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ON HIRATA SEPARABLE EXTENSIONS FOR GROUP RINGS

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Abstract. Let RG be a group ring of a group G over a ring R with 1, R_0 the center of R, C the center of RG, and $\overline{G}$ the inner automorphism group of the group ring RG induced by the elements of G. Characterizations are given for RG which is a Hirata separable extension of $(RG)^{\overline{G}}$ and C is a direct summand of R_0G as a C-module. Thus an Azumaya group ring RG can be characterized in terms of Hirata separable extensions. Moreover, it is shown that RG is neither a Hirata separable extension of R nor a Galois extension of $(RG)^{\overline{G}}$ with Galois group $\overline{G}$.

1 INTRODUCTION Let RG be a group ring of a group G over a ring R with 1, R_0 the center of R, and C the center of RG. F.R. DeMeyer and G.J. Janusz ([2]) characterized RG which is an Azumaya algebra. It is well known that a Hirata separable extension of a ring is a generalization of an Azumaya algebra and that the center C of an Azumaya algebra A is a direct summand of A as a C-module. Let $\overline{G}$ be the inner automorphism group of the group ring RG induced by the elements of G. Then $(RG)^{\overline{G}} = RC$. In the present paper, we are interested in the group ring RG which is a Hirata separable extension of $(RG)^{\overline{G}}$ and C is a direct summand of R_0G as a C-module. We shall show the following equivalent statements: (1) RG is a Hirata separable extension of $(RG)^{\overline{G}}$ and C is a direct summand of R_0G as a C-module. (2) R_0G is an Azumaya C-algebra. (3) The center of G has a finite index and the order of the commutator subgroup of G is a finite integer and invertible in R. Thus an Azumaya group ring RG as studied in ([2]) can be characterized in terms of Hirata separable extensions. Moreover, it will be shown that RG is neither a Hirata separable extension of R nor a Galois extension of $(RG)^{\overline{G}}$ with an inner Galois group $\overline{G}$ induced by the elements of G. This paper was revised under the suggestions of the referee and written under the support of a Caterpillar Fellowship at Bradley University. The authors would like to thank the referee for the valuable suggestions and Caterpillar Inc. for the support.

2 BASIC DEFINITIONS AND NOTATIONS Let B be a ring with 1 and A a subring of B with the same identity 1. Then B is called a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, ..., k$ for some integer $k\}$ such that $\sum a_ib_i = 1$ and $\sum xa_i \otimes b_i = \sum a_i \otimes b_ix$ for all x in B where $\otimes$ is over A. In particular, B is called an Azumaya algebra if it is a separable extension over its center. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule. Let G be a finite automorphism group of B and $B^G = \{x \in B \mid g(x) = x$ for all $g \in G\}$. Then B is called a Galois extension of B^G with Galois group G if there exist elements $\{c_i, d_i$

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in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum c_i d_i = 1$ and $\sum c_i g(d_i) = 0$ for each $g \neq 1$ in G.

Throughout this paper, let R be a ring with identity 1 and G a group. Then RG denotes the group ring of G over R, and RG_f a projective group ring with a factor set $f : G \times G \longrightarrow \{$units in the center of $R\}$ such that $f(gh, l)f(g, h) = f(g, hl)f(h, l)$ if RG_f is a free R-module with a basis $\{U_g \,|\, g \in G\}$ such that $U_g U_h = f(g, h)U_{gh}$; in particular, when R is commutative, a projective group ring RG_f is called a projective group algebra. For more about projective group rings RG_f, see [5] and [6].

3 CHARACTERIZATIONS

Let RG be a group ring of a group G over a ring R with 1, R_0 the center of R, C the center of RG, and $\overline{G}$ the inner automorphism group of the group ring RG induced by the elements of G. We shall give characterizations of RG which is a Hirata separable extension of $(RG)^{\overline{G}}$ and C is a direct summand of R_0G as a C-module. We begin with two useful lemmas.

Lemma 3.1. ([4], Proposition 1.3-(1))
Let B be a Hirata separable extension of A and A is a direct summand of B as a A-bimodule. Then, $V_B(A)$ is separable over the center of B and $V_B(V_B(A)) = A$ where $V_B(A)$ is the commutator subring of A in B.

The proof of the following lemma is straightforward.

Lemma 3.2.
If A is an Azumaya C-algebra, then for any C-algebra D, $D \otimes_C A$ is a Hirata separable extension of D.

Proof. Since A is an Azumaya C-algebra, $A \otimes_C A$ is a direct summand of a finite direct sum of A as a A-bimodule. Hence $D \otimes_C A \otimes_C A \cong (D \otimes_C A) \otimes_D (D \otimes_C A)$ is a direct summand of a finite direct sum of $D \otimes_C A$ as a $D \otimes_C A$-bimodule; that is, $D \otimes_C A$ is a Hirata separable extension of D.

