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ON SOME MAPS CONCERNING gp-CLOSED SETS AND RELATED GROUPS

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Abstract. In the present paper we first introduce three classes of maps called approximately s.M-precontinuity, approximately s.M-preclosedness and contra-preirresoluteness. We obtain their basic properties and their relationships with other types of mappings between topological spaces. Finally, we construct and investigate some groups of some mappings corresponding to a topological space. We show some group structures for a subspace of the digital line and the digital plane.

1 Introduction The concept of closedness of subsets is fundamental with respect to the investigation of general topological spaces. In 1970, Levine [26] initiated the study of *generalized closed sets* (briefly, the so-called *g-closed sets*) and he generalized the concept of closedness (cf. [26, Definition 2.1]). Moreover, he introduced and investigated the concept of $T_{1/2}$-*spaces* which is properly placed between T_0-*spaces* and T_1-*spaces* ; a topological space is called $T_{1/2}$ if every g-closed set is closed; every closed set is g-closed (cf. [26, Definition 5.1, Theorem 5.3, Corollary 5.6]). In 1977, Dunham [12, Theorem 2.5] proved that a topological space is $T_{1/2}$ if and only if every singleton is open or closed (cf. [13, Theorem 3.7], [27, Theorem 3.9]). The *digital line*, so-called *Khalimsky line* [21], is a typical and geometric example of $T_{1/2}$-spaces; the concept was published in Russia by E. Khalimsky in 1970 [21] (cf. [22, p.7, line −6], the end of Section 2 below). In 1990, Khalimsky, Kopperman and Meyer [22] developed the work of [21] and studied a finite analog of the Jordan curve theorem motivated by a problem in computer graphics (cf. [23], [24]).

The notion of a *generalized preclosed set* (briefly, gp-*closed set*) was introduced by Noiri et al.[36, Definition 2] (cf. [29, Definition 2.11], [10], [3]). One of the aims of the present paper is to introduce and investigate two classes of maps called *approximately s.M-precontinuous maps* (Definition 3.2) and *approximately s.M-preclosed maps* (Definition 3.3) by using gp-closed sets and studying some of their basic properties (cf. Section 3). This definition enables us to obtain conditions under which inverse maps and maps preserve gp-closed sets (Theorem 3.8, Theorem 3.9). The class of approximately s.M-precontinuous maps is a generalization of the class of *strongly M-precontinuous maps* introduced by Abd El-Monsef et al. [2] (cf. Definition 2.3(ii), Remark 3.6 below). Reilly and Vamanamurthy [39] introduced the notion of *preirresoluteness* of maps and several generalizations of preirresoluteness have been developed recently (cf. [6], [15], [35], [38], [28]). In Section 3 of the present paper, we present the notion of a new related class of preirresoluteness [39] called *contra-preirresolute* (Definition 2.3(iv)); we define this last class of map by requirement that the inverse image of each preopen set in the codomain is preclosed in the domain. This notion is a stronger form of approximately s.M-precontinuity (Remark 3.6). The final purpose of the present paper is to construct some groups corresponding to a topological space using preirresolute bijections and contra-preirresolute bijections (Definition 4.2, Theorem 4.3) and subgroups

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corresponding to a subspace of the topological space (Definition 5.1, Theorem 5.7). In Section 6, we try to study examples of groups corresponding to a subspace of the digital line. The *digital plane* is the topological product of two copies of the digital lines (eg., [24, Definition 4], [23, p.907], [11], [18], [17], [7], [9], [41], [42]; cf. Section 6 of the present paper); the digital plane or the rectungular portion of the plane as its subspace is a mathematical model of the computer screen. We also investigate characterizations of some preirresolute functions on digital planes (Theorem 6.8), an example of subgroups (Theorem 6.10) and properties of some functions on the digital planes (Theorem 6.12, Corollary 6.15).

2 Preliminalies Throughout the present paper, $(X,\tau),(Y,\sigma)$ and (Z,η) represent nonempty topological spaces on which no separation axioms are assumed unless otherwise mentioned and $\mathbb{Z}$ denotes the set of all integers. For a subset of a topological space, the closure and interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. We first need the following definition.

Definition 2.1 (i) A subset A of a topological space (X,τ) is said to be

(a) *preopen* [30] if $A \subseteq Int(Cl(A))$ holds in (X,τ),

(b) *semi-open* [25] if there exists an open set O such that $O \subseteq A \subseteq Cl(O)$ or equivalently if $A \subseteq Cl(Int(A))$ holds in (X,τ),

(c) *α-open* [32] if $A \subseteq Int(Cl(Int(A)))$ holds in (X,τ).

(ii) A subset F of (X,τ) is said to be *preclosed* (resp. *semi-closed*, *α-closed*) if $X \setminus F$ is preopen (resp. semi-open, α-open) in (X,τ).

(**Notation**) (1) The collection of all preopen (resp. semi-open, α-open) subsets in (X,τ) is denoted by $PO(X,\tau)$ (resp. $SO(X,\tau),\tau^{\alpha}$).

(2) The collection of all preclosed (resp. semi-closed) subsets in (X,τ) is denoted by $PC(X,\tau)$ (resp. $SC(X,\tau)$).

Remark 2.2 **(i)** The intersection of all preclosed subsets containing A is called the *preclosure* [16, Definition 2.1] of A and is denoted by $pCl(A)$. It is well known [16, Lemmas 2.2, 2.3] that for a point $x \in X$ and a subset A of $(X,\tau), x \in pCl(A)$ if and only if $A \cap V \neq \emptyset$ for every $V \in PO(X,\tau)$ containing x; a subset F is preclosed in (X,τ) if and only if $F = pCl(F)$ holds; if $A \subseteq B$ then $pCl(A) \subseteq pCl(B)$ holds; $pCl(pCl(A)) = pCl(A)$ holds for every subset A of (X,τ); $pCl(A)$ is preclosed in (X,τ) for every subset A of (X,τ).

(ii) $pCl(A) = A \cup Cl(Int(A))$ holds for every subset A of (X,τ) ([5, Theorem 1.5 (e)]).

(iii) The *preinterior* [1] of A is the union of all preopen subsets of (X,τ) contained in A and is denoted by $pInt(A)$. It is well known that $pInt(A)$ is preopen in (X,τ) for every subset A of (X,τ). We observe that $pInt(A) = A \cap Int(Cl(A))$ holds for every subset A of (X,τ).

We need the following definition on some maps between topological spaces.

Definition 2.3 A map $f : (X,\tau) \to (Y,\sigma)$ is said to be:

(i) *precontinuous* provided that for every $V \in \sigma$, $f^{-1}(V) \in PO(X,\tau)$ (Mashhour, Abd El-Monsef and El-Deeb [30]),

(ii) *strongly M-precontinuous* (shortly, *s.M-precontinuous*) provided that for every $V \in PO(Y,\sigma)$, $f^{-1}(V)$ is open in (X,τ) (Abd El-Monsef, Mahmoud and Nasef [2], e.g., [33]),

(iii) *preirresolute* provided that for every $V \in PO(Y,\sigma)$, $f^{-1}(V) \in PO(X,\tau)$ (Reilly and Vamanamurthy [39]),

(iv) *contra-preirresolute* provided that for every $V \in PO(Y,\sigma)$, $f^{-1}(V) \in PC(X,\tau)$,

(v) *perfectly preirresolute* provided that for every $V \in PO(Y,\sigma)$, $f^{-1}(V) \in PC(X,\tau) \cap PO(X,\tau)$,

(vi) *M-preopen* (resp. *M-preclosed*) provided that for every $U \in PO(X,\tau)$ (resp. $U \in PC(X,\tau)$), $f(U) \in PO(Y,\sigma)$ (resp. $f(U) \in PC(Y,\sigma)$) (Mashhour, Abd El-Monsef, Hasanein and Noiri [31]),

(vii) *strongly M-preclosed* (shortly, s.*M-preclosed*) provided that for every $F \in PC(X,\tau)$, $f(F)$ is closed in (Y,σ),

(viii) *contra-M-preopen* provided that for every $U \in PO(X,\tau)$, $f(U) \in PC(Y,\sigma)$ (Baker [6, Defnition 3.7]),

(ix) *contra-M-preclosed* provided that for every $U \in PC(X,\tau)$, $f(U) \in PO(Y,\sigma)$.

