ON FILTERS IN BE-ALGEBRAS

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Abstract. In this paper we first give a procedure by which we generate a filter by a subset in a transitive BE-algebra, and give some characterizations of Noetherian and Artinian BE-algebras. Next we give the construction of quotient algebra X/F of a transitive BE-algebra X via a filter F of X. Finally we discuss properties of Noetherian (resp. Artinian) BE-algebras on homomorphisms and prove that let X and Y be transitive BE-algebras, a mapping $f: X \to Y$ be an epimorphism. If X is Noetherian (resp. Artinian), then so does Y. Conversely suppose that Y and $Ker(f)$ (as a subalgebra of X) are Noetherian (resp. Artinian), then so does X. Let X be a transitive BE-algebra and F a filter of X. If X is Noetherian (resp. Artinian), then so does the quotient algebra X/F.

1 Introduction

The study of BCK/BCI-algebras was initiated by K. Iséki in 1966 as a generalization of propositional logic (see[4, 5, 6]). There exist several generalizations of BCK/BCI-algebras, as such BCH-algebras[3], dual BCK-algebras[11], d-algebras[10], etc. Especially, H. S. Kim and Y. H. Kim[7] introduced the notion of BE-algebras which was deeply studied by S. S. Ahn and Y. H. Kim[1], S. S. Ahn and K. S. So[2],H. S. Kim and K. J. Lee[8], A. Walendziak[12], B. L. Meng[9]. In this paper we deeply study filter theory in BE-algebras. We first give a procedure by which we generate a filter by a subset in a transitive BE-algebra, and give some characterizations of Northerian and Artinian BE-algebras. Next we give the construction of quotient algebra X/F of a transitive BE-algebra X via a filter F of X. Finally we discuss properties of Noetherian (resp. Artinian) BE-algebras on homomorphisms and prove that let X and Y be transitive BE-algebras, a mapping f : $X \to Y$ be an epimorphism. If X is Noetherian (resp. Artinian), then so does Y. As consequences, we have that let F be a filter of a transitive BE-algebra X. If X is Noetherian (resp. Artinian), then so does the quotient algebra X/F. Conversely suppose that Y and $Ker(f)$ (as a subalgebra of X) are Noetherian (resp. Artinian), then so does X. In the sequel, let $\mathbb{N}$ denote the set of all positive integers. For any $a_1, \cdots, a_n, x \in X$, we denote $\prod_{i=1}^{n} a_i * x = a_n * (\cdots * (a_1 * x) \cdots)$.

2 Preliminaries

Definition 2.1[7]. An algebra $(X; *, 1)$ of type (2,0) is said to be a BE-algebra if it satisfies the following:

(BE1) $x * x = 1$,
(BE2) $x * 1 = 1$,
(BE3) $1 * x = x$,
(BE4) $x * (y * z) = y * (x * z)$.

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Lemma 2.2. If $(X;\ *,\ 1)$ is a BE-algebra, then for all $x,\ y \in X$
(1) $x * (y * x) = 1$[1],
(2) $x * [(x * y) * y] = 1$[9].

In a BE-algebra, one can introduce a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 1$.

Definition 2.3[1]. A BE-algebra X is said to be *transitive* if for all $x, y, z \in X$,

$$(y * z) * [(x * y) * (x * z)] = 1, \text{ or equivalently, } y * z \leq (x * y) * (x * z).$$

Proposition 2.4. If a BE-algebra X is transitive then for all $x, y, z \in X$,
(1) $y \leq z$ implies $x * y \leq x * z$,
(2) $y \leq z$ implies $z * x \leq y * x$,
(3) $1 \leq x$ implies $x = 1$.

Proof. If $y \leq z$, then $(x * y) * (x * z) = (y * z) * [(x * y) * (x * z)] = 1$, so (1) is true. Let $y \leq z$. Since $(z * x) * (y * x) = (z * x) * [(y * z) * (y * x)] = 1$, (2) holds. (3) follows from (BE3). The proof is completed.

Definition 2.5[8]. Let X be a BE-algebra and F a nonempty subset of X. F is said to be a filter of X if it satisfies: (F1) $1 \in I$, (F2) $x \in F$ and $x * y \in F$ imply $y \in F$.

