^* \cap f_O^{-1}[0, 1/(j+1)]$ are closed in F^* and functionally separated in X. Hence there is a partition $P^*_{Oj} \in \lambda^*$ between them in F^*. Then $P^\wedge_{Oj} = F \cap P^*_{Oj} \in \lambda^\wedge$ is a partition between $(F^\wedge_{Oj})' = F \cap (F^*_{Oj})' = F \cap f_O^{-1}[1/j, 1]$ and $(F^\wedge_{Oj})'' = F \cap (F^*_{Oj})'' = F \cap f_O^{-1}[0, 1/(j+1)]$ in F. Since $P^*_{Oj} \subset O^*$, we have that $P^\wedge_{Oj} \subset O^\wedge$. Since O is closed-open in $O^\wedge$, $P_{Oj} = O \cap P^\wedge_{Oj} \in \lambda^\wedge$ is a partition between $F'_{Oj} = O \cap (F^\wedge_{Oj})' = O \cap F \cap f_O^{-1}[1/j, 1]$ and $F''_{Oj} = (F \setminus O) \cup (O \cap (F^\wedge_{Oj})'') = (F \setminus O) \cup (O \cap F \cap f_O^{-1}[0, 1/(j+1)])$ in F. Hence $F \setminus P_{Oj}$ is the disjoint union of open sets G_{Oj} and H_{Oj} in F such that $F'_{Oj} \subset G_{Oj}$ and $F''_{Oj} \subset H_{Oj}$ and so $G_{Oj} \subset cl\, G_{Oj} \subset G_{Oj} \cup P_{Oj} \subset F \setminus H_{Oj} \subset F \setminus F''_{Oj} = F \setminus ((F \setminus O) \cup (O \cap F \cap f_O^{-1}[0, 1/(j+1)])) = O \cap (F \setminus (O \cap f_O^{-1}[0, 1/(j+1)])) = (O \cap F) \setminus (O \cap f_O^{-1}[0, 1/(j+1)]) = (O \cap F) \setminus f_O^{-1}[0, 1/(j+1)]) = (O \cap F) \cap f_O^{-1}(1/(j+1), 1] \subset F \cap O_{j+1} \subset O_{j+1}$. Hence 1) $bd\, G_{Oj} \subset P_{Oj}$ and so $bd\, G_{Oj} \in \lambda^\wedge$ and 2) (see $(**)$) all $\omega_{lj} = \{G_{Oj} : O \in \Omega_l\}$ are ULF in A and in F. It follows from the definition of F'_{Oj} that $F \cap O_j = F \cap O \cap f_O^{-1}(1/j, 1] \subset F'_{Oj} \subset G_{Oj}$ and so $F \cap (\cup\Omega_{lj}) \subset \cup\omega_{lj}$, $l \in \mathbb{N}$, $j = 2, 3, \dots$ Since Ω_{lj} is FOLF in A, the union $\cup\Omega_{lj}$ is functionally open in A.

By $(***)$ and our construction, $\omega = \{G_{Oj} : j = 2, 3, \dots, O \in \Omega\}$ is an open cover of F; every member of ω is disjoint from either C or D; $\omega = \cup\{\omega_{lj} : l \in \mathbb{N}, j = 2, 3, \dots\}$, where all ω_{lj} are ULF in A; the cover $\Omega' = \{\cup\Omega_{lj} : l \in \mathbb{N}, j = 2, 3, \dots\}$ of A is countable, functionally open and $F \wedge \Omega' = \{F \cap (\cup\Omega_{lj}) : l \in \mathbb{N}, j = 2, 3, \dots\}$ refines the cover $\omega' = \{\cup\omega_{lj} : l \in \mathbb{N}, j = 2, 3, \dots\}$ of F. Since every countable functionally open cover of a space has a (countable) FOLF refinement, by Lemma 5 from [25] ($\equiv$ Lemma 4 from [27]), there exists a closed and ULF family $\varkappa$ in A, such that for any $K \in \varkappa$, there is $G_{Oj} \in \omega$ with $K \subset bd\, G_{Oj} \in \lambda^\wedge$ and $P = \cup\varkappa$ is a partition between C and D in F. Since $P \in \lambda$, we have proved that λ breaks $\mu^\wedge$. $\square$

Theorem 16 ([23]). *If a subset A of a space X is d-right (in particular, d-posed), then*

$$\operatorname{Id} A \leq \operatorname{Id} X. \tag{9}$$

Proof. Let $\operatorname{Id} X = n$, $n = 0, 1, 2, \dots$ Take closed families σ^*_i, $i = -1, 0, \dots, k \leq n$, in X that satisfy conditions (a)-(d) from Definition 9.

