ORDER AMONG POWER MEANS OF POSITIVE OPERATORS, II

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Abstract. As a continuation of our previous research [Sci. Math. Japon. **61** (2005), 25–46.], we discuss order among power means of positive operators with a unital n-tuple of positive linear maps.

1 Introduction.

We assume that H and K are Hilbert spaces and $\mathcal{B}(H)$ and $\mathcal{B}(K)$ are C*-algebras of all bounded linear operators on the appropriate Hilbert space. We say that an n-tuple $(\Phi_1, \dots, \Phi_n)$ of positive linear maps $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$ is *unital* if $\sum_{i=1}^n \Phi_i(\mathbf{1}) = \mathbf{1}$.

Recently F. Hansen, J. Pečarić and I. Perić in [4, 5] gave a general formulation of Jensen's operator inequality for unital fields of positive linear mappings and its converses. Their main results from [4] are presented in the following two theorems [4, Theorem 1 and Theorem 2].

Theorem A (Generalization of discrete Jensen's operator inequality) *Let $(\Phi_1, \dots, \Phi_n)$ be a unital n-tuple of positive linear maps $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$. If $f : I \to \mathbf{R}$ is an operator convex function, then the inequality*

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \le \sum_{i=1}^n \Phi_i\left(f(A_i)\right) \tag{1}$$

holds for every n-tuple $(A_1, \dots, A_n)$ of self-adjoint operators in $\mathcal{B}(H)$ with spectra in I.

The inequality (1) should be compared to Jensen's type inequality [1, 3, 8], cf. also [2]. The following converses of (1) in a general form should be compared to [2, Theorem 2.3]. For a function $f : [m, M] \to \mathbf{R}$ we use the standard notation:

$$\mu_f = \frac{f(M) - f(m)}{M - m} \quad \text{and} \quad \nu_f = \frac{Mf(m) - mf(M)}{M - m}.$$

Theorem B (Generalization of converse of discrete Jensen's operator inequality) *Let $(A_1, \dots, A_n)$ be an n-tuple of self-adjoint operators in $\mathcal{B}(H)$ with spectra in $[m, M]$ and let $(\Phi_1, \dots, \Phi_n)$ be a unital n-tuple of positive linear maps $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$. Let $f, g : [m, M] \to \mathbf{R}$ and $F : U \times V \to \mathbf{R}$ be functions such that $f([m, M]) \subset U$ and $g([m, M]) \subset V$. If F is operator monotone in the first variable and f is convex on $[m, M]$, then*

$$F\left[\sum_{i=1}^n \Phi_i(f(A_i)), g\left(\sum_{i=1}^n \Phi_i(A_i)\right)\right] \le \max_{m \le t \le M} F[\mu_f t + \nu_f, g(t)]\,\mathbf{1}. \tag{2}$$

In the dual case (when f is concave) the opposite inequality holds in (2) *with* min *instead of* max.

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In this paper, we consider a generalization of the weighted power means:

$$\widetilde{M}_n^{[r]}(\mathbf{A},\Phi,\omega) := \left(\sum_{i=1}^{n}\omega_i\ \Phi_i(A_i^r)\right)^{1/r} \qquad \text{if} \quad r\in\mathbf{R}\backslash\{0\}, \tag{3}$$

at these conditions: $(A_1,\dots,A_n)$ is an n-tuple of positive operators in $\mathcal{B}(H)$ with spectra in $[m,M]$ for some scalars $0<m<M$, $(\Phi_1,\dots,\Phi_n)$ is an n-tuple of positive linear maps $\Phi_i:\mathcal{B}(H)\to\mathcal{B}(K)$ and $(\omega_1,\dots,\omega_n)$ is an n-tuple of positive real numbers such that $(\omega_1\Phi_1,\dots,\omega_n\Phi_n)$ is unital.

This construction is inspired by generalizations of Jensen's inequality and its converses given in Theorem A and Theorem B. Our main purpose is to discuss the operator order among generalized power means (3), and improvements over the previous results [2, Theorem 4.4 and Theorem 4.7], cf. also [6].

2 Some preliminary results.

In this section we give some results whose follow from Theorem A and Theorem B.

Using Theorem B, we obtain the following corollary, which should be compared to [2, Theorem 2.4] for a convex case.

Corollary 1 *Let $(A_1,\dots,A_n)$ be an n-tuple of self-adjoint operators in $\mathcal{B}(H)$ with spectra in $[m,M]$, $(\Phi_1,\dots,\Phi_n)$ be an n-tuple of positive linear maps $\Phi_i:\mathcal{B}(H)\to\mathcal{B}(K)$ and $(\omega_1,\dots,\omega_n)$ be an n-tuple of positive real numbers such that $(\omega_1\Phi_1,\dots,\omega_n\Phi_n)$ is unital. Let $\alpha\neq 0$ be a real number. Let $g:[m,M]\to\mathbf{R}$ be a strictly convex differentiable function if $\alpha>0$ or g be a strictly concave differentiable if $\alpha<0$. If $f:[m,M]\to\mathbf{R}$ is convex function, then the inequality*

$$\sum_{i=1}^{n}\omega_i\Phi_i(f(A_i)) \le \alpha g\left(\sum_{i=1}^{n}\omega_i\Phi_i(A_i)\right)+\beta\ \mathbf{1} \tag{4}$$

holds for $\beta=\mu_f t_0+\nu_f-\alpha g(t_0)$, where

$$t_0=\begin{cases}\text{the unique solution of } g'(t)=\frac{\mu_f}{\alpha} & \text{if} \quad \alpha g'(m)\le\mu_f\le\alpha g'(M),\\ M & \text{if} \quad \mu_f>\alpha g'(M),\\ m & \text{if} \quad \alpha g'(m)>\mu_f,\end{cases}$$

In the dual case (when f and αg are concave) the opposite inequality holds in (4) with the same β but the opposite condition while determining t_0.

Proof. We only prove the convex case. Applying Theorem B for $F(u,v)=u-\alpha v$ ($\alpha\neq 0$) and replacing $\Phi_i(A_i)$ by $\omega_i\Phi_i(A_i)$, we have

$$\sum_{i=1}^{n}\omega_i\Phi_i(f(A_i)) \le \alpha g\left(\sum_{i=1}^{n}\omega_i\Phi_i(A_i)\right)+\beta\mathbf{1}$$

for $\beta=\max_{m\le t\le M}\left\{\mu_f t+\nu_f-\alpha g(t)\right\}$. Then by the common differential calculus we determine β. □

Let $(A_1,\dots,A_n)$ be an n-tuple of positive operators in $\mathcal{B}(H)$ and $(\omega_1\Phi_1,\dots,\omega_n\Phi_n)$ be a unital n-tuple of positive linear maps $\Phi_i:\mathcal{B}(H)\to\mathcal{B}(K)$. For readers' convenience we put

$$\Phi\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{pmatrix} = \sum_{i=1}^{n} \omega_i \Phi_i(A_i). \tag{5}$$

Then we have

$$\Phi\begin{pmatrix} A_1^r & & & 0 \\ & A_2^r & & \\ & & \ddots & \\ 0 & & & A_n^r \end{pmatrix}^{1/r} = \left(\sum_{i=1}^{n} \omega_i \Phi_i(A_i)^r\right)^{1/r}$$

and instead of observing the order among power means (3), we observe the order among $\Phi(A^r)^{1/r}$.

Applying Theorem A and Corollary 1 for a function $f(t) \equiv g(t) = t^p$ we obtain inequalities for different and ratio of the power function given in Lemma 2 and Lemma 5. These results in a general form should be compared to [6, Lemma 10 and Lemma 12].

