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# THE STUDY OF TRADE-OFF IN CONJOINT ANALYSIS

H.Kono[†], H.Ishii[†] and S.Shiode[‡]

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Abstract.

Conjoint analysis is a scaling method originally developed in mathematical psychology. Today it is used in many of the social sciences and applied sciences including marketing, product management, and operations research. It is used for measuring each factorfs contribution to the whole evaluation of products consisting of some factors, for example, in testing customer acceptance of new product designs, in assessing the appeal of advertisements and in service design. TRADE-OFF is a method of conjoint analysis used for measuring the part worth value of factors in the total evaluation, exclusively using when evaluations is non-metrical data. The part worth values obtained by TRADE-OFF give an approximate comparison of each factor's contribution to the total evaluation, but it is impossible to utilize their contributions for statistical use since they are usually obtained by numerical solution. Moreover, their solution isn't correct and unique. And it is known that the part worth value obtained by In this paper, we show the problems of TRADE-OFF and then propose a method to obtain its definite solution.

**1 Introduction** Conjoint analysis is a scaling method originally developed in mathematical psychology. Today it is used in many of the social sciences and applied sciences including marketing, product management, and operations research. It is used for measuring each factor's contribution to the whole evaluation of products consisting of some factors, for example, in testing customer acceptance of new product designs, in assessing the appeal of advertisements and in service design.

In Conjoint Analysis, many numbers of algorithms are used to estimate utility functions. Green and Srinivasan(1978) who developed conjoint analysis classified those estimation methods in three categories. First, they described methods that the total evaluation is assumed to be ordinal scaled. In that case estimation methods like *MONANOVA*(MONotone ANalysis Of Variance) ,*TRADE-OFF* or *LINMAP* can be used. Second, when it is assumed to be interval scaled, *OLS*(Ordinary Least Squares) regression techniques can be used. Third, for the paired comparison data in a choice context, the binary *Logit* or *Probit* model can be used.

With respect to TRADE-OFF, even with the method as the representative one of conjoint analysis, one cannot always measure theoretically the definite part worth values from the ranking data. As a result their obtained scores cannot by default be applied in statistical methods. And they also showed very little difference between OLS ,MONANOVA and TRADE-OFF in terms of recovery of parameters, based on simulated rank order data. But since it is theoretically unacceptable to use OLS on ranks, it may for this reason be better to collect ratings, if interval-scales can be assumed. This particular undesirability of TRADE-OFF is well known; nevertheless, the method has been widely used, since it enables users to specify values of factors to a certain degree, simply from the ranking data.

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However, there are very few researchers to measure definite values directly by elaborating on TRADE-OFF.

Therefore, we would like to propose the method to obtain the formularization of TRADE-OFF's part worth values. This paper is organized as follows. In Section 2, we introduce TRADE-OFF and the traditional approach to obtain the part worth values. In Section 3, we propose the method to obtain the formularization of TRADE-OFF's part worth values. Finally in Section 4, we summarize our results.

**2 TRADE-OFF** In this section, we introduce TRADE-OFF. We use the following notation throughout the paper. $Y$ is the order preserving transformation of ordinal scale, where $m$ is the number of samples,

$$Y = [Y_1, Y_2, \cdots, Y_m]^T \tag{1}$$

$D$ is the 0-1 design matirx indicating each level of factors of samples, where $n$ is the number of levels of each factor

$$D = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_j \\ \vdots \\ D_m \end{bmatrix} = \begin{bmatrix} d_{11} & d_{21} & \cdots & d_{i1} & \cdots & d_{n1} \\ d_{12} & d_{22} & \cdots & d_{i2} & \cdots & d_{n2} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{1j} & d_{2j} & \cdots & d_{ij} & \cdots & d_{nj} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ d_{1m} & d_{2m} & \cdots & d_{im} & \cdots & d_{nm} \end{bmatrix} \tag{2}$$

