THE CHROMATIC NUMBER OF THE SIMPLE GRAPH ASSOCIATED WITH A COMMUTATIVE RING

JUNRO SATO* AND KIYOSHI BABA**

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ABSTRACT. Let R be a Noetherian ring and let $Z(R)$ be the set of all zero-divisors of R. We denote by $G(R)$ the simple graph whose vertices are elements of R and in which two distinct vertices x and y are joined by an edge if $x-y$ is in $Z(R)$. Let $\chi(R)$ be the chromatic number of the graph $G(R)$. If $\chi(R)$ is finite, then R is an integral domain or R is a finite Artin ring. In the former case we have $\chi(R)=1$ and in the latter case we get $\chi(R)=\max\{|M_1|,\dots,|M_t|\}$ where $M_1,\dots,M_t$ are all maximal ideals of R and $|M_i|$ denotes the number of elements of the set M_i for $i=1,\dots,t$.

Let R be a commutative ring with the identity element. An element x of R is called a zero-divisor of R if there exists a non-zero element y of R satisfyihg $xy=0$. We denote $Z(R)$ the set of all zero-divisors of R. We consider the simple graph $G(R)$ whose vertices are elements of R and in which distinct two vertices x and y are joined by an edge if $x-y$ is in $Z(R)$. We color the vertices of $G(R)$ so that no two joined vertices have the same color. If we color the vertices, we call it a coloring of $G(R)$. The chromatic number $\chi(R)$ of the graph $G(R)$ is the minimum number of colors of colorings of $G(R)$. We denote by $V(R)$ the vertices of a graph $G(R)$.

Our notation is standard and for unexplained terms, our general reference to commutative algebra is [1], [3] and our general reference to graph theory is [2].

Example 1. Let $\mathbf{Z}$ be the ring of all integers and let R be the residue class ring $\mathbf{Z}/6\mathbf{Z}$. We denote by i the residue class of $i+6\mathbf{Z}$ for $i=0,1,\dots,5$ because no fear of confusion. Therefore $R=\{0,1,2,3,4,5\}$. Then $V(R)=\{0,1,2,3,4,5\}$ and $Z(R)=\{0,2,3,4\}$. We color vertices 0 and 5 as red, 1 and 2 as blue, 3 and 4 as yellow. This is a coloring of $G(R)$ with three colors. The triangle of vertices 0, 2, 4 needs three colors. Hence $\chi(R)=3$.

Let C be a non-empty subset of $V(R)$. We call C a clique of $G(R)$ if every pair of distinct two elements of C is joined by an edge. The clique number $C(R)$ of $G(R)$ is the maximum number of elements of cliques of $G(R)$.

Example 2. The clique number $C(R)$ of R in Example 1 is 3.

Lemma 3. *The following inequality holds:*

$$C(R) \leqq \chi(R).$$

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Proof. In the case that $\chi(R)$ is finite, let C be an arbitrary clique of $G(R)$. Then every vertix of C must be colored with different color because C is a clique of $G(R)$. Moreover, $G(R)$ needs at least $|C|$ colors because C is a subset of $G(R)$ where $|C|$ denotes the number of elements of C. Hence $C(R) \leqq \chi(R)$. In the case $\chi(R)$ is not finite, then $C(R) \leqq \chi(R)$ also holds. Q.E.D.

The symbol $\coprod$ denotes the disjoint union of sets.

Lemma 4. *Let*

$$V(R) = V_1 \coprod V_2 \coprod \cdots \coprod V_t.$$

is a disjoint union of $V(R)$ such that no pair of distinct two elements of V_i is joined by an edge for $i = 1, 2, \ldots, t$. Then $\chi(R) \leqq t$.

Proof. We color all vertices of V_i by the same color and we color the vertices of V_i and the vertices of V_j by different colors for $i \neq j$. It is a coloring of $G(R)$. We need t kinds of colors. Hence $\chi(R) \leqq t$. Q.E.D.

Remark 5. If $\chi(R) = n$ and $c_1, \ldots, c_n$ are colors of minimum coloring of $G(R)$, then we set

$$V_i = \{x \in V(R); x \text{ is colored by a color } c_i\}$$

Then

$$V(R) = V_1 \coprod V_2 \coprod \cdots \coprod V_n.$$

is a disjoint union of $V(R)$ such that no pair of distinct two elements of V_i is joined by an edge.