For a function $f(t) = t^p$ we use the short notation: $\mu := \mu_{t^p}$ and $\nu := \nu_{t^p}$, i.e.

$$\mu = \frac{M^p - m^p}{M - m} \quad \text{and} \quad \nu = \frac{Mm^p - mM^p}{M - m}.$$

Lemma 2 *Let A be a positive operator in $\mathcal{B}(H)$ with spectra in $[m, M]$ for some scalars $0 < m < M$, $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ be a unital positive linear map.*

(a) If $0 < p \le 1$, then

$$K(m, M, p)\, \Phi(A)^p \le \Phi(A^p) \le \Phi(A)^p.$$

(b) If $-1 \le p < 0$ or $1 \le p \le 2$, then

$$\Phi(A)^p \le \Phi(A^p) \le K(m, M, p)\, \Phi(A)^p.$$

(c) If $p < -1$ or $p > 2$, then

$$K(m, M, p)^{-1}\Phi(A)^p \le \Phi(A^p) \le K(m, M, p)\, \Phi(A)^p,$$

where a generalized Kantorovich constant $K(m, M, p)$ [2, §2.7] is defined as

$$(*) \qquad K(m, M, p) := \frac{mM^p - Mm^p}{(p-1)(M-m)} \left(\frac{p-1}{p} \frac{M^p - m^p}{mM^p - Mm^p}\right)^p = \frac{\nu}{p-1}\left(\frac{p-1}{p}\frac{\mu}{\nu}\right)^p \qquad \text{for all } p \in \mathbf{R}.$$

Proof. Since $f(t) = t^p$ is the operator concave function for $0 < p \le 1$, then by Theorem A we have the right hand inequality in (a). Using the dual case of Corollary 1 for $f(t) \equiv g(t) = t^p$, $0 < p \le 1$ and choosing $\alpha > 0$ such that $\beta = 0$, we obtain $\Phi(A^p) \ge \alpha\, \Phi(A_i)^p$, when $\alpha = \min_{m \le t \le M} \left\{\frac{m^p + \mu(t-m)}{t^p}\right\} = K(m, M, p)$. Then we obtain the left hand inequality in (a). Since $f(t) = t^p$ is the operator convex function if $-1 \le p < 0$ or $1 \le p \le 2$, then by Theorem A we have the left hand inequality in (b). Since $f(t) = t^p$ is the convex function for $p<0$ or $p>1$, then again using Corollary 1, we obtain $\Phi(A^p) \le \alpha\, \Phi(A)^p$, where $\alpha = \max_{m\le t\le M}\left\{\frac{m^p+\mu(t-m)}{t^p}\right\} = K(m,M,p)$. Then we have the right hand inequality in (b) and (c). Finally, we prove the left hand inequality in (c). Since $f(t)=t^p$ is convex in the case $p<-1$ or $p>2$, then for each $y\in[m,M]$, $t^p \ge y^p + py^{p-1}(t-p)$ for all $t\in[m,M]$. It follows that $A^p \ge y^p\mathbf{I} + py^{p-1}(A-y\mathbf{I})$. Applying the positive linear map Φ to above inequality, we obtain

$$\Phi(A^p) \ge y^p\mathbf{1} + py^{p-1}\left(\Phi(A)-y\mathbf{1}\right) \qquad \text{for each } y\in[m,M].$$

Now, applying Theorem B for functions $F(u,v)=v^{-1/2}uv^{-1/2}$ $(v>0)$ and $f(t)\equiv g(t)=t^p$, we have:

$$\Phi(A^p) \ge \min_{m\le t\le M}\left\{\frac{y^p+py^{p-1}(t-y)}{t^p}\right\}\Phi(A)^p \qquad \text{for each } y\in[m,M].$$

The function $h(t)=\frac{y^p+py^{p-1}(t-y)}{t^p}$ is increasing for $t<y$ and decreasing for $t>y$. It follows that h attains a minimum at m or M. We choose $y=y_0$ which is the unique solution of $\frac{y^p+py^{p-1}(m-y)}{m^p} = \frac{y^p+py^{p-1}(M-y)}{M^p}$. It follows that $y_0=\frac{\nu p}{\mu(1-p)}$ and

$$\min_{m\le t\le M}\left\{\frac{y_0^p+py_0^{p-1}(t-y_0)}{t^p}\right\} = y_0^{p-1}\,\frac{y_0(1-p)+pm}{m^p} = \left(\frac{\nu p}{\mu(1-p)}\right)^p\frac{1-p}{\nu} = K(m,M,p)^{-1}.$$

Then we have the left hand inequality in (c). □

The following lemma is needed to prove Lemma 5.

Lemma 3 *Assume that the conditions of Lemma 2 hold.*
If $0<p\le 1$, *then*

$$\mu\,\Phi(A)+\nu\mathbf{1} \le \Phi(A^p) \le \Phi(A)^p, \tag{6}$$

if $-1\le p<0$ *or* $1\le p\le 2$, *then*

$$\Phi(A)^p \le \Phi(A^p) \le \mu\Phi(A)+\nu\mathbf{1}, \tag{7}$$

while if $p<-1$ *or* $p>2$, *then*

$$py^{p-1}\Phi(A)+(1-p)y^p\mathbf{1} \le \Phi(A^p) \le \mu\,\Phi(A)+\nu\mathbf{1} \qquad \textit{for every } y\in[m,M]. \tag{8}$$

Proof. The right hand inequality in (6) and the left hand inequality in (7) are proven in Lemma 2. The left hand inequality of (6) and the right hand inequalities in (7) and (8) follow from Corollary 1 if we put $f(t)=t^p$, $g(t)=t$ and $\alpha=\mu$.

Next we prove the left hand inequality in (8). Since $f(t)=t^p$ is convex in the case $p<-1$ or $p>2$, then for each $y\in[m,M]$, $t^p\ge py^{p-1}t+(1-p)y^p$ holds for all $t\in[m,M]$. By using a functional calculus we obtain that $py^{p-1}\Phi(A)+(1-p)y^p\mathbf{1}\le\Phi(A^p)$ holds for every $y\in[m,M]$. □

Remark 4 *Setting* $y=(\mu/p)^{1/(p-1)}$ *the inequality* (8) *gives*

$$\mu\,\Phi(A)+(1-p)\left(\mu/p\right)^{p/(p-1)}\mathbf{1} \le \Phi(A^p) \le \mu\,\Phi(A)+\nu\mathbf{1} \tag{9}$$

for $p<-1$ *or* $p>2$.

Furthermore, setting $y = m$ or $y = M$ the inequality (8) *gives*

$$pm^{p-1}\Phi(A) + (1-p)m^p\mathbf{1} \le \Phi(A^p) \le \mu\,\Phi(A) + \nu\mathbf{1} \tag{10}$$

or

$$pM^{p-1}\Phi(A) + (1-p)M^p\mathbf{1} \le \Phi(A^p) \le \mu\,\Phi(A) + \nu\mathbf{1}. \tag{11}$$

We remark that the operator in LHS of (10) *is positive for $p > 0$, since*

$$0 < m^p\mathbf{1} \le pm^{p-1}\Phi(A) + (1-p)m^p\mathbf{1} \le (pm^{p-1}M + (1-p)m^p)\mathbf{1}$$

and the operator in LHS of (11) *is positive for $p < 0$, since*

$$0 < M^p\mathbf{1} \le pM^{p-1}\Phi(A) + (1-p)M^p\mathbf{1} \le (pM^{p-1}m + (1-p)M^p)\mathbf{1}.$$

In addition, to prove next Lemma 5, we need the Löwner-Heinz theorem, which asserts that the function $f(t) = t^p$ is operator monotone for $p \in [0, 1]$, and the operator order given in the following theorem (see [7, 9]):

Theorem C *[6, Corollary 1] Let A, B be positive operators in $\mathcal{B}(H)$.*
If $A \ge B > 0$ and the spectrum $\mathsf{Sp}(B) \subseteq [m, M]$ for some scalars $0 < m < M$, then

$$A^p + C(m, M, p)\mathbf{1} \ge B^p \qquad \text{for all} \quad p \ge 1.$$

But, if $A \ge B > 0$ and the spectrum $\mathsf{Sp}(A) \subseteq [m, M]$, $0 < m < M$, then

$$B^p + C(m, M, p)\mathbf{1} \ge A^p \qquad \text{for all} \quad p \le -1,$$

where a constant $C(m, M, p)$ [2, §2.7, Lemma 2.59] is defined as

$$C(m, M, p) = (p-1)\left(\tfrac{1}{p}\tfrac{M^p - m^p}{M - m}\right)^{p/(p-1)} + \tfrac{Mm^p - mM^p}{M - m} \qquad \text{for all } p \in \mathbf{R}. \tag{$**$}$$

Finally, in the following lemma we give some inequalities for power functions.