Theorem 3.3.
By keeping the notations in this section, then the following statements are equivalent:

(1) RG is a Hirata separable extension of RC and C is a direct summand of R_0G as a C-module.

(2) R_0G is an Azumaya C-algebra.

(3) The center of G has a finite index and the order of the commutator subgroup of G is a finite integer and invertible in R.

Proof. (1) $\Longrightarrow$ (2) Since $RG \cong R \otimes_{R_0} R_0G$ and $C =$ the center of $RG =$ the center of R_0G, $RG \cong (R \otimes_{R_0} C) \otimes_C R_0G$ by the multiplication map. By hypothesis, C is a direct summand of R_0G as a C-bimodule, so $(R \otimes_{R_0} C)$ is a direct summand of RG as a $(R \otimes_{R_0} C)$-bimodule. Hence RG is a Hirata separable extension of RC which is a direct summand of RG as a RC-bimodule. Thus $V_{RG}(RC)$ is a separable C-algebra by Lemma 3.1. Noting that $V_{RG}(RC) = V_{RG}(R) = R_0G$, we have that R_0G is an Azumaya C-algebra.

(2) $\Longrightarrow$ (1) Since $RG \cong RC \otimes_C R_0G$ as given in the proof of (1) $\Longrightarrow$ (2), that R_0G is an Azumaya C-algebra implies that RG is a Hirata separable extension of RC by Lemma 3.2. Also, since R_0G is an Azumaya C-algebra, C is a direct summand of R_0G as a C-module ([1], Lemma 3.1, page 51).

(2) $\Longleftrightarrow$ (3) is a consequence of Theorem 1 in [2] for R_0 is commutative.

As given by Theorem 3.3, we next show a similar equivalent condition for RG which is a Hirata separable extension of R_0G and C is a direct summand of RC as a C-module.

Theorem 3.4.
RG is a Hirata separable extension of R_0G and C is a direct summand of RC as a C-module if and only if R is an Azumaya R_0-algebra.

Proof. ($\Longrightarrow$) Since $RG \cong RC \otimes_C R_0G$ by multiplication map as given in the proof of Theorem 3.3-((1) $\Longrightarrow$ (2)) and C is a direct summand of RC as a C-module by hypothesis, R_0G is a direct summand of RG as a R_0G-bimodule. Moreover, RG is a Hirata separable extension of R_0G by hypothesis, so $V_{RG}(R_0G) = V_{RG}(G) = RC$ is a separable C-algebra by Lemma 3.1. Hence $R \otimes_{R_0} C$ ($\cong RC$) is an Azumaya C-algebra. Since R_0G is a group algebra, the center C is a free R_0-module with generators $\{c(g) \,|\, c(g) =$ sum of distinct conjugate elements of g for g having finite number of distinct conjugate elements in $G\}$. Thus R_0 is a direct summand of C. This implies that R is a separable R_0-algebra ([1], Corollary 1.8, page 44). Therefore R is an Azumaya R_0-algebra (for R_0 is the center of R).

($\Longleftarrow$) Since R is an Azumaya R_0-algebra, RC ($\cong R \otimes_{R_0} C$) is an Azumaya C-algebra ([1], Corollary 1.7, page 44). Noting that $RG \cong RC \otimes_C R_0G$, we conclude that RG is a Hirata separable extension of R_0G by Lemma 3.2.

By Theorem 3.3 and Theorem 3.4, we derive two characterizations for an Azumaya group ring RG as studied in [2] in terms of Hirata separable extensions.

Theorem 3.5.
By keeping the notations in Theorem 3.4, the following statements are equivalent:

(1) RG is an Azumaya C-algebra.

(2) RG is a Hirata separable extension of RC and R is an Azumaya R_0-algebra.

(3) RG is a Hirata separable extension of R_0G and R_0G is an Azumaya C-algebra.

Proof. (1) $\Longrightarrow$ (2) Since RG is an Azumaya C-algebra by hypothesis, R is an Azumaya R_0-algebra and CG is an Azumaya C-algebra by Lemma 2.2 in [2]. Noting that $CG = R_0G$, we have that R_0G is an Azumaya C-algebra. Thus RG is a Hirata separable extension of RC by Theorem 3.3-((2) $\Longrightarrow$ (1)).