Proposition 6. *If R is an integral domain, then $\chi(R) = 1$*

Proof. Let x and y be elements of $V(R)$. If $x - y$ is a zero-divisor, then $x = y$ because R is an integral domain and 0 is only one zero-divisor of R. Hence $G(R)$ has no edge and $G(R)$ is colored by a single color. This means that $\chi(R) = 1$. Q.E.D.

Primary decompositions of a Noetherian ring is reffered to [3].

Lemma 7. *Let R be a Noetherian ring and*

$$(0) = Q_1 \bigcap Q_2 \bigcap \cdots \bigcap Q_t$$

be an irredundant primary decomposition of (0). Set $P_i = \sqrt{Q_i}$ for $i = 1, 2, \ldots, t$. Then the following hold:

(1) $Z(R) = \bigcup_{i=1}^{t} P_i$

(2) $\mathrm{Ass}_R(R) = \{P_1, P_2, \ldots .P_t\}$.

Under the notations in Lemma 7, P_i is a clique of $G(R)$ for $i = 1, 2, \ldots, t$ because $x - y$ is in $P_i \subset Z(R)$ for all x and y in P_i. Therefore by Lemma 3, we have

$$\max\{|P|; P \in \mathrm{Ass}_R(R)\} \leqq C(R) \leqq \chi(R).$$

Theorem 8. *Let R be a Noethrian ring. If $\chi(R)$ is finite, then R is an integral domain or R is a finite ring.*

Proof. Assume that R is not an integral domain. We will show that R is a finite ring. The ideal (0) is not a prime ideal of R because R is not an integral domain. Let P be an arbitrary element of $\mathrm{Ass}_R(R)$, then $P \neq (0)$. Since $\chi(R)$ is finite, we know that $|P|$ is finite by the previous argument. Furtheremore there exists a non-zero element a of P. Then $aR \subset P$, hence $|aR|$ is finite. Let $\mathrm{Ann}_R(a) = \{x \in R; ax = 0\}$ be the annihilator ideal of a. Then we know that there is an R-module isomorphism $aR \cong R/\mathrm{Ann}_R(a)$. Note that $\mathrm{Ann}_R(a) \subset Z(R) = \bigcup_{P \in \mathrm{Ass}_R(R)} P$. Hence $|\mathrm{Ann}_R(a)| \leqq \sum_{P \in \mathrm{Ass}_R(R)} |P| < \infty$. Since $|aR|$ is finite, we have $|R/\mathrm{Ann}_R(a)| < \infty$. This means that $|R|$ is finite. Q.E.D.

Note that a commutative ring R is an Artin ring if and only if every family of ideals of R has a minimal element with respect to inclusion relation. Hence if R is a finite ring, then R is an Artin ring.

The following is a structure theorem of Artin rings. $A \times B$ denotes the direct product of sets A and B. If A and B are rings, then we cosider $A \times B$ as a ring.

Lemma 9([1] Theorem 8.7) *Let R be an Artin ring and*

$$(0) = Q_1 \bigcap Q_2 \bigcap \cdots \bigcap Q_t$$

be irredundant primary decomposition of (0). Set $m_i = \sqrt{Q_i}$ for $i = 1, 2, \ldots .t$. Then

(1) *R is isomorphic to a finite direct product of Artin local rings:*

$$R \cong R_1 \times R_2 \times \cdots \times R_t$$

where $R_1 = R/Q_1, R/Q_2, \ldots, R_t = R/Q_t$.

(2) *There exists a one-to-one correspondence between the set of maximal ideals of R and the set of maximal ideals of $R_1 \times R_2 \times \cdots \times R_t$.*

Lemma 10. *Let R_1 and R_2 be commutative rings with the identity element. Let V and W be finite subsets of $V(R_1)$ and $V(R_2)$ respectively such that no pair of distinct two elements of V is joined by an edge and so is W. Then there exist finite subsets $U_1, \ldots, U_r$ of $V \times W$ satisfying the following:*

(1) *$V \times W = U_1 \coprod U_2 \coprod \cdots \coprod U_r$.*

(2) *No pair of distinct two elements of U_i is joined by an edge for $i = 1, 2, \ldots, r$.*

