THE ABU-MUHANNA CONJECTURE ON SUPPORT POINTS

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Abstract. We determine support points of the subordination family of an arbitrary bounded analyltic function, thereby proving the Abu-Muhanna conjecture [J. London Math. Soc. (2), 29 (1984)].

1 Introduction In 1984, Abu-Muhanna conjectured that the support points of a subordination family of any bounded analyltic function F are exactly all compositions of F with the finite Blashke products that are null at zero [1]. We shall prove this conjecture exploiting the membership of $f(z) = z$ among the family of convex mappings, as well as the fact that any bounded function, after a proper normalization, can be considered an element of the subordination family $s(z)$.

The main result, from which the Abu-Muhanna conjecture will follow, shall deal with the support points of the subordination family $s(F)$, where F belongs to the closed convex hull of the family of convex univalent mappings

We begin with an overview of definitions and known results. As usual, let $\Delta = \{z \in C : |z| < 1\}$. $A(\Delta)$ denotes the linear space of functions analytic in Δ with the topology of uniform convergence on compact sets. $A(\Delta)$ is locally convex. Let $A(\Delta)^*$ be the space of continuous linear functionals on $A(\Delta)$.

The Krein-Milman theorem holds for every compact subset F of $A(\Delta)$. If HF denotes the closed convex hull of F and EHF denotes the set of its extreme points then $HF = HEHF$. Furthermore, $F \supset EHF$ and for every functional $J \in A(\Delta)^*$

$$\max_{f \in F} \mathrm{Re} J(f) = \max_{f \in EHF} \mathrm{Re} J(f).$$

Let B_0 denote the class of functions $\varphi \in A(\Delta)$ such that $|\varphi| < 1,\ z \in \Delta$, and $\varphi(0) = 0$. Let $f,\ F \in A(\Delta)$. Then f is said to be subordinate to F if and only if there exists a function $\varphi \in B_0$ such that $f = F \circ \varphi$. The class of functions subordinate to F is denoted by $s(F)$ [9].

A function f is called a support point of a compact subset F of $A(\Delta)$ if $f \in A(\Delta)$ and there exists a functional $J \in A(\Delta)^*$ such that Re $J(f) = \max\{\mathrm{Re}J(g) : g \in F\}$ and ReJ is non-constant on F. The set of support points of F is denoted by supp F. Each $J \in A(\Delta)^*$ is uniquely represented by a sequence of complex numbers $\{b_n\}_{n=0}^{\infty}$ such that $\limsup_{n\to\infty} n\sqrt{|b_n|} < 1$ and $J(f) = \sum_{n=0}^{\infty} b_n a_n$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in Δ [11, p.36].

The set supp B_0 consists of all finite Blashke products in B_0 [3].

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Abu-Muhanna proved in [1] that supp $s(F) \subset \{F\circ\varphi : \varphi \in \text{supp}B_0\}$ for any non-constant F, $F \in A(\Delta)$. There are a few known cases in which equality is attained. For example,

$$\text{supp } s(F) = \{F \circ \varphi : \varphi \in \text{supp } B_0\} \tag{1}$$

for $F \in K$, where K denotes the class of univalent convex mappings on Δ with $F(0) = F'(0) - 1 = 0$ [8]. The equality holds also for any non-constant function F analytic in the closed unit disc [1].

The class of bounded analytic functions is denoted by H^∞ and

$$\|f\|^\infty = \lim_{r\to 1} \max_{0\le\theta\le 2\pi} |f(re^{i\theta})|.$$

(See [9].)

2 Support points of the subordination families with majorants in the closed convex hull of the family of convex univalent mappings.

Let $s(K) = \{f \in A(\Delta) : \exists_{F\in K} f \in s(F)\}$. This is to say, each f in $s(K)$ is subordinate to a univalent convex mapping on Δ, with the standard normalization at 0.

The set of extreme points of the closed convex hull $EHs(K)$ of $s(K)$ consists of the functions $yz/(1 - xz)$, where $|x| = |y| = 1$ [7]. We use this fact to prove that (1) occurs whenever $F \in Hs(K)$.

Theorem 1. *If $F \in Hs(K)$ then supp $s(F) = \{F \circ \varphi \in \text{supp}B_0\}$.*

Proof. We need only prove that if $\varphi \in$ supp B_0 then $F \circ \varphi \in$ supp $s(F)$. Let $\bar\varphi(z) = \overline{\varphi(\bar z)}$ and let $J \in [A(\Delta)]^*$ be given by coefficients of $\bar\varphi$. Then

$$J(f) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\bar\varphi\left(\frac{e^{-i\theta}}{r}\right)d\theta$$

for $r < 1$ sufficiently close to 1. It is easy to see that

$$J(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta})\overline{\varphi(e^{i\theta})}d\theta = 1.$$

Lemma 7.18 in [6] implies that $J(\varphi^n) = 0$ for $n = 2, 3, ...$

Hence, $J(F \circ \varphi) = J(\varphi) = 1$. We also have

$$\max_{f\in s(F)} \text{Re}J(f) \le \max_{f\in s(K)} \text{Re}J(f) \le \max_{f\in Hs(K)} \text{Re}J(f) =$$

$$\max_{f\in EHs(K)} \text{Re}J(f) = \max_{x,y\in\partial\Delta} \text{Re}J\left(\frac{yz}{1-xz}\right).$$