In order to investigate the relationships (cf. Remark 3.7 below) of the above mentioned concepts and new one of Definitions 3.2, 3.3 below, we use partially the concept of the Khalimsky line or the so called digital line due to Khalimsky and digital planes ([21]; cf. [22], [23], [24], [10, Example 4.6], [11], [18], [7], [17], [42], [41], [9]): the *digital line* is the set of the integers, $\mathbb{Z}$, equiped with the topology κ having $\{\{2m-1, 2m, 2m+1\} | m \in \mathbb{Z}\}$ as a subbase. This topological space is denoted by $(\mathbb{Z},\kappa)$. Thus, a set U is open in $(\mathbb{Z},\kappa)$ if and only if whenever $x \in U$ is an even integer, then $x-1, x+1 \in U$. Especially, $\{2m+1\}$ is open and $\{2s\}$ is closed in $(\mathbb{Z},\kappa)$, where m and s are integers. It is well known that $\kappa = PO(\mathbb{Z},\kappa)$ (eg., [17, Theorem 2.1 (i)]).

3 Approximately s.M-precontinuity and approximately s.M-preclosedness

First we introduce a weak form of a strongly M-precontinuous function (cf. Definition 2.3 (ii)), called *approximately s.M-precontinuous* (Definition 3.2 below); secondly we introduce a weak form of a strongly M-preclosed function (cf. Definition 2.3 (vii)), called *approximately s.M-preclosed* (Definition 3.3 below). We need the following concepts of *generalized preclosed sets* and *generalized open sets*.

Definition 3.1 ([36, Definition 2]) A subset A of (X,τ) is called *generalized preclosed* (briefly, *gp-closed*) if $pCl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$. A subset B is called *gp-open* if its complement $X \setminus B$ is gp-closed in (X,τ). Let $GPC(X,\tau)$ (resp. $GPO(X,\tau)$) denote the family of all gp-closed sets (resp. gp-open sets) of (X,τ).

It is shown that $GPO(X,\tau) = \{B | F \subseteq pInt(B)$ holds whenever $F \subseteq B$ and F is closed in $(X,\tau)\}$ holds.

Definition 3.2 A map $f : (X,\tau) \to (Y,\sigma)$ is said to be *approximately s.M-precontinuous* (shortly, *ap-s.M-precontinuous*) if $pCl(A) \subseteq f^{-1}(V)$ holds whenever $V \in PO(Y,\sigma), A \in GPC(X,\tau)$ and $A \subseteq f^{-1}(V)$.

Definition 3.3 A map $f : (X,\tau) \to (Y,\sigma)$ is said to be *approximately s.M-preclosed* (shortly, *ap-s.M-preclosed*) if $f(U) \subseteq pInt(B)$ holds whenever $U \in PC(X,\tau), B \in GPO(Y,\sigma)$, and $f(U) \subseteq B$.

Theorem 3.4 *Let $f : (X,\tau) \to (Y,\sigma)$ be a function.*

(i-1) *Every s.M-precontinuous map* (cf. Definition 2.3 (ii)) *is ap-s.M-precontinuous.*

(i-2) *Every contra-preirresolute map* (cf. Definition 2.3 (iv)) *is ap-s.M-precontinuous.*

(ii-1) *Every s.M-preclosed map* (cf. Definition 2.3 (vii)) *is ap-s.M-preclosed.*

(ii-2) *Every contra-M-preclosed map* (cf. Definition 2.3 (ix)) *is ap-s.M-preclosed.*

Proof. **(i-1)** Suppose that $f : (X,\tau) \to (Y,\sigma)$ is a s.M-precontinuous map. Let $V \in PO(Y,\sigma), A \in GPC(X,\tau)$ and $A \subseteq f^{-1}(V)$. By Definition 2.3(ii) and Definition 3.1, $f^{-1}(V) \in \tau$ and so $pCl(A) \subseteq f^{-1}(V)$. Thus, f is ap-s.M-precontinuous. **(i-2)** Suppose that $f : (X,\tau) \to (Y,\sigma)$ is a contra-preirresolute map. Let $V \in PO(Y,\sigma), A \in GPC(X,\tau)$

and $A \subseteq f^{-1}(V)$. It follows from assumption that $pCl(A) \subset f^{-1}(V) = pCl(f^{-1}(V))$ and so f is ap-s.M-precontinuous. **(ii-1)** Suppose that $f : (X, \tau) \to (Y, \sigma)$ is a s.M-preclosed map. Let $U \in PC(X, \tau), B \in GPO(Y, \sigma)$ and $f(U) \subseteq B$. By Definition 2.3(vii) and Definition 3.2, $f(U)$ is closed in (Y, σ) and so $f(U) \subseteq pInt(B)$. Thus, f is ap-s.M-preclosed. **(ii-2)** Suppose that $f : (X, \tau) \to (Y, \sigma)$ is a contra-M-preclosed map. Let $U \in PC(X, \tau), B \in GPO(Y, \sigma)$ and $f(U) \subseteq B$. It follows from assumption that $f(U) = pInt(f(U)) \subseteq pInt(B)$ and so f is ap-s.M-preclosed. $\square$ Under certain conditions, the converses of Theorem 3.4 (i-2) and (ii-2) are true. We observe the following property:

($*$) If $PO(X, \tau) = PC(X, \tau)$ holds for a topological space (X, τ), then $GPC(X, \tau) = GPO(X, \tau) = P(X)$(=the power set of X).

Theorem 3.5 *For a map $f : (X, \tau) \to (Y, \sigma)$, the following properties hold:*

(i) *Suppose that $PO(X, \tau) = PC(X, \tau)$. A map f is ap-s.M-precontinuous if and only if f is contra-preirresolute.*

(ii) *Suppose that $PO(Y, \sigma) = PC(Y, \sigma)$. A map f is ap-s.M-preclosed if and only if f is contra-M-preclosed* (cf. Definition 2.3 (viii)).

Proof. **(i) (Necessity)** Assume f is ap-s.M-precontinuous. Let $V \in PO(Y, \sigma)$. Put $A := f^{-1}(V)$. Then, by ($*$) for the set A, $A \in GPC(X, \tau)$ and $A \subseteq f^{-1}(V)$. Then, by ap-s.M-precontinuity of f, it is obtained that $pCl(A) \subseteq f^{-1}(V)$. Therefore, $pCl(f^{-1}(V)) = f^{-1}(V)$, i.e., $f^{-1}(V) \in PC(X, \tau)$. Namely, f is contra-preirresolute. **(Sufficiency)** It is obtained by Theorem 3.4 (i-2) without using the assumption that $PO(X, \tau) = PC(X, \tau)$.
(ii) (Necessity) Assume f is ap-s.M-preclosed. Let $U \in PC(X, \tau)$. Put $B = f(U)$. By ($*$) for the set B and (Y, σ), it is obtained that $B \in GPO(Y, \sigma)$. Since f is ap-s.M-preclosed, $f(U) \subseteq pInt(B) = pInt(f(U))$, i.e., $f(U) \in PO(Y, \sigma)$. Namely, f is contra-M-preclosed.
(Sufficiency) It is obtained by Theorem 3.4 (ii-2). $\square$

Remark 3.6 The relationships between classes of mappings in Definition 2.3 and Definitions 3.2, 3.3 are shown in the following diagram (cf. Theorem 3.4 and the definitions); the converses are not true in general by Remark 3.7 below. The symbol,$A \not\rightarrow B$, means that A does not necessarily imply B. Some concepts are independent to each other (cf. Remark 3.7 (ix), (xiii) and (xiv) below).