(3) *$|U_1| = |U_2| = \cdots = |U_r|$.*

(4) *$r = \max\{|V|, |W|\}$.*

Proof. Set $n = |V|, m = |W|, V = \{a_1, \ldots, a_n\}$ and $W = \{b_1, \ldots, b_m\}$ We will prove the case $n \leqq m$. We set

$$\begin{aligned}
U_1 &= \{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}, \\
U_2 &= \{(a_1, b_2), (a_2, b_3), \ldots, (a_n, b_{n+1})\}, \\
&\cdots \\
U_{m-n+1} &= \{(a_1, b_{m-n+1}), (a_2, b_{m-n+2}), \ldots, (a_n, b_m)\}, \\
U_{m-n+2} &= \{(a_1, b_{m-n+2}), (a_2, b_{m-n+3}), \ldots, (a_n, b_1)\}, \\
&\cdots \\
U_m &= \{(a_1, b_m), (a_2, b_1), \ldots, (a_n, b_{n-1})\}.
\end{aligned}$$

Then $V \times W$ is equal to the disjoint union $U_1 \coprod U_2 \coprod \cdots \coprod U_m$ and $m = \max\{|V|, |W|\}$. Furtheremore we have $|U_1| = |U_2| = \cdots = |U_m| = n = /rmmin\{|V|, |W|\}$. We shall show that no pair of distinct two elements of U_i is joined by an edge for $i = 1, 2, \ldots, m$. Let x and y be distinct two elements of U_i and set $x = (x_1, y_1), y = (x_2, y_2)$. Then by the definition of U_i, we see that $x_1 \neq x_2$ and $y_1 \neq y_2$. Hence $x_1 - y_1$ is not a zero-divisor of R_1 because no pair of distinct two elements of V is joined by an edge. Also $x_2 - y_2$ is not a zero-divisor of R_2.This means that $x - y$ is not a zero-divisor of $R_1 \times R_2$. Therefore x and y are not joined by an edge.

We can also prove the case $n > m$. Q.E.D.

Lemma 11. *Let R_1 and R_2 be finite commutative rings with the identity element. Assume that the following hold:*

(1) $V(R_1) = V_1 \coprod V_2 \coprod \cdots \coprod V_{r_1}$.
(2) $V(R_2) = W_1 \coprod W_2 \coprod \cdots \coprod W_{r_2}$.
(3) $V_1, V_2, \ldots, V_{r_1}$ *are finite subsets of* $V(R_1)$.
(4) $W_1, W_2, \ldots, W_{r_2}$ *are finite subsets of* $V(R_2)$.
(5) *No pair of distinct two elements of* V_{i_1} *is joined by an edge for* $i_1 = 1, 2, \ldots, r_1$.
(6) *No pair of distinct two elements of* W_{i_2} *is joined by an edge for* $i_2 = 1, 2, \ldots, r_2$.

Then there exist finite subsets $T_1, T_2, \ldots, T_s$ of $R_1 \times R_2$ satisfying the following conditions:

(a) $V(R_1 \times R_2) = T_1 \coprod T_2 \coprod \cdots \coprod T_s$.
(b) *No pair of distinct two elements of* T_k *is joined by an edge for* $k = 1, 2, \ldots, s$.
(c)

$$s = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \max\{|V_{i_1}|, |W_{i_2}|\}.$$

Proof. We apply Lemma10 to V_{i_1} and W_{i_2}. Then there exist finite subsets $U_1^{(i_1,i_2)}, \ldots, U_{r(i_1,i_2)}^{(i_1,i_2)}$ of $V_{i_1} \times W_{i_2}$ satisfying the following properties:

(1) $V_{i_1} \times W_{i_2} = U_1^{(i_1,i_2)} \coprod \cdots \coprod U_{r(i_1,i_2)}^{(i_1,i_2)}$.
(2) *No pair of distinct two elements of* $U_i^{(i_1,i_2)}$ *is joined by an edge for* $i = 1, 2, \ldots, r(i_1, i_2)$.
(3) $|U_1^{(i_1,i_2)}| = \cdots = |U_{r(i_1,i_2)}^{(i_1,i_2)}|$.
(4) $r(i_1, i_2) = \max\{|V_{i_1}|, |W_{i_2}|\}$.