Obviously any filter F of a BE-algebra X is a subalgebra, i.e., $x, y \in F \Rightarrow x * y \in F$.

Lemma 2.6[9]. Let X be a transitive BE-algebra and A a nonempty subset of X. Then A is a filter of X if and only if A satisfies: for any $a_i \in A$ $(i = 1, \cdots, n)$ and $x \in X$, $\prod_{i=1}^{n} a_i * x = 1$ implies $x \in A$.

3 Filter generated by a set

Proposition 3.1. If X is a BE-algebra and $\{F_\lambda \mid \lambda \in \Lambda\}$ is an indexed set of filters of X where $\Lambda \neq \emptyset$, then $F = \cap\{F_\lambda \mid \lambda \in \Lambda\}$ is a filter of X.

Proof. Trivial.

Definition 3.2. Let X be a BE-algebra and A a nonempty subset of X. If B is the least filter containing A in X, then B is said to be the filter generated by A and is denoted by $(A]$. If A is a finite set of X then $(A]$ is said to be finitely generated.

Since X is always a filter of X containing any filter, it follows from Proposition 3.1 that Definition 3.2 is well defined. $(\{a_1,\ \cdots,\ a_n\}]$ is simply denoted by $(a_1,\ \cdots,\ a_n]$. For convenience, let $(\emptyset]= \{1\}$.

Proposition 3.3. Let X be a BE-algebra. Suppose A and B are two subsets of X. Then the followong hold:
(1) $(1] = \{1\},\ (X] = X$,
(2) $A \subseteq B$ implies $(A] \subseteq (B]$,
(3) if A is a filter of X, then $(A] = A$.

Proof. Trivial.

The following is a fundamental result of this paper.

Proposition 3.4. Let X be a transitive BE-algebra and A a nonempty subset of X. Then

$$(A] = \{x \in X \mid \prod_{i=1}^{n} a_i * x = 1 \ \exists\ a_1, \cdots, \ a_n \in A\}.$$

Proof. Denote $F = \{x \in X \mid \prod_{i=1}^{n} a_i * x = 1 \ \exists\ a_1, \cdots, \ a_n \in A\}$.

Since $a * a = 1$ for all $a \in A$, $A \subseteq F$.

Select $a \in A$. Since $a * 1 = 1$ by (BE2), so $1 \in F$.

If $y*x \in F$ and $y \in F$, then there are $a_1, \cdots, a_n, b_1, \cdots, b_m \in A$ such that $\prod_{i=1}^{n} a_i*(y*x) = 1$, $\prod_{j=1}^{m} b_i * y = 1$. Hence $y \leq \prod_{i=1}^{n} a_i * x$. By using Proposition 2.4(1) we obtain

$$1 = \prod_{j=1}^{m} b_j * y \leq \prod_{j=1}^{m} b_j * (\prod_{i=1}^{n} a_i * x),$$

thus

$$\prod_{j=1}^{m} b_j * (\prod_{i=1}^{n} a_i * x) = 1.$$

This shows that $x \in F$, and so F is a filter of X.

Suppose B is any filter containing A and $x \in F$, then there are $a_1, \cdots, a_n \in A$ such that $\prod_{i=1}^{n} a_i * x = 1$. By $A \subseteq B$ we know $a_1, \cdots, a_n \in B$. It follows from Lemma 2.6 that $x \in B$, i.e., $F \subseteq B$. Hence $F = (A]$. This completes the proof.

Definition 3.5[7]. A BE-algebra X is said to be *self distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

Note that every self distributive BE-algebra is transitive, but the converse is not true in general (see[1]Proposition 3.10).

Proposition 3.6. Suppose X is a self distributive BE-algebras and F is a filter of X. Then

(1) the set $F_a = \{x \mid a * x \in F\}$ is a filter,

(2) $(\{a\} \cup F] = F_a$.

Proof. It is easy to see that F_a satisfies (F1). To verify (F2) for F_a, assume $x*y \in F_a$ and $x \in F_a$. Then $a*(x*y) \in F$ and $a*x \in F$. By self distributivity we have $(a*x)*(a*y) \in F$ and $a * x \in F$. Thus $a * y \in F$, and so $y \in F_a$. Therefore F_a is a filter of X. (1) holds.