· preirresolute$^{(*)}$ $\not\rightarrow$ ·M-preclosed
(cf. Definition 2.3(iii)) $\not\leftarrow$ (cf. Definition 2.3(vi))
$\not\downarrow$ $\uparrow$ $\not\downarrow$ $\uparrow$
· strongly M-precontinuous $\not\rightarrow$ · strongly M-preclosed
(cf. Definition 2.3(ii)) $\not\leftarrow$ (cf. Definition 2.3(vii))
$\downarrow$ $\not\uparrow$ $\downarrow$ $\not\uparrow$
• ap-s.M-precontinuous $\not\rightarrow$ • ap-s.M-preclosed
(cf. Definition 3.2) (cf. Definition 3.3)
$\not\downarrow$ $\uparrow$ $\not\downarrow$ $\uparrow$
· contra-preirresolute $\not\rightarrow$ · contra-M-preclosed
(cf. Definition 2.3(iv)) $\not\leftarrow$ (cf. Definition 2.3(ix))
$\not\downarrow$ $\uparrow$ $\not\swarrow$
· perfectly preirresolute (cf. Definition 2.3(v))
$\not\uparrow$ $\downarrow$
· preirresolute$^{(*)}$

Remark 3.7 **(i)** The converse of Theorem 3.4 (i-1) is not true in general. Namely, the concept of ap. s.M-precontinuity is strictly weaker than one of s.M-precontinuity. Let

(X, τ) and (Y, σ) be the following topological spaces: $X := \{a, b, c\}, Y := \{a, b, c\}$, $\tau := \{\emptyset, \{a\}, \{a, b\}, X\}$, $\sigma := \{\emptyset, \{a, b\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a map defined by $f(a) = f(b) = a$ and $f(c) = b$. Then, f is not s.M-precontinuous. Indeed, there exists a subset $\{b\} \in PO(Y, \sigma)$ and $f^{-1}(\{b\}) = \{c\} \notin \tau$, because $PO(Y, \sigma) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}, Y\}$. It is shown that this map f is ap-s.M-precontinuous. Indeed, $PC(X, \tau) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\} \subseteq GPC(X, \tau) = PC(X, \tau) \cup \{\{a, c\}\}$. Let $V \in PO(Y, \sigma)$ and $A \in GPC(X, \tau)$ such that $A \subseteq f^{-1}(V)$. Put $GPC_V := \{A \in GPC(X, \tau) | A \subseteq f^{-1}(V)\}$ for each $V \in PO(Y, \sigma)$. Then, for $V = \{a\}, f^{-1}(V) = \{a, b\}$ and $GPC_V = \{\{b\}, \emptyset\}$; for $V = \{b\}, f^{-1}(V) = \{c\}$ and $GPC_V = \{\{c\}, \emptyset\}$; for $V = \{a, b\}, f^{-1}(V) = X$ and $GPC_V = GPC(X, \tau)$; for $V = \{b, c\}, f^{-1}(V) = \{c\}$ and $GPC_V = \{\{c\}, \emptyset\}$; for $V = \{a, c\}, f^{-1}(V) = \{a, b\}$ and $GPC_V = \{\{b\}, \emptyset\}$; for $V = \emptyset, f^{-1}(V) = \emptyset$ and $GPC_V = \{\emptyset\}$; for $V = Y, f^{-1}(V) = X$ and $GPC_V = GPC(X, \tau)$. Thus, we show that $pCl(A) \subseteq f^{-1}(V)$ holds, whenever $V \in PO(Y, \sigma), A \in GPC_V$ and $A \subset f^{-1}(V)$. Therefore, f is ap-s.M-precontinuous.

(ii) The converse of Theorem 3.4(i-2) is not true in general. The ap-s.M-precontinuous map f of (i) above is not contra-preirresolute. Indeed, there exists a set $V := \{a\} \in PO(Y, \sigma)$ such that $f^{-1}(V) = \{a, b\} \notin PC(X, \tau)$.

(iii) Every perfectly preirresolute map is contra-preirresolute (cf. Definition 2.3 (v), (iv)). The following example shows that the converse of the above implication is not true in general. Let (X, τ) be a topological space, where $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a\}, X\}$. Let $f : (X, \tau) \to (X, \tau)$ be a map defined by $f(a) = b, f(b) = f(c) = a$. Then, for each set $U \in PO(X, \tau) = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$, $f^{-1}(U) \in PC(X, \tau) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$ and $f^{-1}(\{a\}) = \{b, c\} \notin PO(X, \tau) \cap PC(X, \tau)$.

(iv) Every stongly M-precontinuous map is preirresolute (cf. Definition 2.3 (ii), (iii)). The following example shows that the converse of the above implication is not true in general. Let (X, τ) be the same topological space of (iii) above and $f : (X, \tau) \to (X, \tau)$ be the identity map. Then, for a subset $\{a, b\} \in PO(X, \tau), f^{-1}(\{a, b\}) = \{a, b\} \notin \tau$, i.e., f is not s.M-precontonuous. Obviously, f is preirresolute.

(v) The concept of ap-s.M-preclosedness is strictly weaker than one of s.M-preclosedness, i.e., the converse of Theorem 3.4 (ii-1) is not true in general. Let (X, τ) and (Y, σ) be the same topological spaces of (i) above and $f : (X, \tau) \to (Y, \sigma)$ the same map of (i) above also. Then, f is not s.M-preclosed (cf. Definition 2.3 (vii)). Indeed, there exists a subset $\{b\} \in PC(X, \tau)$ and $f(\{b\}) = \{a\}$ is not closed in (Y, σ). It is shown that this map f is ap-s.M-preclosed (cf. Definition 3.3). For subsets $U \in PC(X, \tau)$ and $B \in GPO(Y, \sigma)$ such that $f(U) \subseteq B$, we have that $f(U) \subseteq B = pInt(B)$, because $PO(Y, \sigma) = GPO(Y, \sigma)$ holds.

(vi) The converse of Theorem 3.4 (ii-2) is not true in general. Let (X, τ) and (Y, σ) be topological spaces of (v) above. Let $f : (X, \tau) \to (Y, \sigma)$ be a defined map by $f(x) = x$ for every $x \in X$. Then, f is ap-s.M-preclosed, because $GPO(Y, \sigma) = PO(Y, \sigma)$ holds. The map f is not contra-M-preclosed. Indeed, there exists a subset $U := \{c\} \in PC(X, \tau)$ such that $f(U) = \{c\} \notin PO(Y, \sigma)$.

(vii) Every stongly M-preclosed map is M-preclosed (cf. Definition 2.3 (vi), (vii)). The following example shows that the converse of the above implication is not true in general. Let (X, τ) be the same topological space of (iii) above and $f : (X, \tau) \to (X, \tau)$ be the identity map. Then, for a subset $\{b\} \in PC(X, \tau), f(\{b\}) = \{b\}$ is not closed in (X, τ), i.e., f is not s.M-preclosed. The identity map f is M-preclosed.

(viii) There exist a map f and a topological space (X, τ) such that
$f : (X, \tau) \to (X, \tau)$ is ap-s.M-precontinuous; f is not ap-s.M-preclosed;
Indeed, let $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a\}, X\}$. Define $f : (X, \tau) \to (X, \tau)$ by $f(x) := c$ for every $x \in X$. For the topological space (X, τ), we have $PO(X, \tau)=\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$; $GPC(X, \tau) = P(X) \setminus \{\{a\}\}$. We put $GPC_V := \{A \in GPC(X, \tau) | A \subseteq f^{-1}(V)\}$ for each $V \in PO(X, \tau)$. Then, for each $V \in \{\emptyset, \{a\}, \{a, b\}\}$, $GPC_V = \{\emptyset\}$; for each $V \in$

$\{\{a,c\},X\}, GPC_V = GPC(X,\tau)$. For each $V \in PO(X,\tau)$ above and each $A \in GPC_V$, we show that $pCl(A) \subseteq f^{-1}(V)$ holds. Thus, f is ap-s.M-precontinuous. It is shown that f is not ap-s.M-preclosed, because there exists a subset $U := \{b\} \in PC(X,\tau)$ and subset $B := \{b\} \in GPO(X,\tau)$ such $f(U) = \{b\} \subseteq B = \{b\}$ and $f(U) \not\subseteq pInt(B) = \emptyset$.