Hence we have

$$
\begin{aligned}
V(R_1 \times R_2) &= \coprod_{i_1=1}^{r_1} \coprod_{i_2=1}^{r_2} V_{i_1} \times W_{i_2} \\
&= \coprod_{i_1=1}^{r_1} \coprod_{i_2=1}^{r_2} (U_1^{(i_1,i_2)} \coprod \cdots \coprod U_{r(i_1,i_2)}^{(i_1,i_2)}).
\end{aligned}
$$

Then we can write

$$V(R_1 \times R_2) = T_1 \coprod T_2 \coprod \cdots \coprod T_s$$

with the property (b). Moreover,

$$s = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} r(i_1, i_2) = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \max\{|V_{i_1}|, |W_{i_2}|\}.$$

Q.E.D.

Lemma 12. *Let R_1 and R_2 be finite commutative rings with the identity element. Assume that the following hold:*

(1) $V(R_1) = V_1 \coprod V_2 \coprod \cdots \coprod V_{r_1}$.
(2) $V(R_2) = W_1 \coprod W_2 \coprod \cdots \coprod W_{r_2}$.
(3) $V_1, V_2, \ldots, V_{r_1}$ *are finite subsets of* $V(R_1)$.
(4) $W_1, W_2, \ldots, W_{r_2}$ *are finite subsets of* $V(R_2)$.
(5) $|V_1| = |V_2| = \cdots = |V_{r_1}|$.
(6) $|W_1| = |W_2| = \cdots = |W_{r_2}|$.
(7) *No pair of distinct two elements of* V_{i_1} *is joined by an edge for* $i_1 = 1, 2, \ldots, r_1$.
(8) *No pair of distinct two elements of* W_{i_2} *is joined by an edge for* $i_2 = 1, 2, \ldots, r_2$.

Then there exist finite subsets $T_1, T_2, \ldots, T_s$ of $R_1 \times R_2$ satisfying the following conditions:

(a) $V(R_1 \times R_2) = T_1 \coprod T_2 \coprod \cdots \coprod T_s$.
(b) $|T_1| = |T_2| = \cdots = |T_s|$.
(c) *No pair of distinct two elements of* T_k *is joined by an edge for* $k = 1, 2, \ldots, s$.
(d) $s = \max\{r_1|R_2|, |R_1|r_2\}$.

Proof. Set $t_1 = |V_{i_1}|$ for $i_1 = 1, 2, \ldots, r_1$ and $t_2 = |W_{i_2}|$ for $i_2 = 1, 2, \ldots, r_2$. Under the notations in the proof of Lemma 11, note that

$$|U_1^{(i_1,i_2)}| = \cdots = |U_{r(i_1,i_2)}^{(i_1,i_2)}| = \min\{|V_{i_1}|, |W_{i_2}|\} = \min\{t_1, t_2\}.$$

Hence we know that $|T_1| = |T_2| = \cdots = |T_s|$. On the other hand, we see that $|R_1| = r_1 t_1$ and $|R_2| = r_2 t_2$. This asserts that

$$
\begin{aligned}
s &= \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \max\{|V_{i_1}|, |W_{i_2}|\} = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \max\{t_1, t_2\} \\
&= r_1 r_2 \max\{t_1, t_2\} = \max\{r_1 r_2 t_1, r_1 r_2 t_2\} \\
&= \max\{|R_1| r_2, r_1 |R_2|\}.
\end{aligned}
$$

Q.E.D.

Example 13. Set $R_1 = R_2 = \mathbf{Z}/4\mathbf{Z}$. We denote by i the residue class of $i + 4\mathbf{Z}$ for $i = 0, 1, 2, 3$. Let m_1 and m_2 be the maximal ideals of R_1 and R_2 respectively. Then $m_1 = m_2 = \{0, 2\}$. Set $V_1 = W_1 = \{0, 1\}$ and $V_2 = W_2 = \{2, 3\}$ Then

$$V(R_1) = V_1 \coprod V_2,\ V(R_2) = W_1 \coprod W_2.$$

Set $R = R_1 \times R_2$ and set

$$\begin{aligned} T_1 = \{(0,0),(1,1)\}, T_2 = \{(0,1),(1,0)\}, \\ T_3 = \{(0,2),(1,3)\}, T_4 = \{(0,3),(1,2)\}, \\ T_5 = \{(2,0),(3,1)\}, T_6 = \{(2,1),(3,0)\}, \\ T_7 = \{(2,2),(3,3)\}, T_8 = \{(2,3),(3,2)\}. \end{aligned}$$

Then no pair of distinct two elements of T_k is joined by an edge for $k = 1, 2, \dots, 8$ and $V(R) = T_1 \coprod \cdots \coprod T_8$.