Lemma 5 *Assume that the conditions of Lemma 2 hold.*

(*a*) *If* $r \le s \le -1$ *or* $1 \le s \le -r$ *or* $0 < r \le s \le 2r$, $s \ge 1$, *then*

$$\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le (\tilde{\mu}\,\Phi(A^r) + \tilde{\nu}\mathbf{1})^{1/s}.$$

(*b*) *If* $0 < -r < s$, $s \ge 1$ *or* $0 < 2r < s$, $s \ge 1$, *then*

$$m\left(\frac{s}{r}m^{-r}\Phi(A^r) + \frac{r-s}{r}\mathbf{1}\right)^{1/s} \le \Phi(A^s)^{1/s} \le (\tilde{\mu}\,\Phi(A^r) + \tilde{\nu}\mathbf{1})^{1/s}.$$

(*c*) *If* $r \le s$, $-1 \le s < 0$ *or* $s \le -r$, $0 < s \le 1$ *or* $0 < r \le s \le 2r$, $s \le 1$, *then*

$$\Phi(A^r)^{1/r} - C(m^s, M^s, 1/s)\,\mathbf{1} \le \Phi(A^s)^{1/s} \le (\tilde{\mu}\,\Phi(A^r) + \tilde{\nu}\mathbf{1})^{1/s} + C(m^s, M^s, 1/s)\mathbf{1}.$$

(*d*) *If* $0 < -r < s \le 1$ *or* $0 < 2r < s \le 1$, *then*

$$\begin{aligned} &m\left(\tfrac{s}{r}m^{-r}\Phi(A^r) + \tfrac{r-s}{r}\mathbf{1}\right)^{1/s} - C(m^s, M^s, 1/s)\mathbf{1} \\ &\le \Phi(A^s)^{1/s} \le (\tilde{\mu}\,\Phi(A^r) + \tilde{\nu}\mathbf{1})^{1/s} + C(m^s, M^s, 1/s)\mathbf{1}, \end{aligned}$$

where $\tilde{\mu}$ and $\tilde{\nu}$ are the constants μ_f and ν_f associated with the function $f(t) = t^{s/r}$ on the closed interval joining m^r to M^r, i.e.

$$\tilde{\mu} = \frac{M^s - m^s}{M^r - m^r} \quad \text{and} \quad \tilde{\nu} = \frac{M^r m^s - M^s m^r}{M^r - m^r}.$$

Proof. This lemma follows from Theorem C, the Löwner-Heinz theorem and Lemma 3. We use the same idea from [6, Lemma 10].

Putting $p = s/r$ in (6), (7), (10) and (11) and replacing A by A^r, we obtain the following inequalities.

(I) If $r \le s < 0$, then

$$\tilde{\mu}\,\Phi(A^r) + \tilde{\nu}\mathbf{1} \le \Phi(A^s) \le \Phi(A^r)^{s/r}.$$

(II) If $0 < s \le -r$ or $0 < r \le s \le 2r$, then

$$\Phi(A^r)^{s/r} \le \Phi(A^s) \le \tilde{\mu}\,\Phi(A^r) + \tilde{\nu}\mathbf{1}.$$

(III) If $0 < -r < s$, then by using $M^r \le A^r \le m^r$, we obtain

$$0 < \frac{s}{r} m^{s-r}\Phi(A^r) + \frac{r-s}{r} m^s \mathbf{1} \le \Phi(A^s) \le \tilde{\mu}\,\Phi(A^r) + \tilde{\nu}\mathbf{1}$$

from (11). If $0 < 2r < s$, then by using $m^r \le A^r \le M^r$, we obtain the same inequality from (10).

Using the fact that the function $f(t) = t^{1/s}$ is operator increasing if $s \ge 1$ and operator decreasing if $s \le -1$, we obtain (a) and (b).

Using Theorem C for $p = 1/s > 1$ we obtain (c), since $m^s\mathbf{1} \le \tilde{\mu}\,\Phi(A^r) + \tilde{\nu}\mathbf{1} \le M^s\mathbf{1}$ and $m^s\mathbf{1} \le \Phi(A^s) \le M^s\mathbf{1}$. Similarly, using Theorem C for $p = 1/s < -1$ we obtain (d). □

Remark 6 *Putting* $p = r/s$ *in* (6), (7), (10) *and* (11) *and replacing* A *by* A^s, *we obtain the following inequalities.*

(a_1) *If* $1 \le r \le s$ *or* $-s \le r \le -1$ *or* $2s \le r \le s < 0,\ r \le -1$, *then*

$$\left(\bar{\mu}\Phi(A^s) + \bar{\nu}\mathbf{1}\right)^{1/r} \le \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s}.$$

(b_1) *If* $r < -s < 0,\ r \le -1$ *or* $r < 2s < 0,\ r \le -1$, *then*

$$\left(\bar{\mu}\Phi(A^s) + \bar{\nu}\mathbf{1}\right)^{1/r} \le \Phi(A^r)^{1/r} \le M\left(\frac{r}{s}M^{-s}\Phi(A^s) + \frac{s-r}{s}\mathbf{1}\right)^{1/r}.$$

(c_1) *If* $r \le s,\ 0 < r \le 1$ *or* $-s \le r,\ -1 \le r < 0$ *or* $2s \le r \le s < 0,\ r \ge -1$, *then*

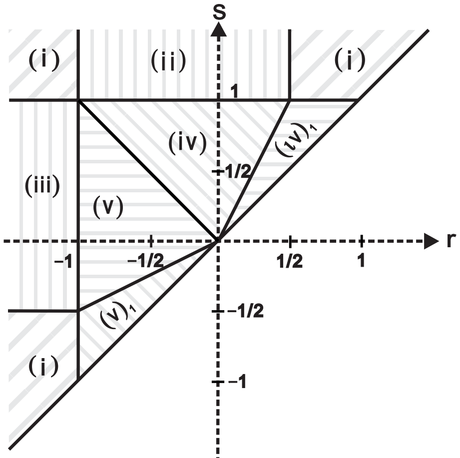
$$\left(\bar{\mu}\Phi(A^s) + \bar{\nu}\mathbf{1}\right)^{1/r} - C\left(m^r, M^r, 1/r\right)\mathbf{1} \le \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} + C(m^r, M^r, 1/r)\mathbf{1}.$$

(d_1) *If* $-1 \le r < -s < 0$ *or* $-1 \le r < 2s < 0$, *then*

$$\begin{aligned}&\left(\bar{\mu}\Phi(A^s) + \bar{\nu^*}\mathbf{1}\right)^{1/r} - C(m^r, M^r, 1/r)\mathbf{1} \le \Phi(A^r)^{1/r}\\ &\le M\left(\tfrac{r}{s}M^{-s}\Phi(A^s) + \tfrac{s-r}{s}\mathbf{1}\right)^{1/r} + C(m^r, M^r, 1/r)\mathbf{1},\end{aligned}$$

where $\bar{\mu}$ *and* $\bar{\nu}$ *are the constants* μ_f *and* ν_f *associated with the function* $f(t) = t^{r/s}$ *on the closed interval joining* m^s *to* M^s, *i.e.*

$$\bar{\mu} = \frac{M^r - m^r}{M^s - m^s} \quad \textit{and} \quad \bar{\nu} = \frac{M^s m^r - M^r m^s}{M^s - m^s}.$$



(i)	r ≤ s, s ∉ (-1,1), r ∉ (-1,1) **or** 1/2 ≤ r ≤ 1 ≤ s **or** r ≤ -1 ≤ s ≤ -1/2,
(ii)	s ≥ 1, -1 < r < 1/2, r ≠ 0,
(iii)	r ≤ -1, -1/2 < s < 1, s ≠ 0,
(iv)	-s ≤ r < s/2, r ≠ 0, 0 < s ≤ 1,
(iv)1	r ≤ s ≤ 2r, 0 < s ≤ 1,
(v)	r/2 < s ≤ -r, s ≠ 0, -1 ≤ r < 0,
(v)1	2s ≤ r ≤ s, -1 ≤ r < 0.

Figure 1: Regions in the (r, s)-plain

3 Main results. In this section we give the usual operator order among the generalized power means defined by (3). Using (5) we can replace $\widetilde{M}_n^{[r]}(\mathbf{A}, \Phi, \omega)$ by $\Phi(A^r)^{1/r}$.

For the sake of convenience, we denote regions in the (r, s)-plain from (i) to (v) as in Figure 1.