(ix) The concepts of preirresoluteness and M-preclosedness are independent to each other. Let (X,τ) be a topological space, where $X := \{a,b,c\}$ and $\tau := \{\emptyset, \{a\}, X\}$ (cf. (viii) above). The following map $f : (X,\tau) \to (X,\tau)$ is preirresolute; it is not M-preclosed: $f(a) = f(b) := a$ and $f(c) := c$. Indeed, $PC(X,\tau) = \{\emptyset, \{c\}, \{b\}, \{b,c\}, X\}$ and so it is obvious that f is preirresolute. The map f is not M-preclosed, because there exists a subset $\{b\} \in PC(X,\tau)$ such that $f(\{b\}) = \{a\} \notin PC(X,\tau)$. The following map $g : (X,\tau) \to (X,\tau)$ is not preirresolute; it is M-preclosed: $g(a) = g(b) := b$ and $g(c) := c$.

(x) It is obvious that a map $f : (X,\tau) \to (Y,\sigma)$ is contra-preirresolute if and only if for every $V \in PC(Y,\sigma), f^{-1}(V) \in PO(X,\tau)$.

(xi) For a perfectly contra-preirresolute map $f : (X,\tau) \to (Y,\sigma)$, we have the following decomposition of the map: f is perfectly contra-preirresolute if and only if f is preirresolute and contra-preirresolute.

(xii) The following map $f : (X,\tau) \to (X,\tau)$ is preirresolute; f is not perfectly preirresolute. Let $X := \{a,b\}$ be the Sierpinski space with the topology $\tau := \{\emptyset, \{a\}, X\}$ and $f : (X,\tau) \to (X,\tau)$ be the identity map.

(xiii) The concepts of s.M-precontinuity and s.M-preclosedness are independent to each other.

(xiii-1) Let $(\mathbb{Z},\kappa)$ be the digital line (cf. Section 2). Let $f : (\mathbb{Z},\kappa) \to (\mathbb{Z},\kappa)$ be a map defined by $f(2m) = f(2m+1) := 2m$ for every $m \in \mathbb{Z}$. Then, f is s.M-preclosed. Indeed, for a subset $F \in PC(\mathbb{Z},\kappa), f(F)$ is expressible as $f(F) = \bigcup\{\{2m_i\} | 2m_i \in \mathbb{Z}, i \in \Lambda_F\}$, where $\Lambda_F \subset \mathbb{Z}$ is an index set determined by the set F. By using [41, Lemma 2.6 (ii)] and a fact that any singleton $\{2m\}$ is closed, where $m \in \mathbb{Z}$, it is shown that $Cl(f(F)) = \bigcup\{Cl(\{2m_i\}) | 2m_i \in \mathbb{Z}, i \in \Lambda_F\} = \bigcup\{\{2m_i\} | 2m_i \in \mathbb{Z}, i \in \Lambda_F\} = f(F)$ and so $f(F)$ is closed in $(\mathbb{Z},\kappa)$. The map f is not s. M-precontinuous, because there exists a subset $U := \{-1,0,1\} \in PO(\mathbb{Z},\kappa)$ such that $f^{-1}(U) = \{0,1\} \notin \kappa$.

(xiii-2) The following map $f : (X,\tau) \to (X,\tau)$ is s.M-precontinuous; f is not s.M-preclosed. Let (X,τ) be the Sierpinski topological space of (xii) above and f be a map defined by $f(a) = f(b) := a$. Then, f is s.M-precontinuous, because $f^{-1}(U) \in \tau$ for any $U \in PO(X,\tau) = \tau$. Since a singleton $\{b\}$ is preclosed in (X,τ) and $f(\{b\}) = \{a\}$ is not closed in (X,τ), f is not s.M-preclosed.

(xiv) The concepts of contra M-preclosedness and contra preirresoluteness are independent to each other.

(xiv-1) The following map $f : (X,\tau) \to (X,\tau)$ is contra preirresolute; f is not contra M-preclosed. Let (X,τ) be the Sierpinski topological space of (xii) above and f be a map defined by $f(a) = f(b) := b$. Then, f is contra preirresolute, because $f^{-1}(U) = \emptyset$ or $X \in PC(X,\tau)$ for any $U \in PO(X,\tau) = \tau$. Since a singleton $\{b\}$ is preclosed in (X,τ) and $f(\{b\}) = \{b\} \notin PO(X,\tau) = \tau$, f is not contra M-preclosed.

(xiv-2) The following map $f : (\mathbb{Z},\kappa) \to (\mathbb{Z},\kappa)$ is contra M-preclosed; f is not contra preirresolute. Define f as follows: $f(x) := 4m+1$ if $x \in \{4m, 4m+1, 4m+2\}$; $f(x) := 4m+3$ if $x = 4m+3$, where $m \in \mathbb{Z}$. By the definition of f, for a subset B of $(\mathbb{Z},\kappa), f(B)$ is expressible as $f(B) = \bigcup\{\{2s+1\} | s \in \Lambda_B\}$, where $\Lambda_B \subset \mathbb{Z}$ is an index set determined by the set B. Thus, if $B \in PC(\mathbb{Z},\kappa)$, then $f(B) \in \kappa = PO(\mathbb{Z},\kappa)$, i.e., f is contra M-preclosed. There exists a subset $U := \{4m+1, 4m+3\} \in \kappa = PO(\mathbb{Z},\kappa)$ such that $f^{-1}(U) = \{4m+1, 4m+2, 4m+3, 4m+4\} \notin PC(\mathbb{Z},\kappa)$. Thus, f is not contra preirresolute.

In [3], Abd El-Monsef and one of the present author discussed the preirresolute maps and M-preclosed maps. We strengthen this result slightly by replacing M-preclosed requirement with ap-preclosed.

Theorem 3.8 *If a map $f:(X,\tau)\to(Y,\sigma)$ is preirresolute and ap-s.M-preclosed, then for every subset $A\in GPC(Y,\sigma)$* (resp. $GPO(Y,\sigma)$), $f^{-1}(A)\in GPC(X,\tau)$ (resp. $GPO(X,\tau)$) holds.

Proof. Let $A\in GPC(Y,\sigma)$. Suppose that $f^{-1}(A)\subseteq V$, where $V\in\tau$. Then, $f(X\setminus V)\subseteq Y\setminus A$ and $Y\setminus A\in GPO(Y,\sigma)$ and $X\setminus V\in PC(X,\tau)$. Since f is ap-s.M-preclosed, $f(X\setminus V)\subseteq pInt(Y\setminus A)$ and so $X\setminus V\subseteq f^{-1}(pInt(Y\setminus A))=X\setminus f^{-1}(pCl(A))$, i.e., $f^{-1}(pCl(A))\subseteq V$. Since f is preirresolute, $f^{-1}(pCl(A))\in PC(X,\tau)$. Therefore, we have $pCl(f^{-1}(A))\subseteq pCl(f^{-1}(pCl(A)))=f^{-1}(pCl(A))\subseteq V$. Namely, we have $f^{-1}(A)\in GPC(X,\tau)$. It is obvious that inverse images of gp-open sets are gp-open by definition and the above result. $\square$

The following theorem is replacing the precontinuous requirement in [3] with ap-s.M-precontinuity.