It follows that $\sum_{i=1}^{n} d_{ij} = l$, respectively, at $j = 1 \cdots m$, where $l$ is the number of factors. $B$ is the set of part worth values to be estimated and $B_j (j = 1 \cdots n)$ express the part worth value of the factor $j$. With using the part worth value $B_j$, we can know the inportance rating of the factor $j$

$$B = [B_1, B_2, \cdots, B_j, \cdots, B_n]^T \tag{3}$$

$X(B)$ is the equation of *the conjoint model.*

$$X(B) = [X_1(B), X_2(B), \cdots, X_m(B)]^T \tag{4}$$

We obtains the partworth values $B$ by minimizing the goodness of fit criterion $S$ under the restriction of *conjoint model* $X(B)$ and $Y$. In TRADEOFF, We use $S$ (5)(6) for the goodness of fit criterion

$$S(B) = \frac{\sum_{i<j} E_{ij}(X_i(B) - X_j(B))^2}{\sum_{i<j} (X_i(B) - X_j(B))^2} \tag{5}$$

$$E_{ij} = \begin{cases} 1: & (X_i(B) - X_j(B))(Y_i - Y_j) < 0 \\ 0: & (X_i(B) - X_j(B))(Y_i - Y_j) \geq 0 \end{cases} \tag{6}$$

and additive model $X_i(B)$ (7) for *conjoint model.*

$$X_i(B) = D_i B \tag{7}$$

The traditional approach to obtain the part worth value $B$ is the following iterative algorithm.

***Algorithm***

**step1:** Chose the initial value of $B^{(k)}$ with the solution of OLS and set $k = 0$. Select a convergence parameter $\epsilon > 0$.

**step2:** Calculate the next equation (8) and get $B^{(k+1)}$.

$$B_j^{(k+1)} = B_j^{(k)} - \alpha \frac{\partial S(B^{(K)})}{\partial B_j} \tag{8}$$

The partial derivative of $S$ with respect to $b_j$ is

$$\frac{\partial S}{\partial B_i} = \frac{\sum_{i<j} 2(E_{ij} - S)(X_i(B) - X_j(B))(D_i - D_j)}{\sum_{i<j} (X_i(B) - X_j(B))^2} \tag{9}$$

**step3:** If $S(B^{(K)}) - S(B^{(K+1)}) < \epsilon$ , then stop the iteration process as $B^{(k)}$ is a minimum point of $S(B)$. Otherwise, Set $k = k + 1$ and go to Step 2.

However using this numerical approach, their solution isn't unique and depended on the starting point $B^{(0)}$. Moreover, they suffer from local optimum problems. As a result we cannot apply especially them as statistical methods.

## 3 PROPOSAL METHODS

With using the follow equations, we obtain the part worth values $\hat{B}$ without using iterative algorithm.

$$\begin{cases} \sum_{i=1}^{m} \sum_{j=1}^{m} (E_{ij}(B^*) - \lambda^*)(d_{ik} - d_{jk})(D_i - D_j)B^* = 0 \quad (k = 1, \cdots, n) \\ \sum_{i=1}^{m} \sum_{j=1}^{m} (E_{ij}(B^*) - \lambda^*)(D_i B^* - D_j B^*)^2 = 0 \end{cases} \tag{10}$$

***Proof.***

We define the value of $\lambda (0 \leq \lambda \leq 1)$ and the equation of $g(\lambda, B)$ (11)

$$g(\lambda, B) = \sum_{i<j} \{(E_{ij} - \lambda)(X_i(B) - X_j(B))^2\} \tag{11}$$

and $G(\lambda)$(12) , as follow

$$G(\lambda) = \min_B [g(\lambda, B)] = g(\lambda, B^*) \tag{12}$$

If

$$G(\lambda^*) = 0 \tag{13}$$

then

$$\lambda^* = \min_B [S(B)] = S(B^*) \tag{14}$$

For $E_{ij} = E_{ji}$ and $(X_i(B) - X_j(B))^2 = (X_j(B) - X_i(B))^2$, we have

$$g(\lambda, B) = \sum_{i<j} \{(E_{ij} - \lambda)(X_i(B) - X_j(B))^2\} \tag{15}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \{(E_{ij} - \lambda)(X_i(B) - X_j(B))^2\}/2 \tag{16}$$

For $\lambda_1 \leq \lambda_2$, we have

$$\begin{aligned}
&(17)\ 2G\{t\lambda_1 + (1-t)\lambda_2\} \\
&= \min_B \sum_{i=1}^{m} \sum_{j=1}^{m} [(E_{ij} - \{t\lambda_1 + (1-t)\lambda_2\}](X_i(B) - X_j(B))^2\} \\
&= \min_B \sum_{i=1}^{m} \sum_{j=1}^{m} [tE_{ij}(X_i - X_j)^2 + (1-t)E_{ij}(X_i - X_j)^2 - \{t\lambda_1 + (1-t)\lambda_2\}(X_i - X_j)^2] \\
&= t \min_B \sum_{i=1}^{m} \sum_{j=1}^{m} \{(E_{ij} - \lambda_1)(X_i - X_j)^2\} + (1-t) \min_B \sum_{i=1}^{m} \sum_{j=1}^{m} \{(E_{ij} - \lambda_2)(X_i - X_j)^2\}
\end{aligned}$$