3.1 Difference type inequalities. Our first result about the order among the power means is given in the following theorem, which should be compared with [2, Theorem 4.7], cf. also [6, Theorem 8].

Theorem 7 *Let $(A_1, \ldots, A_n)$ be an n-tuple of positive operators in $\mathcal{B}(H)$ with spectra in $[m, M]$ for some scalars $0 < m < M$, $(\Phi_1, \ldots, \Phi_n)$ be an n-tuple of positive linear maps $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$ and $(\omega_1, \ldots, \omega_n)$ be an n-tuple of positive real numbers such that $(\omega_1\Phi_1, \ldots, \omega_n\Phi_n)$ is unital. Let $r, s \in \mathbf{R}$, $r \leq s$ and $rs \neq 0$.*
If (r, s) in (i)*, then*

$$0 \leq \widetilde{M}_n^{[s]}(\mathbf{A}, \Phi, \omega) - \widetilde{M}_n^{[r]}(\mathbf{A}, \Phi, \omega) \leq \tilde{\Delta}\mathbf{1},$$

if (r, s) in (ii)*, then*

$$\left(m \left(\frac{s}{r} \frac{M^r}{m^r} + 1 - \frac{s}{r} \right)^{1/s} - M \right) \mathbf{1} \leq \widetilde{M}_n^{[s]}(\mathbf{A}, \Phi, \omega) - \widetilde{M}_n^{[r]}(\mathbf{A}, \Phi, \omega) \leq \tilde{\Delta}\mathbf{1},$$

if (r, s) in (iii)*, then*

$$\left(m - M \left(\frac{r}{s} \frac{m^s}{M^s} + 1 - \frac{r}{s} \right)^{1/r} \right) \mathbf{1} \leq \widetilde{M}_n^{[s]}(\mathbf{A}, \Phi, \omega) - \widetilde{M}_n^{[r]}(\mathbf{A}, \Phi, \omega) \leq \tilde{\Delta}\mathbf{1},$$

if (r, s) in (iv)*, then*

$$\begin{aligned} &\max\{m \left(\tfrac{s}{r} \tfrac{M^r}{m^r} + \tfrac{r-s}{r} \right)^{1/s} - M - \; C(m^s, M^s, 1/s), -C\left(m^r, M^r, 1/r\right)\}\mathbf{1} \\ &\leq \widetilde{M}_n^{[s]}(\mathbf{A}, \Phi, \omega) - \widetilde{M}_n^{[r]}(\mathbf{A}, \Phi, \omega) \leq \left(\tilde{\Delta} + C\left(m^s, M^s, 1/s\right) \right) \mathbf{1}, \end{aligned}$$

if (v) *or* $(\text{iv})_1$ *or* $(\text{v})_1$, *then*

$$-C\left(m^s, M^s, 1/s\right)\mathbf{1} \le \widetilde{M}_n^{[s]}(\mathbf{A}, \Phi, \omega) - \widetilde{M}_n^{[r]}(\mathbf{A}, \Phi, \omega) \le \left(\tilde{\Delta} + C\left(m^s, M^s, 1/s\right)\right)\mathbf{1},$$

where a constant $\tilde{\Delta} \equiv \tilde{\Delta}(m, M, r, s)$ *is*

$$\tilde{\Delta} = \max_{\theta\in[0,1]} \left\{[\theta M^s + (1-\theta)m^s]^{1/s} - [\theta M^r + (1-\theta)m^r]^{1/r}\right\}$$

and a constant $C(m, M, p)$ *defined by* (∗∗).

T. Yamazaki [10, 11] gave a collection of basic properties of the constant $C(m, M, p)$, see also [2, Lemma 2.59]. In order to prove Theorem 7, we shall need some properties of this constant (see Figure 2):

Lemma 8 *Let* $M > m > 0$ *and* $r \in \mathbf{R}$ *and*

$$C(m^r, M^r, 1/r) := \frac{1-r}{r}\left(r\frac{M-m}{M^r-m^r}\right)^{1/(1-r)} + \frac{M^r m - m^r M}{M^r - m^r}$$

has the following property:

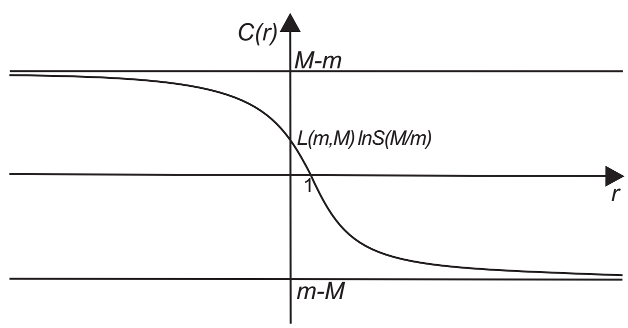


Figure 2: Function $C(r) \equiv C(m^r, M^r, 1/r)$

1. *A function* $C(r) \equiv C(m^r, M^r, 1/r)$ *is strictly decreasing for all* $r \in \mathbf{R}$,

2. $\lim_{r\to1} C(m^r, M^r, 1/r) = 0$ *and* $\lim_{r\to0} C(m^r, M^r, 1/r) = L(m, M)\ln S(M/m)$, *where* $L(m, M)$ *is the logarithmic mean:*

$$L(m, M) = \frac{M-m}{\ln M - \ln m}\ (M \ne m) \quad \text{and} \quad L(m, m) = m,$$

$S(h)$ *is the Specht ratio defined for* $h > 0$ *as*

$$(***) \qquad S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e\ln h}\ (h \ne 1) \quad \text{and} \quad S(1) = 1,$$

3. $\lim_{r\to\infty} C(m^r, M^r, 1/r) = m - M$ *and* $\lim_{r\to-\infty} C(m^r, M^r, 1/r) = M - m$.

Proof.

1. We have by a differential calculation

$$\begin{aligned}\frac{d}{dr}C(r) &= \left(r\frac{M-m}{M^r-m^r}\right)^{1/(1-r)}\left(\frac{m^r\ln m - M^r\ln M}{r(M^r-m^r)}+\frac{1}{r(1-r)}\ln\frac{r(M-m)}{M^r-m^r}\right)\\ &+ \frac{M^r m^r (M-m)\ln(m/M)}{(M^r-m^r)^2}.\end{aligned}$$

Both of functions

$$r\mapsto \frac{m^r\ln m - M^r\ln M}{r(M^r-m^r)}+\frac{1}{r(r-1)}\ln\frac{M^r-m^r}{r(M-m)}$$

and

$$r\mapsto \frac{M^r m^r (M-m)\ln(m/M)}{(M^r-m^r)^2}$$

are negative for all $r\neq 0,1$. So $\frac{d}{dr}C(r)<0$ and the function C is strictly decreasing.

2. We have by L'Hospital's theorem

$$\lim_{r\to 1}\frac{\ln\left(r(M-m)/(M^r-m^r)\right)}{1-r} = -1+\frac{M\ln M - m\ln m}{M-m},$$

so

$$\lim_{r\to 1}\frac{1-r}{r}\left(r\frac{M-m}{M^r-m^r}\right)^{1/(1-r)} = 0\cdot e^{-1+(M\ln M-m\ln m)/(M-m)}=0.$$

Also,

$$\lim_{r\to 1}\frac{M^r m - m^r M}{M^r-m^r} = \lim_{r\to 1} m\frac{h^r-h}{h^r-1}=0, \qquad h=\frac{M}{m}>1.$$

Then, $\lim_{r\to 1} C(m^r,M^r,1/r)=0$.

Using [2, Lemma 2.59 (iv)], we have

$$\lim_{r\to 0} C(m^r,M^r,p/r) = L(m^p,M^p)\ln S(h^p)\quad \text{for all } p\in\mathbf{R} \text{ and } h=M/m,$$

so we obtain $\lim_{r\to 0} C(m^r,M^r,1/r)=L(m,M)\ln S(M/m)$.