Theorem 3.9 *If a map $f:(X,\tau)\to(Y,\sigma)$ is M-preclosed and ap-s.M-precontinous, then for every subset $F\in GPC(X,\tau), f(F)\in GPC(Y,\sigma)$ holds.*

Proof. Let $F\in GPC(X,\tau)$. Suppose that $f(F)\subseteq U$, where $U\in\sigma$. Then, $F\subseteq f^{-1}(U), F\in GPC(X,\tau)$ and $U\in\sigma\subseteq PO(Y,\sigma)$. Since f is ap-s.M-precontinuous, $pCl(F)\subseteq f^{-1}(U)$ and so $f(pCl(F))\subseteq U$. Since f is M-preclosed, $f(pCl(F))\in PC(Y,\sigma)$. Therefore, we have $pCl(f(F))\subseteq pCl(f(pCl(F)))=f(pCl(F))\subseteq U$ and so $f(F)\in GPC(Y,\sigma)$. $\square$

Remark 3.10 (cf. Definition 3.1) In the concept of ap-s.M-precontinuity (Definition 3.2), it is not meaningful to replace $GPC(X,\tau)$ by $PGC(X,\tau)$ ([29, Definition 2.11 (i)]), because $PGC(X,\tau)=PC(X,\tau)$[29, Theorem 2.27 (ii)] holds for any topological space (X,τ). We recall the following concept ([29, Definition 2.11 (i)]): a subset A of (X,τ) is called a *pre generalized closed set* (shortly, *pg-closed set*), if $pCl(A)\subseteq U$ holds whenever $A\subset U$ and $U\in PO(X,\tau)$. Let $PGC(X,\tau)$ denote the family of all pg-closed sets of (X,τ). By [29, Theorem 2.27 (ii)], it was shown that $PGC(X,\tau)=PC(X,\tau)$ holds in general; it was proved by preparing characterizations of pre-$T_{1/2}$ spaces. In the present remark, we have an alternative proof of $PGC(X,\tau)=PC(X,\tau)$ as follows; we have obviously $PC(X,\tau)\subseteq PGC(X,\tau)$ holds. We prove that $PGC(X,\tau)\subseteq PC(X,\tau)$ holds. Let $A\in PGC(X,\tau)$ and $x\in pCl(A)$. If $\{x\}$ is open, then $\{x\}\in PO(X,\tau)$ and $\{x\}\cap A\neq\emptyset$, i.e., $x\in A$. If $\{x\}$ is not open, then $Int(\{x\})=\emptyset$ and so $Cl(Int(\{x\}))=\emptyset\subseteq\{x\}$, i.e., $\{x\}$ is preclosed in (X,τ) and $X\setminus\{x\}\in PO(X,\tau)$. For this point x, suppose $x\notin A$. Since $A\subseteq X\setminus\{x\}$, we have $pCl(A)\subseteq X\setminus\{x\}$; $x\in X\setminus\{x\}$ and this is a contradiction. Thus, we have $x\in A$. For the both cases, we have that $x\in A$ holds for a point $x\in pCl(A)$, i.e., $pCl(A)=A$ and so $A\in PC(X,\tau)$.

4 Groups of pre.c-homeomorphisms and contra-pre.c-homeomorphisms The purpose of this section is to construct some groups corresponding to a topological space (cf. Definition 4.2, Theorem 4.3, Definition 5.1, Theorem 5.7) and investigate their fundamental properties (cf. Theorems 4.3, 4.4, 5.7). First, we observe that the composition of two contra-preirresolute maps need not be contra-preirresolute. Indeed, let $X=\{a,b\}$ be the Sierpinski space and $\tau=\{\emptyset,\{a\},X\}$ and $\sigma=\{\emptyset,\{b\},X\}$. The identity maps $f:(X,\tau)\to(X,\sigma)$ and $g:(X,\sigma)\to(X,\tau)$ are both contra-preirresolute but their composition $g\circ f:(X,\tau)\to$

(X,τ) is not contra-preirresolute. However, the composition of two preirresolute maps is preirresolute and the composition of a preirresolute map and a contra-preirresolute map is contra-preirresolute. To construct the groups, we need the following theorem.

Theorem 4.1 *Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\eta)$ be two maps; $1_X:(X,\tau)\to(X,\tau)$ be the identity map on (X,τ). Then, for the composite $g\circ f:(X,\tau)\to(Z,\eta)$ we have the following properties*:

(i) (i-1) *If f and g are preirresolute, then $g\circ f$ is preirresolute.*

(i-2) *The identity map $1_X:(X,\tau)\to(X,\tau)$ is preirresolute.*

(i-3) *If f and g are contra-preirresolute, then $g\circ f$ is preirresolute.*

(ii) (ii-1) *If f is contra-preirresolute and g is preirresolute, then $g\circ f$ is contra-preirresolute.*

(ii-2) *If f is preirresolute and g is contra-preirresolute, then $g\circ f$ is contra-preirresolute.*

(iii)(iii-1) *If f is preirresolute and g is perfectly preirresolute, then $g\circ f$ is perfectly preirresolute.*

(iii-2) *If f is perfectly preirresolute and g is preirresolute, then $g\circ f$ is perfectly preirresolute.*

(iii-3) *If f is contra-preirresolute and g is perfectly preirresolute, then $g\circ f$ is perfectly preirresolute.*

(iii-4) *If f is perfectly preirresolute and g is contra-preirresolute, then $g\circ f$ is perfectly preirresolute.*

(iii-5) *If f and g are perfectly preirresolute, then $g\circ f$ is perfectly preirresolute.*

(iv) (iv-1) *If f is M-preclosed and g is ap-s.M-preclosed, then $g\circ f$ is ap-s.M-preclosed.*

(iv-2) *If f is ap-s.M-preclosed and g is M-preopen, preirresolute and ap-s.M-preclosed, then $g\circ f$ is ap-s.M-preclosed.*

(v) *If f is ap-s.M-precontinuous and g is preirresolute, then $g\circ f$ is ap-s.M-precontinuous.*

Proof. **(i)-(iii)** The proofs are obvious. **(iv) (iv-1)** Suppose that $B\in PC(X,\tau)$ and $A\in GPO(Z,\eta)$ for which $(g\circ f)(B)\subseteq A$. Then, it follows from assumptions that $f(B)\in PC(Y,\sigma)$ and $g(f(B))\subseteq pInt(A)$. This implies that $g\circ f$ is ap-s.M-preclosed. **(iv-2)** Suppose that $B\in PC(X,\tau)$ and $A\in GPO(Z,\eta)$ for which $(g\circ f)(B)\subseteq A$. Then, we have that $f(B)\subseteq g^{-1}(A)$ and, by Theorem 3.8 for the map g and the subset $A, g^{-1}(A)\in GPO(X,\tau)$. Since f is ap-s.M-preclosed, $f(B)\subseteq pInt(g^{-1}(A))$ holds. Thus, we have that $(g\circ f)(B)\subseteq g(pInt(g^{-1}(A)))\subseteq pInt(gg^{-1}(A))\subseteq pInt(A)$. This implies that $g\circ f$ is ap-s.M-preclosed. **(v)** Suppose that $F\in GPC(X,\tau)$ and $U\in PO(Z,\eta)$ for which $F\subseteq(g\circ f)^{-1}(U)=f^{-1}(g^{-1}(U))$. Then, $g^{-1}(U)\in PO(Y,\sigma)$ and $pCl(F)\subseteq f^{-1}(g^{-1}(U))$. Hence $g\circ f$ is ap-s.M-precontinuous. □

We shall construct some families of maps from a topological space onto itself; we construct a new group of maps.

Definition 4.2 A map $f:(X,\tau)\to(Y,\sigma)$ is said to be :

(i) a *pre.c-homeomorphism*, if f is bijective preirresolute and f^{-1} is also preirresolute (cf. Definition 2.3 (iii));

(ii) a *contra-pre.c-homeomorphism*, if f is bijective contra-preirresolute and f^{-1} is also contra-preirresolute (cf. Definition 2.3 (iv));

(iii) a *perfectly pre.c-homeorphism*, if f is bijective perfectly preirresolute and f^{-1} is also perfectly preirresolute (cf. Definition 2.3 (v));

(iv) a *pre-homeomorphism*, if f is bijective precontinuous and f^{-1} is also precontinuous (cf. Definition 2.3 (i)).

We use the following notation on familes of two maps above:

$pch(X;\tau) := \{f | f : (X,\tau) \to (X,\tau)$ is a pre.c-homeomorphism$\}$;

$contpch(X;\tau) := \{f | f : (X,\tau) \to (X,\tau)$ is a contra-pre.c-homeomorphism$\}$.