(2) $\Longrightarrow$ (1) Since R is an Azumaya R_0-algebra by hypothesis, $R \otimes_{R_0} C$ is an Azumaya $R_0 \otimes_{R_0} C$-algebra; and so RC is an Azumaya C-algebra. But RG is a Hirata separable extension of RC by hypothesis, so RG is a separable extension of RC. Thus RG is a separable C-algebra by the transitivity property of separable extensions; that is, RG is an Azumaya C-algebra.

(1) $\Longrightarrow$ (3) Since RG is an Azumaya C-algebra, R_0G ($= CG$) is an Azumaya C-algebra and R is an Azumaya R_0-algebra by Lemma 2.2 in [2]. Hence RG is a Hirata separable extension of R_0G by the sufficiency of Theorem 3.4. Thus statement (3) holds.

(3) $\Longrightarrow$ (1) Since Hirata separable extensions and Azumaya algebras are separable extensions, condition (3) implies that RG is an Azumaya C-algebra by the transitivity property of separable extensions.

Remark.

To show (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) in Theorem 3.5, we may employ a theorem for an Azumaya algebra due to S. Ikehata ([3], Theorem 1): Let B be an Azumaya algebra over C and A a subalgebra of B. If B is projective over A, then B is a Hirata separable extension of A. In Theorem 3.5, $B = RG$, $A = RC$ or $A = R_0G$. Since RG is an Azumaya C-algebra, both RC and R_0G are Azumaya C-algebras. Hence RG is projective over RC and R_0G, respectively, by the Zelinsky-Rosenburg Theorem ([1], Proposition 2.3, page 48). Thus, by Theorem 1 in [3], RG is a Hirata separable extension of RC and R_0G, respectively. Therefore (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) hold.

4 TWO PROPERTIES

By keeping the notations in section 3, we shall show that RG is neither a Hirata separable extension of R nor a Galois extension of $(RG)^{\overline{G}}$ with Galois group $\overline{G}$.

Theorem 4.1.
Assume that $G \neq \{1\}$. Then RG is not a Hirata separable extension of R.

Proof. Suppose RG is a Hirata separable extension of R. Noting that R is a direct summand of RG as a R-bimodule, we have that $V_{RG}(V_{RG}(R)) = R$ by Lemma 3.1. But $V_{RG}(V_{RG}(R)) = V_{RG}(R_0G) = V_{RG}(G) = RC \neq R$, so the contradiction completes the proof.

In case G is abelian, $\overline{G} = \{1\}$, and hence RG is a Galois extension of $(RG)^{\overline{G}} = RG$ with a trivial Galois group $\{1\}$. Next, we show that in case G is not abelian, RG is not a Galois extension of $(RG)^{\overline{G}}$ with inner Galois group $\overline{G}$ induced by the elements of G.

Theorem 4.2.
Assume that G is not abelian. Then RG is not a Galois extension of $(RG)^{\overline{G}}$ with inner Galois group $\overline{G}$ induced by the elements of G.

Proof. Assume that RG is a Galois extension of $(RG)^{\overline{G}}$ with inner Galois group $\overline{G}$ induced by the elements of G. Then RG contains a projective group algebra $C\overline{G}_f$ of $\overline{G}$ over C with a factor set $f : \overline{G} \times \overline{G} \longrightarrow$ units of C ([6], Theorem 2.1). Hence $C\overline{G}_f$ is a free C-module with basis $\{\overline{g} | \, \overline{g} \in \overline{G}\}$; and so $R_0\overline{G}_f$ is a free R_0-module with basis $\{\overline{g} | \, \overline{g} \in \overline{G}\}$ (for $R_0 \subset C$). But $C \subset R_0\overline{G}_f$, so $C\overline{G}_f \subset R_0\overline{G}_f$. Thus $C\overline{G}_f = R_0\overline{G}_f$, where $R_0 \subset C$, with basis $\{\overline{g} | \, \overline{g} \in \overline{G}\}$. Therefore $C = R_0$. This contradicts to that $C \neq R_0$ because G is a nonabelian group. Hence RG is not a Galois extension of $(RG)^{\overline{G}}$ with inner Galois group $\overline{G}$ induced by the elements of G.

We conclude the present paper with an example of RG which is a Hirata separable extension of $(RG)^{\overline{G}}$ and C is a direct summand of R_0G as a C-module, but RG is not an Azumaya algebra.

Example.
Let Q be the rational field and G a finite nonabelian group. Then QG is an Azumaya algebra by Theorem 1 in [2]. Let A be the direct product of infinite copies of quaternion Q-algebra. Then AG is a Hirata separable extension of AC ($\cong A \otimes_Q C$) by Lemma 3.2 where C is the center of QG, but AG is not an Azumaya algebra because A is not an Azumaya algebra by Lemma 2.2 in [2].

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