3. We have by L'Hospital's theorem

$$\lim_{r\to\infty}\frac{\ln\left(r(M-m)/(M^r-m^r)\right)}{1-r} = \lim_{r\to\infty}\frac{M^r m - m^r M}{M^r-m^r}=\ln M,$$

so

$$\lim_{r\to\infty}\frac{1-r}{r}\left(r\frac{M-m}{M^r-m^r}\right)^{1/(1-r)} = -1\cdot e^{\ln M}=-M.$$

Also,

$$\lim_{r\to\infty}\frac{M^r m - m^r M}{M^r-m^r} = \lim_{r\to\infty} m\frac{h^r-h}{h^r-1}=m, \qquad h=\frac{M}{m}>1.$$

Then, $\lim_{r\to\infty} C(m^r,M^r,1/r)=m-M$. Similarly, we obtain $\lim_{r\to-\infty} C(m^r,M^r,1/r) = M-m$. □

Proof of Theorem 7. For the reader's convenience we prove Theorem 7 by means of the order among $\Phi(A^r)^{1/r}$. Using Lemma 5 we have the following inequalities, where T_r denotes the closed interval joining m^r to M^r.

(a) If $r \leq s \leq -1$ or $1 \leq s \leq -r$ or $0 < r \leq s \leq 2r$, $s \geq 1$, then

$$0 \leq \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \leq \max_{t \in T_r} \left\{ (\tilde{\mu}\, t + \tilde{\nu})^{1/s} - t^{1/r} \right\} \mathbf{1}$$

holds. Setting $t = \theta M^r + (1-\theta) m^r$ for some $\theta \in [0,1]$, we obtain $\max_{t \in T_r} \left\{ (\tilde{\mu}\, t + \tilde{\nu})^{1/s} - t^{1/r} \right\} = \tilde{\Delta}$. Then,

$$0 \leq \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \leq \tilde{\Delta}\mathbf{1}. \tag{12}$$

(b) If $0 < -r < s$, $s \geq 1$ or $0 < 2r < s$, $s \geq 1$, then

$$\begin{aligned} \min\nolimits_{t \in T_r} \left\{ m \left(\tfrac{s}{r} m^{-r} t + \tfrac{r-s}{r} \right)^{1/s} - t^{1/r} \right\} \mathbf{1} &\leq \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \\ &\leq \max\nolimits_{t \in T_r} \left\{ (\tilde{\mu}\, t + \tilde{\nu})^{1/s} - t^{1/r} \right\} \mathbf{1} \end{aligned}$$

holds. A function $h(t) = m \left(\frac{s}{r} m^{-r} t + \frac{r-s}{r} \right)^{1/s} - t^{1/r}$ is increasing on $[M^r, m^r]$ for $0 < -r < s$, $s \geq 1$. Really, a function $t \mapsto t^{\frac{1}{r}-1}$ is decreasing imply that $(m^r)^{\frac{1}{r}-1} < t^{\frac{1}{r}-1}$, which imply that $h'(t) = \frac{1}{r} \left(m^{1-r} \left(\frac{s}{r} m^{-r} t + \frac{r-s}{r} \right)^{\frac{1}{s}-1} - t^{\frac{1}{r}-1} \right) \geq \frac{1}{r} \left(m^{1-r} - t^{\frac{1}{r}-1} \right) > 0$. Then $\min_{t \in [M^r, m^r]} h(t) = h(M^r)$ and we obtain

$$\left(m \left(\frac{s}{r} \frac{M^r}{m^r} + \frac{r-s}{r} \right)^{1/s} - M \right) \mathbf{1} \leq \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \leq \tilde{\Delta}\mathbf{1}. \tag{13}$$

Similarly, h is a decreasing function on $[m^r, M^r]$ for $0 < 2r < s$, $s \geq 1$. Then $\min_{t \in [m^r, M^r]} h(t) = h(M^r)$ and we obtain (13) again.

(c) If $r \leq s$, $-1 \leq s < 0$ or $s \leq -r$, $0 < s \leq 1$ or $0 < r \leq s \leq 2r$, $s \leq 1$, then

$$-\, C\left(m^s, M^s, 1/s\right) \mathbf{1} \leq \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \leq \left(\tilde{\Delta} + C(m^s, M^s, 1/s) \right) \mathbf{1}. \tag{14}$$

(d) If $0 < -r < s \leq 1$ or $0 < 2r < s \leq 1$, then, we obtain

$$\begin{aligned} \left(m \left(\tfrac{s}{r} \tfrac{M^r}{m^r} + \tfrac{r-s}{r} \right)^{1/s} - M - C(m^s, M^s, 1/s) \right) \mathbf{1} &\leq \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \\ &\leq \left(\tilde{\Delta} + C(m^s, M^s, 1/s) \right) \mathbf{1}. \end{aligned} \tag{15}$$

Next, using Remark 6, we obtain the following inequalities, where T_s denotes the closed interval joining m^s to M^s.

(a_1) If $1 \leq r \leq s$ or $-s \leq r \leq -1$ or $2s \leq r \leq s < 0$, $r \leq -1$, then

$$0 \leq \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \leq \max_{t \in T_s} \left\{ t^{1/s} - (\bar{\mu}\, t + \bar{\nu})^{1/r} \right\} \mathbf{1}$$

holds. Setting $t = \theta M^s + (1-\theta) m^s$ for some $\theta \in [0,1]$, we obtain $\max_{t \in T_s} \left\{ t^{1/s} - (\bar{\mu}\, t + \bar{\nu})^{1/r} \right\} = \tilde{\Delta}$. Then we obtain (12) in this case too.

(b_1) If $r < -s < 0,\ r \le -1$ or $r < 2s < 0,\ r \le -1$, then

$$\min_{t\in T_s}\left\{t^{1/s} - M\left(\tfrac{r}{s}M^{-s}t + \tfrac{s-r}{s}\right)^{1/r}\right\}\mathbf{1} \le \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \le \max_{t\in T_s}\left\{t^{1/s} - \left(\bar\mu\, t + \bar\nu\right)^{1/r}\right\}\mathbf{1}$$

holds. A function $h(t) = t^{1/s} - M\left(\frac{r}{s}M^{-s}t + \frac{s-r}{s}\right)^{1/r}$ is decreasing on $[M^s, m^s]$ for $r < 2s < 0,\ r \le -1$. Then $\min\limits_{t\in[M^s,m^s]} h(t) = h(m^s)$ and we obtain

$$\left(m - M\left(\frac{r}{s}\frac{m^s}{M^s} + \frac{s-r}{s}\right)^{1/r}\right)\mathbf{1} \le \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \le \tilde\Delta\mathbf{1}. \tag{16}$$

Similarly, h is a increasing function on $[m^s, M^s]$ for $r < -s < 0,\ r \le -1$. Then $\min\limits_{t\in[m^s,mM^s]} h(t) = h(m^s)$ and we obtain (16) again.

(c_1) If $r \le s,\ 0 < r \le 1$ or $-s \le r,\ -1 \le r < 0$ or $2s \le r \le s < 0,\ r \ge -1$, then

$$-\,C\left(m^r, M^r, 1/r\right)\mathbf{1} \le \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \le \left(\tilde\Delta + \,C\left(m^r, M^r, 1/r\right)\right)\mathbf{1}. \tag{17}$$

(d_1) If $-1 \le r < -s < 0$ or $-1 \le r < 2s < 0$, then

$$\left(m - M\left(\tfrac{r}{s}\tfrac{m^s}{M^s} + \tfrac{s-r}{s}\right)^{1/r} - \,C\left(m^r, M^r, 1/r\right)\right)\mathbf{1} \le \Phi(A^s)^{1/s} - \Phi(A^r)^{1/r} \le \left(\tilde\Delta + \,C\left(m^r, M^r, 1/r\right)\right)\mathbf{1}. \tag{18}$$

Finally, (12) holds in cases (a) and (a_1). If we put $r = 1$ in (12) in the case (a_1) and $s = -1$ in the case (a), then we obtain that

$$\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s}$$

holds in the region (i). Consequently, we obtain that (12) holds in the region (i), (13) holds in the region (ii) and (16) holds in the region (iii).