For the homeomorphisms group $h(X;\tau)$, we have $h(X;\tau) \subseteq pch(X;\tau)$ (cf. Theorem 4.3 (iii) below). Indeed, we recall $h(X;\tau) := \{a | a : (X,\tau) \to (X,\tau)$ is a homeomorphism$\}$. For every homeomorphism $f : (X,\tau) \to (Y,\sigma)$, every subset $F \in PC(Y,\sigma)$ and for every subset $V \in PC(X,\tau)$, it is shown that $f^{-1}(F) \in PC(X,\tau)$ and $f(V) \in PC(Y,\sigma)$ hold; f and f^{-1} are preirresolute. Thus, we have that $f \in pch(X;\tau)$ holds for every $f \in h(X;\tau)$ and hence $h(X;\tau) \subseteq pch(X;\tau)$. Let (X,τ) be a topological space, where $X := \{a,b,c\}$ and $\tau := \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, X\}$. Then, it is shown that $h(X;\tau) = \{1_X\} = pch(X;\tau)$ and $contpch(X;\tau) = \{h_b\}$, where $1_X : (X;\tau) \to (X;\tau)$ is the identity map and $h_b : (X;\tau) \to (X;\tau)$ is a map defined by $h_b(b) := \{b\}, h_b(a) := c, h_b(c) := a$. These properties show that $pch(X,\tau)$ and $pch(X;\tau) \cup contpch(X,\tau) = \{1_X, h_b\}$ form groups under the composition of maps.

Theorem 4.3 *Let (X,τ) be a topological space.*

(i) *The union of two families $pch(X;\tau)$ and $contpch(X;\tau)$, i.e., $pch(X;\tau) \cup contpch(X;\tau)$, forms a group under the composition of maps.*

(ii) *The family $pch(X;\tau)$ forms a subgroup of $pch(X;\tau) \cup contpch(X;\tau)$.*

(iii) *Fortunately, the group $h(X;\tau)$ is a subgroup of $pch(X;\tau)$ and $h(X;\tau)$ is also a subgroup of $pch(X;\tau) \cup contpch(X;\tau)$.*

Proof. Set $\mathcal{H}_X := pch(X;\tau) \cup contpch(X;\tau)$ for a topological space (X,τ) throughout this proof. **(i)** A binary operation $\omega : \mathcal{H}_X \times \mathcal{H}_X \to \mathcal{H}_X$ is defined by $\omega(a,b) := b \circ a$, where $a, b \in \mathcal{H}_X$ and $b \circ a$ denotes the composite of two maps a and b defined by $(b \circ a)(x) = b(a(x))$ for any $x \in X$. By Theorem 4.1(i) and (ii), it is shown that $\omega(a,b) = b \circ a \in \mathcal{H}_X$ for any $a, b \in \mathcal{H}_X$. We observe that the axioms of group are satisfied. The identity map $1_X : (X,\tau) \to (X,\tau)$ is the identity element of the group $\mathcal{H}_X$ (cf. Theorem 4.1(i)(i-2)). **(ii)** Let $a \in pch(X;\tau)$ and $b \in pch(X;\tau)$. Then, it is shown that $pch(X;\tau) \neq \emptyset$ because $1_X \in pch(X;\tau)$, and $\omega(a, b^{-1}) = b^{-1} \circ a \in pch(X;\tau)$ by Theorem 4.1(i) and Definition 4.2(i), where $\omega : \mathcal{H}_X \times \mathcal{H}_X \to \mathcal{H}_X$. Thus, $pch(X;\tau)$ is a subgroup of $\mathcal{H}_X = pch(X;\tau) \cup contpch(X;\tau)$ under the binary operation $\omega_X := \omega|(pch(X;\tau) \times pch(X;\tau))$. **(iii)** First recall that $h(X,\tau) \subseteq pch(X;\tau)$ and $h(X;\tau) \neq \emptyset$. Moreover, $\omega_X(a, b^{-1}) = b^{-1} \circ a \in h(X;\tau)$ for any $a, b \in h(X;\tau)$ (cf. the proof of (ii) above). □

Theorem 4.4 *Let $(X,\tau), (Y,\sigma)$ and (Z,η) be topological spaces.*

(i) *If $f : (X,\tau) \to (Y,\sigma)$ is a pre.c-homeomorphism* (resp. *contra-pre.c-homeomorphism*), *then the map f induces an isomorphism $f_* : pch(X;\tau) \cup contpch(X;\tau) \to pch(Y;\sigma) \cup contpch(Y;\sigma)$, where f_* is defined by $f_*(a) := f \circ a \circ f^{-1}$ for any $a \in pch(X;\tau) \cup contpch(X;\tau)$. Moreover,*

($*$-1) *$(g \circ f)_* = g_* \circ f_* : pch(X;\tau) \cup contpch(X;\tau) \to pch(Z;\eta) \cup contpch(Z;\eta)$ holds, where $g : (Y,\sigma) \to (Z,\eta)$ is a pre.c-homeomorphism* (resp. *contra-pre.c-homeomorphism*),

($*$-2) *$(1_X)_* = 1 : pch(X;\tau) \cup contpch(X;\tau) \to pch(X;\tau) \cup contpch(X;\tau)$ is the identity isomorphism,*

($*$-3) *$f_*(pch(X;\tau)) = pch(Y;\sigma)$, $f_*(h(X;\tau)) \subseteq pch(Y;\sigma)$ and $f_*(contpch(X;\tau)) = contpch(Y;\sigma)$ hold.*

(ii) *Especially, if a map $f : (X,\tau) \to (Y,\sigma)$ is a homeomorphism and $g : (Y,\sigma) \to (Z,\eta)$ is a homeomorphism, then the induced maps $f_* : pch(X;\tau) \cup contpch(X;\tau) \to pch(Y;\sigma) \cup contpch(Y,\sigma)$ and $g_* : pch(Y;\sigma) \cup contpch(Y,\sigma) \to pch(Z;\eta) \cup contpch(Z;\eta)$ are isomorphisms* (cf. (i) *above*). *Moreover, they have the same property of* ($*$-1),($*$-2) *and* ($*$-3) *in* (i) *above. We note that, in* ($*$-3), *$f_*(h(X;\tau)) = h(Y;\sigma)$ holds precisely.*

Proof. Let $\mathcal{H}_X := pch(X;\tau) \cup contpch(X;\tau)$ for a topological space (X,τ) throughout this proof.

(i) By using Theorem 4.1 (i-1), (i-2), (ii) (resp. Theorem 4.1 (i-2), (i-3), (ii)) , it is shown that the map f_* is well defined and $f_* : \mathcal{H}_X \to \mathcal{H}_Y$ is an isomorphism of groups.

Proofs of $(*\text{-}1)$ *and* $(*\text{-}2)$: For an element $a \in \mathcal{H}_X$, $(g\circ f)_*(a) = (g\circ f)\circ a \circ (g\circ f)^{-1} = g\circ(f\circ a\circ f^{-1})\circ g^{-1} = g_*(f_*(a))$ and $f_*(1_X) = f\circ 1_X \circ f^{-1} = 1_X$ hold.