Suppose B is any filter containing $\{a\} \cup F$. Assume $x \in F_a$, then $a * x \in F \subseteq B$, so $x \in B$. Hence $(\{a\} \cup F] = F_a$, (2) holds. The proof is complete.

4 Noetherian and Artinian BE-algebras

Definition 4.1. A BE-algebra X is said to be Noetherian if every filter of X is finitely generated. We say that X satisfy the ascending (resp. descending) chain condition if for every ascending sequence $F_1 \subseteq F_2 \subseteq \cdots$ (resp. every descending sequence $F_1 \supseteq F_2 \supseteq \cdots$) of filters of X there is $n \in \mathbb{N}$ such that $F_n = F_k$ for $k \geq n$. X is said to satisfy the maximal (resp. minimal) condition if every nonempty set of filters of X has a maximal (resp. minimal) element.

Proposition 4.2. Let X be a BE-algebra. Then the following conditions are equivalent:
(1) X is Noetherian,
(2) X satisfies the ascending chain condition,
(3) X satisfies the maximal condition.

Proof. (1)⇒(2) Let X be Noetherian. Take any ascending sequence $F_1 \subseteq F_2 \subseteq \cdots$ of filters of X. Denote

$$F = \bigcup_{i=1}^{\infty} F_i \ .$$

It is easy to verify that F is a filter of X and hence F is finiely generated. So there are $a_i \in F(i = 1, \ \cdots, \ m)$ such that $F = (a_1, \ \cdots, \ a_m]$. This means $a_i \in F_{n_i}$ for some $n_i \in \mathbb{N}$. Let $k = \max\{n_1, \cdots, n_m\}$. Thus $\{a_1, \ \cdots, \ a_m\} \subseteq F_k$, and so $F_k = F_{k+1} = F_{k+2} = \cdots$, i.e., (2) is true.

(2)⇒(3) Let (2) be true. If (3) is not true, then there is a nonempty set $\mathbb{F}$ of filters of X such that $\mathbb{F}$ has no maximal element. Select $F_1 \in \mathbb{F}$. Since F_1 is not maximal element of $\mathbb{F}$, there is $F_2 \in \mathbb{F}$ such that $F_1 \subset F_2$. Repeating the above process we obtain an infinitely ascending sequence $F_1 \subset F_2 \subset \cdots$, a contradiction. Therefore (3) is true.

(3)⇒(1) Suppose (3) is true. Let F be any filter of X. We denote by $\mathcal{F}$ the set of all finitely genereted filters of X which are contained in F. Obviously $\{1\} \in \mathcal{F}$, hence $\mathcal{F} \neq \emptyset$. By (3) $\mathcal{F}$ has a maximal element, for example, F_0. Then F_0 is finitely genereted, let $F_0 = (a_1, \cdots, a_n]$. If $F_0 \neq F$, then there is $a \in F$ such that $F_0 \subset (a_1, \cdots, a_n, a] \subseteq F$, which is contrary to the maximality of F_0. Therefore $F = F_0$, i.e., F is finitely generated, (1) holds. This completes the proof.

Definition 4.3. Let X be a BE-algebra. X is said to be *Artinian* if X satisfies the descending chain condition.

Proposition 4.4. Let X is a BE-algebra. Then X is Artinian if and only if X satisfies the minimal condition.

Proof. (⇒) It can be proved by a similar argument used in Proposition 4.2 (2)⇒ (3).
(⇐) It is easy and is omitted.

5 Quotient BE-algebra induced by a filter

Throughout this section X always means a transitive BE-algebra without otherwise mentioned

Let F be a filter of X. A binary relation $\sim$ on X can be defined as follows, for all $x, y \in X$, $x \sim y$ if and only if $x * y \in F$ and $y * x \in F$.

Lemma 5.1. $\sim$ is an equivalent relation on X.

Proof Since $x * x = 1 \in F$, $x \sim x$.

By the definition of $\sim$, $x \sim y$ implies $y \sim x$.

If $x \sim y$ and $y \sim z$ then $x * y \in F$, $y * x \in F$, $y * z \in F$ and $z * y \in F$. By transitivity $(x * y) * [(y * z) * (x * z)] = 1$ and $(y * x) * [(z * y) * (z * x)] = 1$, it follows from Lemma 2.6 that $x * z \in F$ and $z * x \in F$, i.e., $x \sim z$. Thus $\sim$ is an equivalent relation on X.