In the region (iv) inequalities (15) and (17) hold. Using Lemma 8 (1), we obtain that $\tilde\Delta + \,C\left(m^s, M^s, 1/s\right)$ is a better upper bound. A lower bound is

$$\max\left\{m\left(\frac{s}{r}\frac{M^r}{m^r} + \frac{r-s}{r}\right)^{1/s} - M - \,C(m^s, M^s, 1/s), -C\left(m^r, M^r, 1/r\right)\right\}.$$

In the region (v) inequalities (14) and (18) hold. Using Lemma 8 (1), we have that $\tilde\Delta + \,C\left(m^s, M^s, 1/s\right)$ is a better upper bound. A lower bound is

$$\max\left\{m - M\left(\frac{r}{s}\frac{m^s}{M^s} + \frac{s-r}{s}\right)^{1/r} - \,C\left(m^r, M^r, 1/r\right), -C\left(m^s, M^s, 1/s\right)\right\} = -\,C\left(m^s, M^s, 1/s\right),$$

since $m < M\left(\frac{r}{s}\frac{m^s}{M^s} + \frac{s-r}{s}\right)^{1/r}$ in this case and $-C\left(m^r, M^r, 1/r\right) < -C\left(m^s, M^s, 1/s\right)$.

In the regions $(\text{iv})_1$ and $(\text{v})_1$ inequalities (14) and (17) hold. We have a better bounds in LHS of (14). □

We can obtain better bounds in the regions (ii) and (iii) then appropriate bounds in Theorem 7 under additional conditions.

Theorem 9 *Let $(A_1,\dots,A_n)$ be an n-tuple of positive operators in $\mathcal{B}(H)$ with spectra in $[m,M]$ for some scalars $0<m<M$, $(\Phi_1,\dots,\Phi_n)$ be an n-tuple of positive linear maps $\Phi_i:\mathcal{B}(H)\to\mathcal{B}(K)$ and $(\omega_1,\dots,\omega_n)$ be an n-tuple of positive real numbers such that $(\omega_1\Phi_1,\dots,\omega_n\Phi_n)$ is unital.*

Let (r,s) be in (ii) *and T_r be the closed interval joining m^r to M^r. If $Y\subseteq T_r$ such that $\frac{s}{r}y^{-r}m^r+1-\frac{s}{r}>0$ for every $y\in Y$, then*

$$\max_{y\in Y}\min\left\{y\left(\tfrac{s}{r}y^{-r}m^r+1-\tfrac{s}{r}\right)^{1/s}-m,\,y\left(\tfrac{s}{r}y^{-r}M^r+1-\tfrac{s}{r}\right)^{1/s}-M\right\}\mathbf{1} \le \widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega)-\widetilde{M}_n^{[r]}(\mathbf{A},\Phi,\omega). \tag{19}$$

Let (r,s) be in (iii) *and T_s be the closed interval joining m^s to M^s. If $Y\subseteq T_s$ such that $\frac{r}{s}y^{-s}M^s+1-\frac{r}{s}>0$ for every $y\in Y$, then*

$$\max_{y\in Y}\min\left\{m-y\left(\tfrac{r}{s}y^{-s}m^s+1-\tfrac{r}{s}\right)^{1/r},\,M-y\left(\tfrac{r}{s}y^{-s}M^s+1-\tfrac{r}{s}\right)^{1/r}\right\}\mathbf{1} \le \widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega)-\widetilde{M}_n^{[r]}(\mathbf{A},\Phi,\omega). \tag{20}$$

Proof. In the region (ii) we use the same technique as in the proof of Theorem 7 in the same region. If $0<s\le -r$ or $0<r\le s\le 2r$, then we have

$$0<\left(\frac{s}{r}y^{s-r}m^r+\frac{r-s}{r}y^s\right)\mathbf{1}\le \frac{s}{r}y^{s-r}\Phi(A^r)+\frac{r-s}{r}y^s\mathbf{1}\le\left(\frac{s}{r}y^{s-r}M^r+\frac{r-s}{r}y^s\right)\mathbf{1}.$$

Putting $p=s/r$ in (8) in Lemma 3 and replacing t by t^r and y by y^r, we obtain

$$\min_{t\in T_r}\left\{\left(\frac{s}{r}y^{s-r}t+\left(1-\frac{s}{r}\right)y^s\right)^{1/s}-t^{1/r}\right\}\le \Phi(A^s)^{1/s}-\Phi(A^r)^{1/r}.$$

A function $h(t)=\left(\frac{s}{r}y^{s-r}t+\left(1-\frac{s}{r}\right)y^s\right)^{1/s}-t^{1/r}$ is concave, since it is obviously that $h''(t)<0$ for (s,r) in (ii). It follows

$$\min\left\{y\left(\frac{s}{r}y^{-r}m^r+1-\frac{s}{r}\right)^{1/s}-m,\,y\left(\frac{s}{r}y^{-r}M^r+1-\frac{s}{r}\right)^{1/s}-M\right\}\mathbf{1}\le \Phi(A^s)^{1/s}-\Phi(A^r)^{1/r}$$

and we obtain (19). Similarly we can prove (20). □

Applying Theorem 9 for some special y, we obtain the following corollary.

Corollary 10 *Assume that $(A_1,\dots,A_n)$, $(\Phi_1,\dots,\Phi_n)$, m and M as in Theorem 9.*

Let (r,s) be in the region (ii) *and I_r be the open interval joining m^r to M^r. If $\min\{\tilde{\mu}m^r+\tilde{\nu^*},\tilde{\mu}M^r+\tilde{\nu^*}\}>0$ and $\left(\frac{r}{s}\tilde{\mu}\right)^{1/(s-r)}\in I_r$, where we denote $\tilde{\mu}=\frac{M^s-m^s}{M^r-m^r}$, $\tilde{\nu^*}=\left(1-\frac{s}{r}\right)\left(\frac{r}{s}\tilde{\mu}\right)^{s/(s-r)}$, then*

$$\min\left\{\left(\tilde{\mu}m^r+\tilde{\nu^*}\right)^{1/s}-m,\left(\tilde{\mu}M^r+\tilde{\nu^*}\right)^{1/s}-M\right\}\mathbf{1}\le \widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega)-\widetilde{M}_n^{[r]}(\mathbf{A},\Phi,\omega). \tag{21}$$

The bound in (21) is better than the bound: $m\left(\frac{s}{r}\frac{M^r}{m^r}+1-\frac{s}{r}\right)^{1/s}-M$ was given in Theorem 7.

Further, let y_0 be a solution of the equation $y\left(\frac{s}{r}y^{-r}m^r+1-\frac{s}{r}\right)^{1/s}-m=y\left(\frac{s}{r}y^{-r}M^r+1-\frac{s}{r}\right)^{1/s}-M$. If $y_0\in I_r$ and $\frac{s}{r}y_0^{-r}m^r+1-\frac{s}{r}>0$, then

$$\left(y_0\left(\frac{s}{r}y_0^{-r}m^r+1-\frac{s}{r}\right)^{1/s}-m\right)\mathbf{1}\le \widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega)-\widetilde{M}_n^{[r]}(\mathbf{A},\Phi,\omega) \tag{22}$$

and this bound in LHS is better than the lower bound in (21).

Similarly, let (r,s) *be in the region* (iii) *and* I_s *be the open interval joining* m^s *to* M^s. *If* $\min\{\bar{\mu}m^s+\bar{\nu^*},\bar{\mu}M^s+\bar{\nu^*}\}>0$ *and* $\left(\frac{s}{r}\bar{\mu}\right)^{1/(r-s)}\in I_s$, *where we denote* $\bar{\mu}=\frac{M^r-m^r}{M^s-m^s}$, $\bar{\nu^*}=\left(1-\frac{r}{s}\right)\left(\frac{s}{r}\bar{\mu}\right)^{r/(r-s)}$, *then*

$$\min\left\{m-\left(\bar{\mu}m^s+\bar{\nu^*}\right)^{1/r},M-\left(\bar{\mu}M^s+\bar{\nu^*}\right)^{1/r}\right\}\mathbf{1}\le\widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega)-\widetilde{M}_n^{[r]}(\mathbf{A},\Phi,\omega). \tag{23}$$

The bound in (23) *is better than the bound:* $m-M\left(\frac{r}{s}\frac{m^s}{M^s}+1-\frac{r}{s}\right)^{1/r}$ *was given in Theorem 7.*

Further, let y_0 *be a solution of the equation* $m-y\left(\frac{r}{s}y^{-s}m^s+1-\frac{r}{s}\right)^{1/r}=M-y\left(\frac{r}{s}y^{-s}M^s+1-\frac{r}{s}\right)^{1/r}$. *If* $y_0\in I_s$ *and* $\frac{r}{s}y_0^{-s}M^s+1-\frac{r}{s}>0$, *then*

$$\left(m-y_0\left(\frac{r}{s}y_0^{-s}m^s+1-\frac{r}{s}\right)^{1/r}\right)\mathbf{1}\le\widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega)-\widetilde{M}_n^{[r]}(\mathbf{A},\Phi,\omega) \tag{24}$$

and this bound in LHS is better than the lower bound in (23).