($X_i, X_j$ means $X_i(B), X_j(B)$)

Here, We denote $B^*$ is an optimal solution of $S(B)$.

$$G\{t\lambda_1 + (1-t)\lambda_2\} \leq tG(\lambda_1) + (1-t)G(\lambda_2). \tag{18}$$

and

$$\begin{aligned}
2G(\lambda_1) &= \min_B \sum_{i=1}^{m} \sum_{j=1}^{m} (E_{ij} - \lambda_1)(X_i(B) - X_j(B))^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} (E^*_{ij} - \lambda_1)(X_i(B^*) - X_j(B^*))^2 \\
&\geq \sum_{i=1}^{m} \sum_{j=1}^{m} (E^*_{ij} - \lambda_2)(X_i(B^*) - X_j(B^*))^2 \geq \min_B \sum_{i=1}^{m} \sum_{j=1}^{m} (E_{ij} - \lambda_2)(X_i(B) - X_j(B))^2
\end{aligned}$$

$$G(\lambda_1) \geq G(\lambda_2) \tag{19}$$

Hence , $G(\lambda)$ is monotone decreasing and concave function of $\lambda$, Furthermore,

$$G(\lambda = 0) = \min_B \sum_{i=1}^{m} \sum_{j=1}^{m} E_{ij}(X_i(B) - X_j(B))^2 \geq 0 \tag{20}$$

$$G(\lambda = 1) = \min_B \sum_{i=1}^{m} \sum_{j=1}^{m} (E_{ij} - 1)(X_i(B) - X_j(B))^2 \leq 0 \tag{21}$$

$$G(\lambda = 0) > G(\lambda = 1) \tag{22}$$

so there exists a unique $\lambda^*$ such that $G(\lambda^*) = 0$; that is,

$$G(\lambda^*) = g(\lambda^*, B^*) = \sum_{i=1}^{m} \sum_{j=1}^{m} (E_{ij}(B^*) - \lambda^*)(X_i(B^*) - X_j(B^*))^2 = 0 \tag{23}$$

Since

$$\begin{aligned} 0 &= \sum_{i=1}^{m}\sum_{j=1}^{m}(E_{ij}(B^*)-\lambda)(X_i(B^*)-X_j(B^*))^2 \\ &\leq \sum_{i=1}^{m}\sum_{j=1}^{m}(E_{ij}(B)-\lambda)(X_i(B)-X_j(B))^2 \end{aligned} \tag{24}$$

for any $B$, and

$$\begin{aligned} \lambda^* &= \frac{\sum_{i=1}^{m}\sum_{j=1}^{m}E_{ij}(B^*)(X_i(B^*)-X_j(B^*))^2}{\sum_{i=1}^{m}\sum_{j=1}^{m}(X_i(B^*)-X_j(B^*))^2} \\ &\leq \frac{\sum_{i=1}^{m}\sum_{j=1}^{m}E_{ij}(B)(X_i(B)-X_j(B))^2}{\sum_{i=1}^{m}\sum_{j=1}^{m}(X_i(B)-X_j(B))^2} = S(B) \end{aligned} \tag{25}$$

We have that $B^*$ is an optimal solution of $S(B)$.

When we use the sigmoid function $E_{ij}(B)(26)$,

$$E_{ij}(B) = \frac{1}{1+\exp\left[\alpha(Y_i-Y_j)(D_i-D_j)B\right]} \quad (\alpha \to \infty) \tag{26}$$

we obtain $\partial g(\lambda, B)/\partial B_k$ and $\partial^2 g(\lambda, B)/\partial B_k^2$ as follow.

$$g(\lambda, B) = 2\sum_{i=1}^{m}\sum_{j=1}^{m}(E_{ij}(B)-\lambda)(X_i(B)-X_j(B))^2 \tag{27}$$

$$\frac{\partial g(\lambda, B)}{\partial B_k} = 2\sum_{i=1}^{m}\sum_{j=1}^{m}(E_{ij}(B)-\lambda)(d_{ik}-d_{jk})(X_i(B)-X_j(B)) \tag{28}$$

$$\frac{\partial^2 g(\lambda, B)}{\partial B_k^2} = 2\sum_{i=1}^{m}\sum_{j=1}^{m}(E_{ij}(B)-\lambda)(d_{ik}-d_{jk})^2 \tag{29}$$