Assume now that $\psi \in A(\bar\Delta)$. Then, for $r < 1$ and sufficiently close to 1 and for $x \in \partial\Delta$,

$$\psi(x) = \frac{1}{2\pi i}\int_{|\xi|=1/r} \frac{\psi(\xi)}{\xi - x}d\xi = \frac{1}{2\pi i}\int_{2\pi}^{0} \frac{\psi\left(\frac{e^{-i\theta}}{r}\right)}{\frac{e^{-i\theta}}{r} - x}\left(\frac{-ie^{-i\theta}}{r}\right)d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\psi\left(\frac{e^{-i\theta}}{r}\right)\frac{e^{-i\theta}}{r}}{\frac{e^{-i\theta}}{r} - x}d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{\psi\left(\frac{e^{-i\theta}}{r}\right)}{1 - xre^{i\theta}}d\theta.$$

Let $\psi(z) = \frac{\bar{\varphi}(z)}{z}$. Since $\bar{\varphi}(0) = 0$ and $\bar{\varphi} \in A(\bar{\Delta})$ also $\psi \in A(\bar{\Delta})$.
Hence,

$$J\left(\frac{yz}{1-xz}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{yre^{i\theta}}{1-xre^{i\theta}}\bar{\varphi}\left(\frac{e^{-i\theta}}{r}\right)d\theta = y\frac{1}{2\pi}\int_0^{2\pi} \frac{re^{i\theta}\bar{\varphi}\left(\frac{e^{-i\theta}}{r}\right)}{1-xre^{i\theta}}d\theta$$

$$= y\frac{1}{2\pi}\int_0^{2\pi} \frac{\left(\frac{e^{-i\theta}}{r}\right)}{1-xre^{i\theta}}d\theta = y\psi(x) = y\frac{\bar{\varphi}(x)}{x}.$$

It follows that $\max_{x,y\in\partial\Delta} \mathrm{Re}J(\frac{yz}{1-xz}) = \max_{x,y\in\partial\Delta} \mathrm{Re}(y\frac{\bar{\varphi}(x)}{x}) \le \max_{x,y\in\partial\Delta}\left|\frac{y\bar{\varphi}(x)}{x}\right| = 1.$

Therefore, $\max_{f\in s(F)} \mathrm{Re}J(f) \le 1$.

Finally, since $F\circ\varphi^2 \in s(F)$ and $J(F\circ\varphi^2) = J(\varphi^2) = 0$, $\mathrm{Re}J \ne$ const on $s(F)$. Hence $F\circ\varphi \in \mathrm{supp}\ s(F)$. □

Theorem 2. *If F is a bounded function, non-constant and analytic in Δ, then supp $s(F) = \{F\circ\varphi : \varphi \in \mathrm{supp}\ B_0\}$.*

Proof. Let$F \in H^\infty$, and define G in Δ as follows:

$$G(z) = \frac{F(z) - F(0)}{\|F\|^\infty}$$

Note that $G \in s(z)$.

Suppose now that $f \in \mathrm{supp}\ s(F)$. There exists $\psi \in B_0$ and a functional $J \in A(\Delta)^*$, with $\mathrm{Re}J \ne$ const on $s(F)$, such that $f = F\circ\psi$ and

$$\forall\varphi \in B_0 \ \ \mathrm{Re}J(F\circ\varphi) \le \mathrm{Re}J(F\circ\psi).$$

We shall show that $g = G\circ\psi$ is a support point of $s(G)$.
Indeed,

$$\mathrm{Re}J(g) = \mathrm{Re}J(G\circ\psi) = \mathrm{Re}J\left(\frac{F(\psi(z)) - F(0)}{\|F\|^\infty}\right)$$

$$= \frac{1}{\|F\|^\infty}[\mathrm{Re}J(F(\psi(z) - \mathrm{Re}J(F(0))] \ge \frac{1}{\|F\|^\infty}[\mathrm{Re}J(F(\varphi(z) - \mathrm{Re}J(F(0))] \ \ \forall\varphi \in B_0$$

Since the latter equals Re $J(G\circ\varphi)$, we have

$$\forall\varphi \in B_0 \ \ \mathrm{Re}J(g) \ge \mathrm{Re}J(G\circ\varphi).$$

Hence, $g \in \mathrm{supp}\ s(G)$. Since $G \in s(z)$ and $z \in K$, it follows that $G \in Hs(K)$ and, by Theorem 1, supp $s(G) = \{G\circ\varphi : \varphi \in \mathrm{supp}\ B_0\}$.
Therefore, $\psi \in \mathrm{supp}\ (B_0)$.

Conversely, if $\psi \in \mathrm{supp}B_0$, then, by Theorem 1, $g = G\circ\psi \in \mathrm{supp}\ s(G)$. Hence, there is a functional $J \in A(\Delta)^*$, with Re $J \ne$ const on $s(G)$, such that

$$\forall\varphi \in B_0 \ \ \mathrm{Re}J(G\circ\varphi) \le \mathrm{Re}J(g).$$

Furthermore,

$$\text{Re}J(F\circ\psi) = \|F\|^{\infty}\text{Re}J(G\circ\psi) + \text{Re}J(F(0)) \geq \|F\|^{\infty}\text{Re}J(G\circ\varphi) + \text{Re}J(F(0)) \quad \forall\varphi\in B_0$$

Since $\|F\|^{\infty}\text{Re}J(G\circ\varphi)+\text{Re}J(F(0)) = \text{Re}J(F\circ\varphi)$, we have shown that $F\circ\psi \in \text{supp}\, s(F)$ □

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