Proof. Putting $y=\left(\frac{r}{s}\tilde{\mu}\right)^{1/(s-r)}$ in (19) we obtain (21).

Next, using that $h(t)=t^{s/r}$ is a convex function for (r,s) in (ii) and that $\left(\frac{r}{s}\tilde{\mu}\right)^{1/(s-r)}$ is in I_r, we obtain the following relation between their tangent lines

$$\min_{t\in T_r}\left\{\frac{s}{r}m^{s-r}t+\left(1-\frac{s}{r}\right)m^s\right\}<\min_{t\in T_r}\left\{\tilde{\mu}t+\tilde{\nu^*}\right\}.$$

It follows

$$\min_{t\in T_r}\left\{\left(\frac{s}{r}m^{s-r}t+\left(1-\frac{s}{r}\right)m^s\right)^{1/s}\right\}<\min_{t\in T_r}\left\{\left(\tilde{\mu}t+\tilde{\nu^*}\right)^{1/s}\right\}\quad\text{for } s>0,$$

which gives that the bound in (21) is better than the bound $m\left(\frac{s}{r}\frac{M^r}{m^r}+1-\frac{s}{r}\right)^{1/s}-M$.

Further, we can prove that the bound in (24) is better than the bound in (23) by contraposition. □

3.2 Ratio type inequalities. Our second main result about the order among the power means is given in the following theorem, which improve [2, Theorem 4.4], cf. also [6, Theorem 11].

Theorem 11 *Assume that the conditions of Theorem 7 hold.*
If (r,s) *in* (i), *then*

$$\Delta(h,r,s)^{-1}\,\widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega)\le\widetilde{M}_n^{[r]}(\mathbf{A},\Phi,\omega)\le\widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega),$$

if (r,s) *in* (ii) *or* (iii), *then*

$$\Delta(h,r,s)^{-1}\,\widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega)\le\widetilde{M}_n^{[r]}(\mathbf{A},\Phi,\omega)\le\Delta(h,r,s)\,\widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega),$$

if (r,s) *in* (iv), *then*

$$\begin{aligned}\Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}\,\widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega)&\le\widetilde{M}_n^{[r]}(\mathbf{A},\Phi,\omega)\\&\le\min\{\Delta(h,r,1),\Delta(h,s,1)\Delta(h,r,s)\}\,\widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega),\end{aligned}$$

if (r,s) *in* (v) *or* $(\text{iv})_1$ *or* $(\text{v})_1$*, then*

$$\Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}\,\widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega) \le \widetilde{M}_n^{[r]}(\mathbf{A},\Phi,\omega) \le \Delta(h,s,1)\,\widetilde{M}_n^{[s]}(\mathbf{A},\Phi,\omega),$$

where a generalized Specht ratio $\Delta(h,r,s)$ *[2, § 2.7] is defined as*

$$\Delta(h,r,s) = \left\{\frac{r(h^s-h^r)}{(s-r)(h^r-1)}\right\}^{1/s}\left\{\frac{s(h^r-h^s)}{(r-s)(h^s-1)}\right\}^{-1/r}, \qquad h=\frac{M}{m}.$$

In order to prove Theorem 11, we need the operator order given in the following theorem.

Theorem D *[6, Corollary 3] If* $A, B \in \mathcal{B}_+(H)$*,* $A \ge B > 0$ *such that* $\mathsf{Sp}(A) \subseteq [n,N]$ *and* $\mathsf{Sp}(B) \subseteq [m,M]$ *for some scalars* $0<n<N$ *and* $0<m<M$*, then*

$$\begin{aligned} K(n,N,p)\ \ A^p \ge B^p > 0 &\quad \text{for all} \quad p>1,\\ K(m,M,p)\ \ A^p \ge B^p > 0 &\quad \text{for all} \quad p>1,\\ K(n,N,p)\ \ B^p \ge A^p > 0 &\quad \text{for all} \quad p<-1,\\ K(m,M,p)\ \ B^p \ge A^p > 0 &\quad \text{for all} \quad p<-1. \end{aligned}$$

In [2, Lemma 2.62] we gave a collection of basic properties of the generalized Specht ratio $\Delta(h,r,s)$. We shall need some properties of the constant $\Delta(h,r) \equiv \Delta(h,r,1)$ (see Figure 3).

Lemma 12 *Let* $M>m>0$ *and* $r \in \mathbf{R}$ *and*

$$\Delta(h,r,1) = \frac{r(h-h^r)}{(1-r)(h^r-1)}\left(\frac{h^r-h}{(r-1)(h-1)}\right)^{-1/r}, \qquad h=\frac{M}{m}.$$

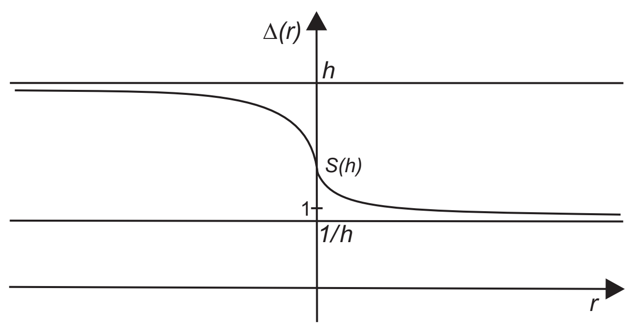


Figure 3: Function $\Delta(r) \equiv \Delta(h,r,1)$

1. *A function* $\Delta(r) \equiv \Delta(h,r,1)$ *is strictly decreasing for all* $r \in \mathbf{R}$*,*

2. $\lim_{r\to 1}\Delta(h,r,1) = 1$ *and* $\lim_{r\to 0}\Delta(h,r,1) = S(h)$*,*
   where $S(h)$ *is the Specht ratio defined by* $(***)$*,*

3. $\lim_{r\to\infty}\Delta(h,r,1) = 1/h$ *and* $\lim_{r\to-\infty}\Delta(h,r,1) = h$*.*

Proof.

1. We write

(25)
$$\Delta(r) = \Delta_1(r) \cdot \Delta_2(r), \quad \Delta_1(r) = \frac{r(h^r - h)}{(r-1)(h^r-1)}, \quad \Delta_2(r) = \left(\frac{h^r - h}{(r-1)(h-1)}\right)^{-1/r}.$$

By using differential calculus we shall prove that a function Δ_1 is strictly decreasing for all $r \neq 0, 1$. We have

$$\begin{aligned} \tfrac{d}{dr}\Delta_1(r) &= \tfrac{-1}{(r-1)^2(h^r-1)^2}\left((h^r-1)(h^r-h) - (r-1)rh^r(h-1)\ln h\right) \\ &= -\tfrac{h^r(h-1)\ln h}{(r-1)^2(h^r-1)^2} f(r), \quad \text{where } f(r) = \tfrac{(h^r-1)(h^r-h)}{h^r(h-1)\ln h} - (r-1)r. \end{aligned} \tag{26}$$

Stationary points of the function f are 0, 0.5, 1 and it is a strictly decreasing function on $(-\infty, 0) \cup (0.5, 1)$ and strictly increasing on $(0, 0.5) \cup (1, \infty)$. Also, $f(0) = f(1) = 0$. So, $f(r) > 0$ for all $r \neq 0, 1$. (More exactly, $f'''(r) = \frac{\ln^2 h}{h-1}\left(h^r - h^{1-r}\right)$ imply $f'''(r) > 0$ for $r > 0.5$ and $f'''(r) < 0$ for $r < 0.5$. It follows that the function f'' is strictly increasing on $(0.5, \infty)$ and strictly decreasing on $(-\infty, 0.5)$. Next, $f''(0.5) < 0$, $f''(0) > 0$ and $f''(1) > 0$ imply that f'' has two roots $0 < r_1 < 0.5 < r_2$. It follows that f' is strictly increasing on $(-\infty, r_1) \cup (r_2, \infty)$ and strictly decreasing on (r_1, r_2). Also, $f'(0) = f'(0.5) = f'(1) = 0$. It follows that $f'(r) < 0$ for $r \in (-\infty, 0) \cup (0.5, 1)$ and $f'(r) > 0$ for $r \in (0, 0.5) \cup (1, \infty)$.) Now, using (26) we have that $\frac{d}{dr}\Delta_1(r) < 0$ for all $r \neq 0, 1$ and it follows that Δ_1 is strictly decreasing function.