When $\lambda = \lambda^*$,

$$\frac{\partial^2 g(\lambda, B)}{\partial B_k^2} = \sum_{i=1}^{m}\sum_{j=1}^{m}(E_{ij}(B)-\lambda^*)(d_{ik}-d_{jk})^2 \tag{30}$$

$$= \left\{\frac{\sum_{i=1}^{m}\sum_{j=1}^{m}E_{ij}(B)(d_{ik}-d_{jk})^2}{\sum_{i=1}^{m}\sum_{j=1}^{m}(d_{ik}-d_{jk})^2} - \lambda^*\right\}\sum_{i=1}^{m}\sum_{j=1}^{m}(d_{ik}-d_{jk})^2 \tag{31}$$

Here, $\sum_{i=1}^{m}\sum_{j=1}^{m}(d_{ik}-d_{jk})^2 \geq 0$ and

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{m}E_{ij}(B)(d_{ik}-d_{jk})^2}{\sum_{i=1}^{m}\sum_{j=1}^{m}(d_{ik}-d_{jk})^2} - \lambda^* \tag{32}$$

$$= \frac{\sum_{i=1}^{m}\sum_{j=1}^{m}E_{ij}(B)(d_{ik}-d_{jk})^2}{\sum_{i=1}^{m}\sum_{j=1}^{m}(d_{ik}-d_{jk})^2} - \min_B \left\{ \frac{\sum_{i=1}^{m}\sum_{j=1}^{m}E_{ij}(B)(X_i(B)-X_j(B))^2}{\sum_{i=1}^{m}\sum_{j=1}^{m}(X_i(B)-X_j(B))^2} \right\} \geq 0 \tag{33}$$

So , $\partial g(\lambda, B)/\partial B_k$ is monotone increasing function of $B_k$, Furthermore,

$$g(0, B) = 2\sum_{i=1}^{m}\sum_{j=1}^{m}E_{ij}(B)(X_i(B)-X_j(B))^2 \geq 0 \tag{34}$$

$$g(1, B) = 2\sum_{i=1}^{m}\sum_{j=1}^{m}(E_{ij}(B)-1)(X_i(B)-X_j(B))^2 \leq 0 \tag{35}$$

so there exists a unique $B_k$ such that $\partial g(\lambda, B)/\partial B_k = 0$; that is, $g(\lambda, B)$ is a convex function of $B_k$, so, when we minimize $g(\lambda, B)$ with respect to $B_k$, $\dfrac{\partial g(\lambda^*, B)}{\partial B_i} = 0$.

$$\frac{\partial g(\lambda, B)}{\partial B_k} = 2\sum_{i=1}^{m}\sum_{j=1}^{m}(E_{ij}(B^*)-\lambda^*)(d_{ik}-d_{jk})(D_i-D_j)B^* = 0 \tag{36}$$

$$g(\lambda, B) = 2\sum_{i=1}^{m}\sum_{j=1}^{m}(E_{ij}(B^*)-\lambda^*)(D_iB^*-D_jB^*)^2 = 0 \tag{37}$$

We obtain the part worth values $\hat{B}$ as the follow equations.

$$\begin{cases} \sum_{i=1}^{m}\sum_{j=1}^{m}(E_{ij}(B^*)-\lambda^*)(d_{ik}-d_{jk})(D_i-D_j)B^* = 0 \quad (k = 1, \cdots, n) \\ \sum_{i=1}^{m}\sum_{j=1}^{m}(E_{ij}(B^*)-\lambda^*)(D_iB^*-D_jB^*)^2 = 0 \end{cases} \tag{38}$$

**4 Conclusion** In this paper, we have proposed the method to obtain the formula of TRADE-OFF's part worth value. With this formula, we can easily obtain the part worth value and prevent the local optimum problems.

As future studies, we would like to show the difference and similarity of TRADE-OFF and the other method of conjoint analysis. the score of LINMAP is known similar to that of TRADE-OFF.[11] We show similarity between LINMAP and TRADE-OFF more explicitly, though it is suggested by Green.

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†Department of Information and Physical Sciences
Graduate School of Science and Technology, Osaka University
2-1 Yamada-oka, Suita 565–0871, Japan

‡Faculty of Business Administration, Kobe Gakuin University
518 Arise,Ikawadani-cho,Nishi-ku,Kobe 651-2180 JAPAN, Japan

E-mail : hikouno@ist.osaka-u.ac.jp, ishii@ist.osaka-u.ac.jp, shiode@ba.kobegakuin.ac.jp