Put $\sigma^\wedge_i = A \wedge \sigma^*_i = \{A \cap F^* : F^* \in \sigma^*_i\}$, $i = -1, 0, \dots, k$. Evidently, families $\sigma^\wedge_i$ satisfy conditions (a), (c) and (d) from Definition 9. Let σ_i be uniformly generated by $\sigma^\wedge_i$. Then these families also satisfy conditions (a), (c) and (d) from Definition 9. By Lemma 4, σ_{i-1} brakes $\sigma^\wedge_i$. By Lemma 3 from [25], the family uniformly generated by σ_{i-1} coincides with σ_{i-1}. Hence by Lemma 2 from [25] ($\equiv$ Lemma 2 from [27]), σ_{i-1} breaks σ_i. $\square$

Lemma 5. *If F is a closed subset of a normal subspace A of a space X and A is d-right (d-posed) subset of X, then A is also d-right (d-posed) subset of X.*

Proof. Let Ω be a finite functionally open cover of F. Since A is normal, for any $O \in \Omega$, there is a functionally open set U_O in A such that $O = A \cap U_O$. Then F and $G = A \setminus \cup\{U_O : O \in \Omega\}$ are disjoint closed sets in the normal space A. Hence there exists a functionally open set U in A that contains G and is contained in $A \setminus F$. Then $\Omega' = \{U\} \cup \{U_O : O \in \Omega\}$ is a finite functionally open cover of A and so there exists its σ-locally finite (and functionally open) refinement μ with the following property: For any $V \in \mu$, there is a functionally open set $W(V)$ in X such that V is closed-open in $A \cap W(V)$ (respectively, $V = A \cap W(V)$). It follows from this that $F \wedge \mu = \{F \cap V : V \in \mu\}$ is a σ-locally finite (and functionally open) refinement of Ω with the following property: For any $F \cap V$, $V \in \mu$, there is a functionally open set $W(V)$ in X such that $F \cap V$ is closed-open in $F \cap W(V)$ (respectively, $F \cap V = F \cap W(V)$).

Proof of Theorem 9. If the finite sum theorem for Ind is true in X, then by Proposition 1 and Theorem 16, $\operatorname{Ind} A \leq \operatorname{Id} A \leq \operatorname{Id} X = \operatorname{Ind} X$.

Now let the finite sum theorem for Ind is true in A.

If $\operatorname{Ind} X = -1$, then $\operatorname{Ind} A = -1 \leq \operatorname{Ind} X$. Let (5) be true for $\operatorname{Ind} X < k$, $k > -1$, and let $\operatorname{Ind} X = k$.

Take $\sigma^*_{(i,\mathrm{Ind})} = \{F^* \subset X : F^* \text{ is closed in } X, \operatorname{Ind} F^* \leq i\}$, $i = -1, \dots, k = \operatorname{Ind} X$. Then the families $\sigma^*_{(i,\mathrm{Ind})}$ have properties $(a), (b)$ and (d) from Definition 9 and the families $\sigma^\wedge_{(i,\mathrm{Ind})} = A \wedge \sigma^*_{(i,\mathrm{Ind})}$ (in A) has properties (a) and (d) from Definition 9. By Lemma 5, for any $F^* \in \sigma^*_{(i,\mathrm{Ind})}$, $i < k$, the set $F^\wedge = A \cap F^*$ is d-right in X and (so) in F^*. Then by the inductive hypothesis, $\operatorname{Ind} F^\wedge \leq \operatorname{Ind} F^* \leq i$. Let σ_{k-1} be the family uniformly generated by $\sigma^\wedge_{(k-1,rmInd)}$. Then by Corollary 3, $\operatorname{Ind} F \leq k-1$ for any $F \in \sigma_{k-1}$. By Lemma 3, σ_{k-1} breaks the family $\sigma^\wedge_{(k,\mathrm{Ind})}$. Since $X \in \sigma^*_{(k,\mathrm{Ind})}$, we have that $A \in \sigma^\wedge_{(k,\mathrm{Ind})}$ and so $\operatorname{Ind} A \leq k = \operatorname{Ind} X$. □

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