Proof of $(*\text{-}3)$: Let $a \in phc(X;\tau), b \in h(X;\tau)$ and $c \in contpch(X;\tau)$. Then, $f_*(a) = f\circ a\circ f^{-1} \in pch(Y;\sigma)$ (cf. Theorem 4.1 (i-1) (resp. Theorem 4.1 (ii-1), (i-3))) and so $f_*(pch(X;\tau)) \subseteq pch(Y;\sigma)$. Conversely, for each element $h \in pch(Y;\sigma)$ we take an element $f^{-1}\circ h\circ f \in pch(X;\tau)$ (cf. Theorem 4.1 (i-1) (resp. Theorem 4.1 (ii-1), (i-3))). Thus we have that $h = f_*(f^{-1}\circ h\circ f) \in f_*(pch(X;\tau))$ and so $pch(Y;\sigma) \subseteq f_*(pch(X;\tau))$. Namely, we have that $pch(Y;\sigma) = f_*(pch(X;\tau))$. For the element $b \in h(X;\tau), f_*(b) = f\circ b\circ f^{-1} \in pch(Y;\sigma)$ (cf. Theorem 4.1 (i-1) (resp. Theorem 4.1 (ii-1), (i-3)) and Theorem 4.3 (iii) and so $f_*(h(X;\tau)) \subseteq pch(Y;\sigma)$. For the element $c \in contpch(X;\tau), f_*(c) = f\circ c\circ f^{-1} \in contpch(Y;\sigma)$ (cf. Theorem 4.1 (ii-2), (ii-1) (resp. Theorem 4.1 (i-3), (ii-2))) and so $f_*(contpch(X;\tau)) \subseteq contpch(Y;\sigma)$. Conversely, for any $h \in contpch(Y;\sigma)$ we take an element $f^{-1}\circ h\circ f \in contpch(X,\tau)$ (cf. Theorem 4.1 (ii-2), (ii-1) (resp. Theorem 4.1 (i-3), (ii-2))); $h = f_*(f^{-1}\circ h\circ f) \in f_*(contpch(X;\tau))$. Namely, we have that $contpch(Y;\sigma) = f_*(contpch(X;\tau))$. **(ii)** The proof is obtained by an argument similar to that in the proof of (i) (cf. Theorem 4.1 (i), (ii) and Theorem 4.3 (iii)). □

5 Subgroups of $pch(X;\tau)$

In this section, we investigate some structure of $pch(H;\tau|H)$ for a subspace $(H,\tau|H)$ of (X,τ) using two subgroups of $pch(X;\tau)$, say $pch(X,X\setminus H;\tau)$ and $pch_0(X,X\setminus H;\tau)$ below (cf. Theorem 5.7).

Definition 5.1 For a topological space (X,τ) and subset H of X, we define the following families of maps:

(i) $pch(X,X\setminus H;\tau) := \{a | a \in pch(X;\tau) \text{ and } a(X\setminus H) = X\setminus H\}$;

(ii) $pch_0(X,X\setminus H;\tau) := \{a | a \in pch(X,X\setminus H;\tau) \text{ and } a(x) = x \text{ for every } x \in X\setminus H\}$.

Theorem 5.2 *Let H be a subset of a topological space (X,τ).*

(i) *The family $pch(X,X\setminus H;\tau)$ forms a subgroup of $pch(X,\tau)$ (cf. Definition 5.1, Corollary 4.3) and $pch(X,X\setminus H;\tau) = pch(X,H;\tau)$ holds.* (ii) *The family $pch_0(X,X\setminus H;\tau)$ forms a subgroup of $pch(X,X\setminus H;\tau)$ (cf. Definition 5.1) and hence $pch_0(X,X\setminus H;\tau)$ forms a subgroup of $pch(X;\tau)$.*

Proof. **(i)** It is shown obviously that $pch(X,X\setminus H;\tau)$ is a non-empty subset of $pch(X;\tau)$, because $1_X \in pch(X,X\setminus H;\tau)$ (cf. Definition 5.1 (i)). Moreover, we have that $\omega_X(a,b^{-1}) = b^{-1}\circ a \in pch(X,X\setminus H;\tau)$ for any elements $a,b \in pch(X,X\setminus H;\tau)$, where $\omega_X := \omega|(pch(X,X\setminus H;\tau)\times pch(X,X\setminus H;\tau))$ (cf. ω is the binary operation of the group $pch(X;\tau)$ (Corollary 4.3 (ii))). Thus, $pch(X,X\setminus H;\tau)$ is a subgroup of $pch(X;\tau)$. Evidently, the identity map 1_X is the identity element of $pch(X,X\setminus H;\tau)$. **(ii)** It is shown that $pch_0(X,X\setminus H;\tau)$ is a non-empty subset of $pch(X,X\setminus H;\tau)$, because $1_X \in pch_0(X,X\setminus H;\tau)$ (cf. Definition 5.1 (ii)). We have that $\omega_{X,0}(a,b^{-1}) = b^{-1}\circ a \in pch_0(X,X\setminus H;\tau)$ for any elements $a,b \in pch_0(X,X\setminus H;\tau)$, where $\omega_{X,0} := \omega_X|(pch_0(X,X\setminus H;\tau)\times pch_0(X,X\setminus H;\tau))$ (cf. ω_X is the binary operation of the group $pch(X,X\setminus H;\tau)$ ((i) above)). Thus, $pch_0(X,X\setminus H;\tau)$ is a subgroup of $pch(X,X\setminus H;\tau)$ and the identity map 1_X is the identity element of $pch_0(X,X\setminus H;\tau)$. By using (i) above, $pch_0(X,X\setminus H;\tau)$ is a subgroup of $pch(X;\tau)$. □

Let H and K be subsets of X and Y, respectively. For a map $f : X \to Y$ satisfying a property $K = f(H)$, we define the following map $r_{H,K}(f) : H \to K$ by $r_{H,K}(f)(x) = f(x)$

for every $x \in H$. Then, we have that $j_K \circ r_{H,K}(f) = f|H : H \to Y$, where $j_K : K \to Y$ be an inclusion defined by $j_K(y) = y$ for every $y \in K$ and $f|H : H \to Y$ is a restriction of f to H defined by $(f|H)(x) = f(x)$ for every $x \in H$. Especially, we consider the following case that $X = Y, H = K \subseteq X$ and $a(H) = H, b(H) = H$ for any maps $a, b : X \to X$. Then, $r_{H,H}(b \circ a) = r_{H,H}(b) \circ r_{H,H}(a)$ holds. Moreover, if a map $a : X \to X$ is a bijection such that $a(H) = H$, then $r_{H,H} : H \to H$ is bijective and $r_{H,H}(a^{-1}) = (r_{H,H}(a))^{-1}$.

We recall well known properties on preopen sets of subspace topological spaces etc, i.e., Theorem 5.3 needed later. For a subset H of (X, τ) and a subset $U \subseteq H$, $Int_H(U)$ (resp. $Cl_H(U)$) is the interior (resp. closure) of the set U in a subspace $(H, \tau|H)$ and $pInt(U)$ is the preinterior of U in (X, τ). It is well known that for any subset A of $(X, \tau), pInt(A) = A \cap Int(Cl(A))$ holds.

Theorem 5.3 *For a topological space (X, τ) and subsets H and U of X and $A \subseteq H, V \subseteq H$, $B \subseteq H$, the following properties hold.*

(i) ([1], eg., [4, Section 2 (1),(10)]) *Arbitrary union of preopen sets of (X, τ) is preopen in (X, τ); the intersection of an α-open set of (X, τ) and a preopen set of (X, τ) is preopen in (X, τ).*

(ii) (ii-1) ([37, Lemma 2.3], eg., [16, Lemma 4.1]) *If A is a preopen in (X, τ) and $A \subset H$, then A is preopen in a subspace $(H, \tau|H)$.*

(ii-2) ([16, Lemma 4.2]) *If $H \subseteq X$ is α-open in (X, τ) and a subset $U \subseteq X$ is preopen in (X, τ), then $H \cap U$ is preopen in (X, τ) and hence $H \cap U$ is preopen in a subspace $(H, \tau|H)$.*

(ii-3) ([34, Lemma 2.10]) *If $H \subseteq X$ is semi-open in (X, τ) and a subset $U \subseteq X$ is preopen in (X, τ), then $H \cap U$ is preopen in $(H, \tau|H)$.*

(iii) *Let $V \subseteq H \subseteq X$.*

(iii-1) *If H is preopen in (X, τ), then $Int_H(V) \subseteq pInt(V)$ holds.*

(iii-2) (eg., [4, Section 2, (13)]) *If H is preopen in (X, τ) and V is preopen in a subspace $(H, \tau|H)$, then V is preopen in (X, τ).*

(iv) *Let $B \subseteq H \subseteq X$. If H is preclosed in (X, τ) and B is preclosed in a subspace $(H, \tau|H)$, then B is preclosed in (X, τ).*

(v) ((ii-1), (ii-2) above; cf. [14, Corollary 4]) *Let $V \subseteq H \subseteq X$ and assume that $H \in PO(X, \tau)$ holds. Then, the following properties (1) and (2) are equivalente:*

(1) *$V \in PO(X, \tau)$ holds;*

(2) *$V \in PO(H, \tau|H)$ holds,*

(*the implication of* (1)$\Rightarrow$(2) *is true without the assumption of preopennness of H*).

