ON NAGATA'S CONTRIBUTION TO THEORY OF GENERALIZED METRIC SPACES

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Abstract.

In this article, we summarize the late J. Nagata's contribution to the theory of generalized metric spaces. Especially, the central point of our discussion we give here is the so-called Nagata-Smirnov's metrization theorem and its application. We state how he has influenced us in this field.

1 Introduction It has been just a while meeting him at the annual symposium of topological spaces held at Tokyo Gakugei University. Suddenly, we were informed the very sad news that Dr. J. Nagata died in October, 2007 and finished his eighty two years lifetime. This is the biggest grief to the Japanese researchers of general topology as well as all researchers in the world.

Nagata was one of the biggest figures in general topology. His work ranges from metrization theory, theory of generalized metric spaces and dimension theory in general topology. It is still prominent and nowadays is playing an important role in this research. Many researchers are interested in reading his publications and influenced by his work.

In this article, we review his work only in the area of the theory of generalized metric spaces.

Throughout this article, by a space we mean a topological space. All spaces are assumed to be regular T_1-spaces, unless the contrary stated explicitly. Letters $\mathbb{N}$, $\mathbb{R}$ are all natural numbers, real numbers, respectively. All mappings are continuous. In the sequel, references N$*$ means Nagata's publication, prepared in the last section of this article.

2 Metrizability theory A set X combined with a metric function $d : X \times X \to \mathbb{R}^+ = \{r \in \mathbb{R} \mid r \geq 0\}$ satisfying the following metric axiom is called a metric space.

(1) $d(x, y) = 0 \iff x = y$;

(2) $d(x, y) = d(y, x)$;

(3) (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$

A metric space (X, d) becomes a topological space if for each $x \in X$, the collection $\{B(x; \epsilon) \mid \epsilon > 0\}$ of ϵ-balls forms a local base at x in X. In this case, the topology $\tau(X)$ is said to be induced by the metric function d. A topological space X is called *metrizable* if $\tau(X)$ is compatible with the topology induced by some metric function. So, it is natural to ask when a topological space is metrizable. According to Nagata [N27], metrization theory is a field of topology where the main objective is to study conditions for a given topological space to be metrizable, and propositions stating this result is metrization theorem.

As the metrization theorem before Nagata and Smirnov, the following was established by Alexandroff and Urysohn in 1923. This is one of the types how to give the condition

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of metrizability. This case is to require existence of a sequence of open covers with special properties of a given space.

Theorem 1 (Alexandroff and Urysohn's metrization theorem [3]). *A T_0-space X is metrizable if and only if there exists a sequence $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of open covers of X such that $\mathcal{U}_{n+1}^* < \mathcal{U}_n$ for each $n \in \mathbb{N}$ and for each $x \in X$, the collection $\{St(x,\mathcal{U}_n) \mid n \in \mathbb{N}\}$ is a local base at x in X, where $St(x,\mathcal{U}_n)$ is the union of all members of $\mathcal{U}_n$ containing x.*

The following Moore's metrization theorem is of the same type:

Theorem 2 (Moore's metrization theorem [18]). *A T_0-space X is metrizable if and only if there exists a sequence $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of open covers of X such that for each $x \in X$ and $x \in U \in \tau(X)$, there exists an open set V and $n \in \mathbb{N}$ such that $x \in V \subset St(V,\mathcal{U}_n) \subset U$.*

On the contrary, Nagata's metrization is different. It requires a special type of "open base" for the topology, in place of a sequence of open covers. Smirnov found the same theorem independently in 1951.

Theorem 3 (Nagata-Smironov's metrization theorem [N72, 25]). *A space X is metrizable if and only if there exists a σ-locally finite base for X.*

This is the most influential contribution to the theory of generalized metric spaces. For, as stated later, by this metrization theorem we were given many methods to generalize metric spaces in various manners.

How do we generalize metric spaces ?

Until this, we have been relaxing the condition of metric function $d : X \times X \to \mathbb{R}^+$ stated above, and has obtained quasi-metric spaces, semi-metric spaces, symmetrizable spaces, etc. Here we call these spaces generalized metric space of "the first type".

But now that we are given Nagata-Smirnov's metrization theorem, we can generalize the condition of σ-locally finite base. In fact, various types of generalized metric spaces are flourished. We call these spaces generalized metric space of "the second type".