Further, in the case of a function Δ_2 in (25), we obtain

$$\begin{aligned} &\tfrac{d}{dr}\Delta_2(r) = \tfrac{-1}{(r-1)r^2(h^r-h)}\left(\tfrac{h^r-h}{(r-1)(h-1)}\right)^{-1/r} \\ &\times \left[r(r-1)h^r \ln h - r(h^r - h) + (r-1)(h^r-h)\ln\left(\tfrac{(r-1)(h-1)}{h^r-h}\right)\right]. \end{aligned}$$

By using differential calculus we can prove that a function

$$r \mapsto r(r-1)h^r \ln h - r(h^r - h) + (r-1)(h^r - h)\ln\left(\frac{(r-1)(h-1)}{h^r - h}\right)$$

is positive for all $r \neq 0, 1$. So $\frac{d}{dr}\Delta_2(r) < 0$ for all $r \neq 0, 1$ and it follows that Δ_2 is strictly decreasing function.

2. Using [2, (2.97)], we have $\Delta(h, r, 1) = K(h^r, 1/r)$ if $r \neq 0$. Now, we have $K(h, 1) = 1$ by using [2, Theorem 2.54 (iii)] and $\lim\limits_{r\to 0} K(h^r, 1/r) = S(h)$ by using [2, Theorem 2.56].

3. We have by L'Hospital's theorem

$$\lim_{r\to\infty} \frac{\ln\left((r-1)(h-1)/(h^r-h)\right)}{r} = \lim_{r\to\infty}\left(\frac{1}{r-1} - \frac{h^r \ln h}{h^r - h}\right) = -\ln h.$$

So

$$\lim_{r\to\infty} \Delta(h, r, 1) = \lim_{r\to\infty} \frac{r}{r-1} \cdot \frac{h^r - h}{h^r - 1} \cdot \left(\frac{(r-1)(h-1)}{h^r - h}\right)^{1/r} = e^{-\ln h} = 1/h.$$

Similarly, we obtain $\lim\limits_{r\to-\infty} \Delta(h, r, 1) = h$. □

Proof of Theorem 11. This theorem follows from Lemma 2 by putting $p = s/r$ or $p = r/s$ and then using the Löwner-Heinz theorem, Theorem D and Lemma 12. We give the proof for the sake of completeness.

Putting $p = s/r$ in Lemma 2 and replacing A by A^r and applying the Löwner-Heinz theorem, we obtain the following cases (a) and (b):

(a) If $r \le s \le -1$ or $1 \le s \le -r$ or $0 < r \le s \le 2r,\ s \ge 1$, then

$$\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le \Delta(h,r,s)\ \Phi(A^r)^{1/r}. \tag{27}$$

(b) If $0 < -r < s,\ s \ge 1$ or $0 < 2r < s,\ s \ge 1$, then

$$\Delta(h,r,s)^{-1}\ \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le \Delta(h,r,s)\ \Phi(A^r)^{1/r}. \tag{28}$$

We used equalities $K\left(m^r, M^r, s/r\right)^{1/s} = K\left(M^r, m^r, s/r\right)^{1/s} = \Delta(h,r,s)$ in the above cases.

If $-1 \le s \le 1$, then we apply Theorem D and obtain the following cases (c) and (d):

(c) If $r \le s,\ -1 \le s < 0$ or $s \le -r,\ 0 < s \le 1$ or $0 < r \le s \le 2r,\ s \le 1$, then

$$\Delta(h,s,1)^{-1}\ \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le \Delta(h,s,1)\ \Delta(h,r,s)\ \Phi(A^r)^{1/r}. \tag{29}$$

(d) If $0 < -r < s \le 1$ or $0 < 2r < s \le 1$, then

$$\Delta(h,s,1)^{-1}\ \Delta(h,r,s)^{-1}\ \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s} \le \Delta(h,s,1)\ \Delta(h,r,s)\ \Phi(A^r)^{1/r}, \tag{30}$$

where we used equalities $K\left(m^s, M^s, 1/s\right) = \Delta(h,1,s)^{-1} = \Delta(h,s,1)$.

Similarly, putting $p = r/s$ in Lemma 2 and replacing A by A^s and then, applying the Löwner-Heinz theorem, we obtain the following cases (a_1) and (b_1):

(a_1) If $1 \le r \le s$ or $-s \le r \le -1$ or $2s \le r \le s < 0,\ r \le -1$, then

$$\Delta(h,r,s)^{-1}\ \Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s}. \tag{31}$$

(b_1) If $r < -s < 0,\ r \le -1$ or $r < 2s < 0,\ r \le -1$, then

$$\Delta(h,r,s)^{-1}\ \Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \Delta(h,r,s)\ \Phi(A^s)^{1/s}. \tag{32}$$

We used equalities $K\left(m^s, M^s, r/s\right)^{1/r} = \left(M^s, m^s, r/s\right)^{1/r} = \Delta(h,r,s)^{-1}$ in the above cases.

If $-1 \le s \le 1$, then we apply Theorem D and obtain the following cases (c_1) and (d_1):

(c_1) If $r \le s,\ 0 < r \le 1$ or $-s \le r,\ -1 \le r < 0$ or $2s \le r \le s < 0,\ r \ge -1$, then

$$\Delta(h,r,1)^{-1}\ \Delta(h,r,s)^{-1}\ \Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \Delta(h,r,1)\ \Phi(A^s)^{1/s}. \tag{33}$$

(d_1) If $-1 \le r < -s < 0$ or $-1 \le r < 2s < 0$, then

$$\Delta(h,r,1)^{-1}\ \Delta(h,r,s)^{-1}\ \Phi(A^s)^{1/s} \le \Phi(A^r)^{1/r} \le \Delta(h,r,1)\ \Delta(h,r,s)\ \Phi(A^s)^{1/s}. \tag{34}$$

Finally, we choose better bounds using Lemma 12 and the same technique as in Theorem 7. Really, in cases (a) and (a_1) the inequality (27) holds and in cases (b) and (b_1) the inequality (28) holds. If we put $r = 1$ in (31) for $1 \le r \le s$ and $s = -1$ in (27) for $r \le s \le -1$, then we obtain $\Phi(A^r)^{1/r} \le \Phi(A^s)^{1/s}$ in the region (i). Consequently, we obtain that (27) holds in the region (i) and (28) holds in the regions (ii) and (iii).

In the region (iv) inequalities (30) and (33) hold. Now, using Lemma 12 (1) we obtain

$$\Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1} \ge \Delta(h,r,1)^{-1}\Delta(h,r,s)^{-1}$$

and it follows that $\Delta(h,s,1)^{-1}\Delta(h,r,s)^{-1}$ is a better lower bound. The upper bound is equal

$$\min\{\Delta(h,r,1),\Delta(h,s,1)\Delta(h,r,s)\}.$$

In the region (v) inequalities (29) and (34) hold. Using Lemma 12 (1), we have that $\Delta(h,s,1)^{-1}\ \times\Delta(h,r,s)^{-1}$ is a better lower bound. The upper bound is equal

$$\min\{\Delta(h,s,1),\Delta(h,r,1)\Delta(h,r,s)\}=\Delta(h,s,1),$$

since hold $\Delta(h,r,s)\geq 1$ and $\Delta(h,r,1)\geq\Delta(h,s,1)$. (More exactly, applying properties of the generalized Kantorovich constant [2, §2.7], we have that $K(m,M,p)\geq 1$ for $p\in\mathbf{R}\backslash(0,1)$ and $0<K(m,M,p)\leq 1$ for $p\in[0,1]$. Then we obtain that $\Delta(h,r,s)=K(m^r,M^r,s/r)^{1/s}\geq 1$ for $r\leq s$.)

In the regions $(\mathrm{iv})_1$ and $(\mathrm{v})_1$ inequalities (29) and (33) hold. Analogously in the inequality (29) we have better bounds. □

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