Proposition 14. *Let $R_1, \dots, R_t$ be finite commutative rings with the identity element. Assume that the following hold:*

(1) $V(R_i) = V_1^{(i)} \coprod \cdots \coprod V_{r_i}^{(i)}$ *for $i = 1, 2, \dots, t$.*

(2) $V_1^{(i)}, \dots, V_{r_i}^{(i)}$ *are finite subsets of $V(R_i)$ for $i = 1, 2, \dots, t$.*

(3) $|V_1^{(i)}| = \cdots = |V_{r_i}^{(i)}|$ *for $i = 1, 2, \dots, t$.*

(4) *No pair of distinct two elements of $V_k^{(i)}$ is joined by an edge for $k = 1, 2, \dots, r_i$ and for $i = 1, 2, \dots, t$.*

Then there exist finite subsets $U_1, \dots, U_s$ of $R_1 \times \cdots \times R_t$ satisfying the following conditions:

(a) $V(R_1 \times \cdots \times R_t) = U_1 \coprod \cdots \coprod U_s$.

(b) $|U_1| = \cdots = |U_s|$.

(c) *No pair of distinct two elements of U_j is joined by an edge for $j = 1, 2, \dots, s$.*

(d) $s = \max\{|R_1| \cdot \cdots \cdot |R_{i-1}| r_i |R_{i+1}| \cdot \cdots \cdot |R_t|; i = 1, 2, \dots, t\}$.

Proof. We shall show Proposition 14 by induction on t. The case $t = 1$ is obvious. Suppose that $t \geqq 1$ and the assertion holds for t. Then by the induction hypothesis, we get $V(R_1 \times \cdots \times R_t) = U_1 \coprod \cdots \coprod U_s$ with the properties (a)-(d). Furthermore

$$V(R_{t+1}) = V_1^{(t+1)} \coprod \cdots \coprod V_{r_{t+1}}^{(t+1)}$$

with $|V_1^{(t+1)}| = \cdots = |V_{r_{t+1}}^{(t+1)}|$. By Lemma 12, there exist finite subsets $T_1, \dots, T_{s'}$ *of* $(R_1 \times \cdots \times R_t) \times R_{t+1}$ satisfying the following conditions:

(1) $V((R_1 \times \cdots \times R_t) \times R_{t+1}) = T_1 \coprod \cdots \coprod T_{s'}$.

(2) $|T_1| = \cdots = |T_{s'}|$.

(3) No pair of distinct two elements of T_k is joined by an edge for $k = 1, 2, \dots, s'$.

(4) $s' = \max\{s|R_{t+1}|, |R_1 \times \cdots \times R_t| r_{t+1}\}$.

Hence $s' = \max\{|R_1| \cdot \cdots \cdot |R_{i-1}| r_i |R_{i+1}| \cdot \cdots \cdot |R_{t+1}|; i = 1, 2, \dots, t\}$. We prove the assertion.

Q.E.D.

Proposition 15. *Let (R, m) be finite Artin local ring. Then the following assertions hold:*

(1) *There exist finite subsets* $T_1, T_2, \dots, T_r$ *of* R *satisfying the following conditions:*
(a) $V(R) = T_1 \coprod T_2 \coprod \cdots \coprod T_r$.
(b) $|T_1| \cdots = |T_r| = |R/m|$.
(c) *No pair of distinct two elements of* T_j *is joined by an edge for* $j = 1, 2, \dots, r$.
(d) $r = |m|$.
(2) $\chi(R) = |m|$.

Proof. (1) Let $a_1+m, \dots, a_t+m$ be all residue classes of R/m and set $m = \{x_1, \dots, x_r\}$. Furthermore set $y_{ij} = a_i + x_j$ for $i = 1, 2, \dots, t$ and $j = 1, 2, \dots, r$. Then we have $R = \{y_{ij}; i = 1, 2, \dots, t; j = 1, 2, \dots, r\}$. Set $T_j = \{y_{ij}; i = 1, 2, \dots, t\}$ for $j = 1, 2, \dots, r$. Then we get $V(R) = T_1 \coprod T_2 \coprod \cdots \coprod T_r$ and $|T_1| = \cdots = |T_r| = t = |R/m|$. Let y_{ij} and y_{kj} be distinct two elements of $T_j (j = 1, 2, \dots, r)$. Then

$$y_{ij} - y_{kj} = (a_i + x_j) - (a_k + x_j) = a_i - a_k \notin m = Z(R).$$

Hence no pair of distinct two elements of T_j is joined by an edge for $j = 1, 2, \dots, r$.