It is needless to say that a class $\mathcal{C}$ of generalized metric spaces constructed is required to satisfy the criteria for generalized metric spaces such as

(1) $\mathcal{C}$ is hereditary, i.e., $\mathcal{C}$ is closed under subspaces;

(2) $\mathcal{C}$ is countably productive, i.e., if for each $n \in \mathbb{N}$, $X_n \in \mathcal{C}$, then the product space $\prod_{n\in\mathbb{N}} X_n \in \mathcal{C}$;

(3) $\mathcal{C}$ is closed under perfect or closed mappings.

Looking in this manner, we can stress that Nagata's contribution in essence has been playing the vital role and is indispensable for the later development of generalized metric spaces.

Local finiteness in Nagata-Smirnov's metrization was replaced with a strong condition "discrete" in Bing's metrization, which has the same direction:

Theorem 4 (Bing's metrization [4]). *A space X is metrizable if and only if there exists a σ-discrete base for X.*

In his second paper [N54], he showed the metrization theorem of new style using his original theorem Theorem 3 and Michael's characterization of paracompact spaces in terms of σ-locally finite as follows: A space X is paracompact if and only if every open cover has a σ-locally finite open refinement, [15].

Theorem 5 ([N54, Theorem 1]). *A T_1-space X is metrizable if and only if for each point $x \in X$, we can assign a local base $\{g(n,x) \mid n \in \mathbb{N}\}$ and $\{g_1(n,x), g_2(n,x) \mid n \in \mathbb{N}\}$ satisfying the following:*

(1) $g_2(n,y) \cap g_1(n,x) \neq \emptyset \implies y \in g(n,x)$;

(2) $y \in g_1(n,x) \implies g_2(n,x) \subset g(n,x)$

To state metrizability in terms of the so-called g-functions is a new point. We are naturally derived to characterize various generalized metric spaces in terms of g-functions. This is another contribution to general topology.

Nagata himself gave us a few good surveys of metrizability, [N14, N27]

As an example of generalized metric spaces in the namesake of him, we have "Nagata space" defined by J. Ceder [6, Definition 3.1], which has a Nagata structure

$$< g(n,x), g_1(n,x) >_{x \in X, n \in \mathbb{N}}$$

satisfying the following:

(1) $g_1(n,y) \cap g_1(n,x) \neq \emptyset \implies y \in g(n,x)$;

(2) for each $x \in X$, $\{g(n,x) \mid n \in \mathbb{N}\}$ is a local base at x in X.

3 The generalized metric spaces of the second type J. Ceder generalized Nagata-Smirnov's metrization theorem by relaxing the condition "locally finite" with "closure-preserving" and "base" with "quasi-base", "pair-base" and defined M_i-spaces ($i = 1, 2, 3$), [6].

Definition 1. *A collection $\mathcal{B}$ of subsets of a space X is called closure-preserving (for brevity, CP) in X if for any subcollection $\mathcal{B}'$ of $\mathcal{B}$,*

$$\overline{\bigcup \mathcal{B}'} = \bigcup \{\overline{B} \mid B \in \mathcal{B}'\}$$

The following implication is obvious and not reversible. This is the reason that M_i-spaces are generalizations of metric spaces.

$$\text{discrete} \longrightarrow \text{locally finite} \longrightarrow \text{CP} \longrightarrow \text{cushioned}$$

Definition 2 ([6, Definition 1]). *A space X is called an M_1-space if X has a σ-CP base.*

Definition 3. *A collection $\mathcal{B}$ of subsets of a space X is called a quasi-base for X if whenever $x \in U \in \tau(X)$, then $x \in \mathrm{Int}B \subset B \subset B$ for some $B \in \mathcal{B}$.*

Quasi-bases are generalized to pair-bases:

Definition 4. *Let $\mathcal{P}$ be a collection of pairs $P = (P_1, P_2)$ of an open set P_1 and a closed subset P_2 of X such that $P_1 \subset P_2$. $\mathcal{P}$ is called a pair-base for X whenever $x \in U \in \tau(X)$, then there exists $P \in \mathcal{P}$ such that*

$$x \in P_1 \subset P_2 \subset U.$$

$\mathcal{P}$ is called cushioned in X for any subcollection $\mathcal{P}'$ of $\mathcal{P}$,

$$\overline{\bigcup \{P_1 \mid P \in \mathcal{P}'\}} \subset \bigcup \{P_2 \mid P \in \mathcal{P}'\}.$$

A space X is called an M_3-space if there exists a σ-cushioned pair-base for X.