Lemma 5.2. $\sim$ is a congruence relation on X.

Proof If $x \sim y$ and $u \sim v$, then $x * y \in F$, $y * x \in F$, $u * v \in F$ and $v * u \in F$. By transitivity , we have $(u*v)*[(x*u)*(x*v)] = 1$ and $(v*u)*[(x*v)*(x*u)] = 1$, it follows from Lemma 2.6 that $(x*u)*(x*v) \in F$ and $(x*v)*(x*u) \in F$. Thus $x*u \sim x*v$. By the same argument one can prove that $x*v \sim y*v$. By Lemma 5.1 we obtain $x*u \sim y*v$. Therefore $\sim$ is a congruence relation on X.

X is decomposed by the congruence relation $\sim$. The class containing x is denoted by $[x]_F$. Denote $X/F = \{[x]_F \mid x \in X\}$. We define a binary operation $*$ on X/F by $[x]_F * [y]_F := [x*y]_F$. This definition is well defined by Lemma 5.2.

Lemma 5.3. $[1]_F = F$.

Proof. If $x \in [1]_F$, then $x = 1 * x \in F$, so $[1]_F \subseteq F$. Conversely if $x \in F$, then $x * 1 \in F$ and $1 * x \in F$ since F is a subalgebra and $1 \in F$. Therefore $F \subseteq [1]_F$. This prove that $[1]_F = F$.

Proposition 5.4. $(X/F;\ *,\ [1]_F)$ is a BE-algebra.

Proof. It is immediate.

Definition 5.5. Let $(X;\ *_X,\ 1_X)$ and $(Y;\ *_Y,\ 1_Y)$ be two BE-algebras. A mapping $f:$ $X \to Y$ is called a *homomorphism* from X to Y if for all $x, y \in X$, $f(x *_X y) = f(x) *_Y f(y)$. The set $Ker(f) := \{x \in X \mid f(x) = 1_Y\}$ is called the kernel of f. If, in addition, the mapping f is onto then f is called an *epimorphism*. If f is both an epimorphism and one-to-one, then f is said to be an isomorphism, and we say that X is isomorphic to Y, written $X \cong Y$.

Definition 5.6. Let X be a transitive BE-algebras and F a filter of X. The natural map $\nu_F :\ X \to X/F$ is defined by $\nu_F(x) = [x]_F$ for all $x \in X$. When there is no ambiquity we write simply ν instead of ν_F.

Proposition 5.7. Let X be a transitive BE-algebras and F a filter of X. Then the natural mapping $\nu :\ X \to X/F$ is an epimorphism.

Proof. Trivial.

Proposition 5.8. Let X and Y be transitive BE-algebras, a mapping $f :\ X \to Y$ be an epimorphism. If X is Noetherian, then so does Y.

Proof. For any filter F of Y, $f^{-1}(F)$ is a filter of X. Therefore $f^{-1}(F)$ is finitely generated from Proposition 3.4, for instance, $f^{-1}(F) = (a_1,\ \cdots,\ a_n]$ where $a_1,\ \cdots,\ a_n \in f^{-1}(F)$, hence $F = (f(a_1),\ \cdots,\ f(a_n)]$. This shows that Y is Noetherian, ending the proof.

Corollary 5.9. Let X be a transitive BE-algebra and F a filter of X. If X is Noetherian, then so does X/F.

Proof. Since the natural map $\nu : X \to X/F$ be an epimorphism, it follows from Proposition 5.8 that X/F is Noetherian. This completes the proof.

Proposition 5.10. Let X and Y be BE-algebras, a mapping $f :\ X \to Y$ an epimorphism. If X is Artinian, then so does Y.

Proof. If we are given any descending sequence $F_1 \supseteq F_2 \supseteq \cdots$ of filters of Y, then $f^{-1}(F_1) \supseteq f^{-1}(F_2) \supseteq \cdots$ is a descending sequence of filters of X. Since X is Artinian, there is $n \in \mathbb{N}$ such that $f^{-1}(F_k) = f^{-1}(F_n)$ for all $k \geq n$. Hence $F_k = F_n$ for all $k \geq n$. This shows that Y is Artinian, completing the proof.

Corollary 5.11. Let X be a transitive BE-algebra and F a filter of X. If X is Artinian, then so does X/F.