(2) By the assertion (1) and Lemma 4 we see that $\chi(R) \leqq r$. Note that $m = Z(R)$ because $\mathrm{Ass}_R(R) = \{m\}$. Therefore m is a clique of $G(R)$. By Lemma 3 we have $r \leqq \chi(R)$. This means that $\chi(R) = r = |m|$. Q.E.D.

Theorem 16. *Let* $(R_1, m_1), \dots, (R_t, m_t)$ *be finite Artin local rings. Then the following assertions hold:*

(1) *There exist finite subsets* $U_1, \dots, U_s$ *of* $R_1 \times \cdots \times R_t$ *satisfying the following conditions:*
(a) $V(R_1 \times \cdots \times R_t) = U_1 \coprod U_2 \coprod \cdots \coprod U_s$.
(b) $|U_1| = \cdots = |U_s|$.
(c) *No pair of distinct two elements of* U_j *is joined by an edge for* $j = 1, 2, \dots, s$.
(d) $s = \max\{|R_1| \cdot \cdots \cdot |R_{i-1}||m_i||R_{i+1}| \cdot \cdots \cdot |R_t|; i = 1, 2, \dots, t\}$.
(2) $\chi(R_1 \times \cdots \times R_t) = \max\{|R_1| \cdot \cdots \cdot |R_{i-1}||m_i||R_{i+1}| \cdot \cdots \cdot |R_t|; i = 1, 2, \dots, t\}$.

Proof. (1) By Propositions 14 and 15, we get the assertion (1) noting that $r_i = |m_i|$ under the notation r_i in Proposition 15.

(2) Lemma 4 asserts that

$$\chi(R_1 \times \cdots \times R_t) \leqq \max\{|R_1| \cdot \cdots \cdot |R_{i-1}||m_i||R_{i+1}| \cdot \cdots \cdot |R_t|; i = 1, 2, \dots, t\}.$$

On the other hand $R_1 \times \cdots \times R_{i-1} \times m_i \times R_{i+1} \times \cdots \times R_t$ is a clique of $G(R_1 \times \cdots \times R_t)$. Hence by Lemma 3 we have

$$\max\{|R_1| \cdot \cdots \cdot |R_{i-1}||m_i||R_{i+1}| \cdot \cdots \cdot |R_t|; i = 1, 2, \dots, t\} \leqq \chi(R_1 \times \cdots \times R_t).$$

Hence we get the assertion (2). Q.E.D.

Let R be a Noetherian ring. If $\chi(R)$ is finite, then R is an integral domain or R is a finite ring by Theorem 8.

Theorem 17. *Let* R *be a Noetherian ring. Assume that* $\chi(R)$ *is finite. Then the following assertions hold:*

(1) *If* R *is an integral domain, then* $\chi(R) = 1$.

(2) *If R is a finite ring, then*

$$\chi(R) = \max\{|M_1|, \dots, |M_t|\}$$

where $M_1, \dots, M_t$ are all maximal ideals of R.

Proof. (1) The assertion (1) is clear from Proposition 6.

(2) If R is a finite ring, then R is a finite Artin ring. By Lemma 9, we know that R is isomorphic to a finite direct product of Artin local rings $(R_1, m_1), \dots, (R_t, m_t)$. Moreover, there is a one-to-one correspondence between $\{M_1 \cdot \cdots \cdot M_t\}$ and $\{R_1 \times \cdots \times R_{i-1} \times m_i \times R_{i+1} \times \cdots \times R_t; i = 1, 2, \dots, t\}$ by Lemma 9 (2). Hence Theorem 16 asserts that $\chi(R) = \max\{|M_1|, \dots, |M_t|\}$. Q.E.D.

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*Department of Mathematics, Faculty of Education Kochi University 2-5-1 Akebono-cho, Kochi 780-8520, JAPAN

E-mail: junro@cc.kochi-u.ac.jp

**Department of Mathematics, Faculty of Education and Welfare Science Oita University, Oita 870-1192, JAPAN

E-mail: baba@cc.oita-u.ac.jp