Borges [5] renamed M_3-spaces stratifiable spaces that were defined as follows:

Definition 5 ([5]). *A space X is called stratifiable if it has the stratification $S : \mathbb{N} \times \tau(X) \to \tau(X)$ satisfying the following:*

(1) $\overline{S(n,U)} \subset U$ *for every* $U \in \tau(X),\ n \in \mathbb{N}$*;*

(2) $U = \bigcup_{n\in\mathbb{N}} \overline{S(n,U)}$*;*

(3) $U \subset V$ *implies* $S(n,U) \subset S(n,V)$ *for each* $n \in \mathbb{N}$*.*

¿From the definition, the following are obvious:

metrizable space $\longrightarrow$$M_1$-space$\longrightarrow$$M_2$-space$\longrightarrow$$M_3$ -space=stratifiable space

Whether the reverse implications hold or not has been an open problem in theory of generalized metric spaces. Junnila [12] and Gruenhage [7] independently showed the coincidence of M_2-spaces and M_3-spaces. Finally, whether the implication M_3-space $\Longrightarrow$ M_1-space hold or not remains, and the problem is called M_3 versus M_1 problem. Even today the final solution is not given, and this is considered to be one of the most outstanding problem in general topology.

But, we have a series of partial answers to it. Nagata space is one of them. Ito showed that Nagata space is an M_1-space and gave the positive answer to Ceder's question in [11], where he showed in a more general form that every M_3-space whose every point has a CP open local base is an M_1-space. Recall that Nagata spaces are nothing but first countable M_3-spaces, [6, Theorem 3.1].

It is surprising that Nagata indicated without proof that McAuley space is non-metrizable, first countable, M_1-space [N54, Footnote], before Ceder's paper was published. Ceder himself gave the complete but "lengthy" proof to it [6, Example 9.2].

Under this circumstance, it is quite natural that Watson proposed the problem to give a simple proof to Nagata's statement in the above comparing to Ceder's proof, [26, Problem 3.3.18]. To this problem, Amano and Mizokami gave one positive answer [1].

As for M_3 versus M_1 problem, there is the following Nagata's study: Nagata refined Hyman's result to that every paracomplex in the sense of Hyman is an M_1-space [N38]. Paracomplexes contain all metric spaces and CW-complexes. In [N38] he proposed the problem whether any closed image of paracomplex is an M_1-space. This was solved positively by Mizokami [17].

Until now, we have constructed a series of partial answers to M_3 versus M_1 problem. In it, we have the following classes of M_1-spaces:

Lašnev space[14] $\Longrightarrow$ D-space in the sense of Nagami [21] $\Longrightarrow$ L-space in the sense of Nagami[21] $\Longrightarrow$ free L-space in the sense of Nagami [22] $\Longrightarrow$ M_3-μ-space

For each space of the implication, it is shown that the covering dimension (or large inductive dimension)

$dimX \le n$ $\Longleftrightarrow$ there exists a σ-CP base $\mathcal{W}$ for X such that $\dim B(W) \le n-1$ for each $W \in \mathcal{W}$, where $B(W)$ is the boundary of W in X.

It is quite natural that Nagata proposed whether this is the case for M_1-spaces [N38, Problem 2].

But, in spite of all effort, this remains open. Only some partial answers were obtained. For example, Mizokami studied this problem and defined M-structures and showed that every M_3-space with an M-structure is an M_1-space and the dimension is characterized by the existence of such a base, stated above, giving the partial answer [16]. But, by Junnila and Mizokami, both M_3-spaces with an M-structure and M_3-μ-spaces coincide with each other, [13].

Another relaxation of Nagata-Smirnov's metrization theorem is to generalize base to "network".

Definition 6. *A collection $\mathcal{F}$ of subsets of a space X is called a network for X if every open set of X is the union of members of $\mathcal{F}$, that is, whenever $x \in O \in \tau(X)$, then $x \in F \subset O$ for some $F \in \mathcal{F}$.*

In 1967, Okuyama defined the class $\mathcal{C}$ of σ-spaces and studied the fundamental topological operations for $\mathcal{C}$ [24].

Definition 7 ([24]). *A space X is called a σ-space if it has a σ-locally finite closed network.*

If we replace 'locally finite' with 'CP' or 'discrete', then what kinds of spaces are obtained ? As for the case Nagata-Smirnov's metrization, we have M_1-spaces, and they are different from metric spaces. But this is not the case for σ-spaces.

Most other cases are different from the case of metric spaces as above. Generally speaking, in many definitions and results containing the term "discrete" or "locally finite" in theory of generalized metric spaces, even if we replace "discrete" or "locally finite" with "CP", one gets an equivalence statement.

This is the case for σ-spaces:

Theorem 6 ([N52]). *For a space X, the following are equivalent:*

(1) X is a σ-space;

(2) X has a σ-discrete network;

(3) X has a σ-CP network.