Proof. This is immediate from Proposition 5.10.

Proposition 5.12. Let X and Y be transitive BE-algebras. Let f be an epimorphism from X to Y. Suppose that Y and $Ker(f)$ (as a subalgebra of X) are Noetherian, then so does X.

Proof. Since $X/Ker(f)$ is isomorphic to Y, it is sufficient to prove that if $X/Ker(f)$ and $Ker(f)$ are Noetherian, then so does X. Let $\nu : X \to X/Ker(f)$ be the natural mapping. We are given any ascending sequence $F_1 \subseteq F_2 \subseteq \cdots$ of filters of X, then $F_1 \cap Ker(f) \subseteq F_2 \cap Ker(f) \subseteq \cdots$ and $\nu(F_1) \subseteq \nu(F_2) \subseteq \cdots$ are ascending sequences of filters of $Ker(f)$ and $X/Ker(f)$, respectively. Therefore there exist $k_1 \in \mathbb{N}$ and $k_2 \in \mathbb{N}$ such that $F_i \cap Ker(f) = F_{k_1} \cap Ker(f)$ for all $i \geq k_1$ and $\nu(F_i) = \nu(F_{k_2})$ for all $i \geq k_2$. Denote $k = \max\{k_1,\ k_2\}$. We will prove that $F_i = F_k$ for all $i \geq k$. Given any $i \geq k$. To prove that $F_k = F_i$, take any $x \in F_i$. Then $\nu(x) \in \nu(F_i) = \nu(F_k)$, thus there is $x_0 \in F_k$ such that $\nu(x) = \nu(x_0)$. Hence $\nu(x_0 * x) = \nu(x_0) * \nu(x) = Ker(f)$, and so $x_0 * x \in Ker(f)$. Because $x_0 * x \in F_i$, it follows that $x_0 * x \in F_i \cap Ker(f) = F_k \cap Ker(f)$. Noticing $x_0 \in F_k$ and F_k to be a filter of X, we have $x \in F_k$, i.e., $F_k = F_i$. This completes the proof.

Proposition 5.13. Let X and Y be transitive BE-algebras. Let f be an epimorphism from X to Y. Suppose that Y and $Ker(f)$ (as a subalgebra of X) are Artinian, then so does X.

Proof. Since $X/Ker(f)$ is isomorphic to Y, it is sufficient to prove that if $X/Ker(f)$ and $Ker(f)$ are Artinian, then so does X. Let $\nu : X \to X/Ker(f)$ be the natural mapping. We are given any descending sequence $F_1 \supseteq F_2 \supseteq \cdots$ of filters of X, then $F_1 \cap Ker(f) \supseteq F_2 \cap Ker(f) \supseteq \cdots$ and $\nu(F_1) \supseteq \nu(F_2) \supseteq \cdots$ are descending sequences of filters of $Ker(f)$ and $X/Ker(f)$, respectively. Therefore there exist $k_1 \in \mathbb{N}$ and $k_2 \in \mathbb{N}$ such that $F_i \cap Ker(f) = F_{k_1} \cap Ker(f)$ for all $i \geq k_1$ and $\nu(F_i) = \nu(F_{k_2})$ for all $i \geq k_2$. Denote $k = \max\{k_1,\ k_2\}$. We will prove that $F_i = F_k$ for all $i \geq k$. Given any $i \geq k$. To prove that $F_k = F_i$, take any $x \in F_k$. Then $\nu(x) \in \nu(F_k) = \nu(F_i)$, thus there is $x_0 \in F_i$ such that $\nu(x) = \nu(x_0)$. Hence $\nu(x_0 * x) = \nu(x_0) * \nu(x) = Ker(f)$, and so $x_0 * x \in Ker(f)$. Because $x_0 * x \in F_k$, it follows that $x_0 * x \in F_k \cap Ker(f) = F_i \cap Ker(f)$. Noticing $x_0 \in F_i$ and F_i being a filter of X, we have $x \in F_i$, i.e., $F_i = F_k$. This completes the proof.

As consequences of Propositions 5.12 and 5.13, we have

Corollary 5.14. Let X be a transitive BE-algebra and F a filter of X. If X/F and F (as a subalgebra of X) are Noetherian (resp. Artinian), then so does X.

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