Here, we would like to pay attention to the fact that the method in the proof of the construction of a σ-discrete collection from a given CP collection of subsets of a space with a σ-CP closed network is very influential to the later researchers. For example, Mizokami's M-structures [16], Oka's encircling-network [23], Heath's construction [8] were influenced directly or indirectly.

4 Generalized metric spaces of the third type By relaxing Alexandroff and Urysohn's metrization theorem, we can get many generalized metric spaces. For example, there are Moore spaces (developable spaces), $w\Delta$-spaces, G_δ-diagonals, etc. We call these spaces generalized metric spaces of "the third type". As a typical one, there are the following spaces:

Definition 8 (Morita [19, 20]). *A space X is called an M-space if there exists a sequence $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of open covers of X satisfying the following:*

(1) $\mathcal{U}^*_{n+1} < \mathcal{U}_n$ for each $n \in \mathbb{N}$;

(2) if for each $x \in X$ and $n \in \mathbb{N}$, $x_n \in St(x,\mathcal{U}_n)$, then $\{x_n \mid n \in \mathbb{N}\}$ clusters in X.

Definition 9 (Ishii [10]). *A space X is called an M^*-space if there exists a sequence $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of locally finite open covers of X satisfying the condition (2) in the above definition.*

There are several characterizations of paracompact M-spaces. First, a paracompact M-spaces is characterized to be a perfect pre-image of a metric space, [19]. M-spaces are turn out to be the same as p-spaces in the sense of Arhangel'skii [2] in the presence of paracompactness.

Nagata proves the embedding theorem for paracompact M-spaces as follows:

Theorem 7 (Nagata [N47, Theorem 1]). *A space X is a paracompact M-space if and only if X is homeomorphic to a closed subspace of the product of a metric space and a compact space.*

As the characterization and embedding theorem of M*-spaces, the following is also due to Nagata:

Theorem 8 (Nagata [N41, Theorem 7]). *The following are equivalent:*

(1) Y is an M^*-space;

(2) there exists an M-space X and a compact space C such that X is a closed set in the product space $Y \times C$, and $\pi_Y(X) = Y$, where π_Y denotes the projection from $Y \times C$ onto Y;

(3) there exists an M-space X and a perfect mapping f from X onto Y.

Nagata introduced a new term "half-metric spaces", and then characterized generalized metric spaces:

Definition 10 (Nagata [N41, Definition 7]). *Let $(X.X')$ be a pair of a space X and its subspace X'. If X has a sequence $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of open covers of X satisfying the following condition $(*)$, then (X, X') is called a half-metric space with the metric part X':*
$(*)$ *For every point $x \in X'$ and for every neighborhood V of x in X, there exists $U \in \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ such that $x \in U \subset V$.*

Theorem 9 (Nagata [N41, Theorem 8]). *For a space Y, the following are equivalent:*

(1) Y is a σ-space:

(2) there exists a compact space C and a half-metric space (X, X') such that X is a closed set in the product space $Y \times C$, and $\pi_Y(X') = Y$, where π_Y denotes the projection from $Y \times C$ onto Y;

(3) there exists a half-metric space (X, X') and a perfect mapping f from X onto Y satisfying $f(X') = Y$.

Besides these results, he characterized nice spaces (generalized metric spaces) by nice mappings. For example, there are the following:

Theorem 10 (Nagata [N50, Theorem 1]). *A space Y is a q-space in the sense of Michael if and only if there are an M-space and an open mapping f from X onto Y.*

Theorem 11 (Nagata [N41, Theorem 11]). *A space Y is a Nagata space if there are a subspace X of Baire's zero-dimensional metric space and a mapping f of X onto Y, where f is almost open and q-closed.*

Similar results on stratifiable spaces and semi-stratifiable spaces were obtained in [N41].

In his later years, he has studied generalized metric spaces through g-functions, and those results were published in the journal "Q and A in general Topology", stated in the next section, [N6, N5, N3, 22, 21]. This style as one researcher of General topology continued to his death. we should take him as the role model.

5 The others Finally, we should pay more attention to his good surveys of metrization theory and that of generalized metric spaces [N4, N14, N16, N17, N27, N31, N32, N40] as well as dimension theory. These are giving the researchers good advices and the direction of research.

On the other hand, we, only Japanese researchers, should express thanks to his activity of introducing everywhere the world researchers the up-to-date development in general topology in Japan, [N8, N10, N11, N23, N37].

As another thing worthy of note, we have to mention that in Japan he started the publication of the journal "Questions and Answers in General Topology" and this contributes development of general topology not only in Japan but also in the world.

We are thinking that this is a monument of his contribution to general topology.

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