(vi) (vi-1) *Assume that H is a semi-open subset of (X, τ). Then,*
$PO(X, \tau)|H \subseteq PO(H, \tau|H)$ holds, where $PO(X, \tau)|H := \{W \cap H | W \in PO(X, \tau)\}$.

(vi-2) *Assume that H is a preopen subset of (X, τ). Then,*
$PO(H, \tau|H) \subseteq PO(X, \tau)|H$ holds.

(vi-3) *Under the assumption that H is an α-open set of (X, τ),*
$PO(H, \tau|H) = PO(X, \tau)|H$ holds.

Proof. They are well known; but we give the proofs of (iii), (iv) and (vi). **(iii) (iii-1)** Let $x \in Int_H(V)$. There exists a subset $W(x) \in \tau$ such that $W(x) \cap H \subseteq V$. By (i), $W(x) \cap H \in PO(X, \tau)$. Thus, this shows that $x \in pInt(V)$ and so $Int_H(V) \subset pInt(V)$ holds. **(iii-2)** Since $V \in PO(H, \tau|H), V \subseteq Int_H(Cl_H(V))$ holds in $(H, \tau|H)$. Using (ii-2) and a formula of the preinterior, we have $V \subseteq pInt(Cl_H(V)) = Cl_H(V) \cap Int(Cl(Cl_H(V))) \subseteq Int(Cl(Cl(V) \cap H)) \subseteq Int(Cl(V))$ and hence $V \subseteq Int(Cl(V))$, i.e., $V \in PO(X, \tau)$. **(iv)** It follows from definition that $Cl_H(Int_H(B)) \subseteq B$ holds. Then, it is shown that $Cl_H(Int_H(B)) \supseteq Cl(Int(B))$. Indeed, because $Cl(A) \supseteq pCl(A)$ and $Int_H(E) \supseteq Int(E)$ hold for any subset $A \subseteq X$ and $E \subseteq H$, we have $Cl_H(Int_H(B)) = H \cap Cl(Int_H(B)) \supseteq$

$H \cap Cl(Int(B)) \supseteq Cl(Int(H)) \cap Cl(Int(B)) \supseteq Cl(Int(H) \cap Int(B)) = Cl(Int(H \cap B)) = Cl(Int(B))$. Therefore, we have that $Cl(Int(B)) \subseteq B$ and so B is preclosed in (X, τ). **(v) (1)⇒(2)** Assume that $V \in PO(X, \tau)$ with $V \subseteq H$. By (ii-1) above, it is obtained that $V \in PO(H, \tau|H)$. **(2)⇒(1)** Since $V \in PO(H, \tau|H)$ and $H \in PO(X, \tau)$, we have that $V \in PO(X, \tau)$ holds (cf. (iii-2) above). **(vi) (vi-1)** Let $V \in PO(X, \tau)|H$. For some set $W \in PO(X, \tau), V = W \cap H$ and so we have $W \cap H \in PO(H, \tau|H)$ (cf. (ii-3) above). Hence, $V \in PO(H, \tau|H)$ holds. **(vi-2)** Let $V \in PO(H, \tau|H)$. Since $H \in PO(X, \tau)$, we have $V \in PO(X, \tau)$ by (iii-2) above or (v-2); thus $V = V \cap H \in PO(X, \tau)|H$. **(vi-3)** The property (vi-3) follows from (vi-1) and (vi-2), because H is α-open if and only if H is preopen and semi-open, in general. □

Proposition 5.4 (i) *If $f : (X, \tau) \to (Y, \sigma)$ is preirresolute and a subset H is semi-open in (X, τ), then $f|H : (H, \tau|H) \to (Y, \sigma)$ is preirresolute.*

(ii) *Let* (1) *and* (2) *be properties of two maps $k : (X, \tau) \to (K, \sigma|K)$, where $K \subseteq Y$, and $j_K \circ k : (X, \tau) \to (Y, \sigma)$ as follows:*

(1) *$k : (X, \tau) \to (K, \sigma|K)$ is preirresolute;*

(2) *$j_K \circ k : (X, \tau) \to (Y, \sigma)$ is preirresolute.*

Then, the following implications and an equivalence hold:

(ii-1) *Under the assumption that K is semi-open in (Y, σ),* (1)⇒(2).

(ii-2) *Conversely, under the assumption that K is preopen in (Y, σ),* (2)⇒(1).

(ii-3) *Under the assumption that K is α-open,* (1)⇔(2).

(iii) *If $f : (X, \tau) \to (Y, \sigma)$ is preirresolute and a subset H is semi-open in (X, τ) and $f(H)$ is preopen in (Y, σ), then $r_{H,f(H)}(f) : (H, \tau|H) \to (f(H), \sigma|f(H))$ is preirresolute* (cf. *the contents between Definition 5.1 and Theorem 5.3 for the notation of $r_{H,f(H)}(f)$).*

Proof. **(i)** Let $V \in PO(Y, \sigma)$. Then, we have $(f|H)^{-1}(V) = f^{-1}(V) \cap H$ and $(f|H)^{-1}(V) \in PO(H, \tau|H)$ (cf. Theorem 5.3 (ii-3)). **(ii) (ii-1) (1)⇒(2)** Let $V \in PO(Y, \sigma)$. Since $(j_K \circ k)^{-1}(V) = k^{-1}(V \cap K)$ and $V \cap K \in PO(K, \sigma|K)$ (cf. Theorem 5.3 (ii-3)), we have that $(j_K \circ k)^{-1}(V) \in PO(X, \tau)$ and hence $j_K \circ k$ is preirresolute. **(ii-2) (2)⇒(1)** Let $U \in PO(K, \sigma|K)$. Since $U \in PO(Y, \sigma)$ (cf. Theorem 5.3 (iii-2)), we have $k^{-1}(U) = (j_K \circ k)^{-1}(U) \in PO(X, \tau)$. Thus, k is preirresolute. **(ii-3)** In general, it is well known that a subset A of (Y, σ) is α-open in (Y, σ) if and only if A is semi-open and preopen in (Y, σ). Thus, (ii-3) is obtained by (ii-1) and (ii-2). **(iii)** By (i), $f|H : (H, \tau|H) \to (Y, \sigma)$ is preirresolute. The map $r_{H,f(H)}(f)$ is preirresolute, because $f|H = j_{f(H)} \circ r_{H,f(H)}(f)$ holds. □

Definition 5.5 For an α-open subset H of (X, τ), the following maps $(r_H)_* : pch(X, X \setminus H; \tau) \to pch(H; \tau|H)$ and $(r_H)_{*,0} : pch_0(X, X \setminus H; \tau) \to pch(H; \tau|H)$ are well defined as follows (cf. Proposition 5.4(iii)), respectively:

$(r_H)_*(f) := r_{H,H}(f)$ for every $f \in pch(X, X \setminus H; \tau)$;

$(r_H)_{*,0}(g) := r_{H,H}(g)$ for every $g \in pch_0(X, X \setminus H; \tau)$.

Indeed, in Proposition 5.4 (iii), we assume that $X = Y, \tau = \sigma$ and $H = f(H)$. Then, under the assumption that H is semi-open and preopen in (X, τ), it is obtained that $r_{H,H}(f) \in pch(H; \tau|H)$ holds for any $f \in pch(X, X \setminus H; \tau)$ (resp. $f \in pch_0(X, X \setminus H; \tau)$).

We need the following lemma and then we prove that $(r_H)_*$ and $(r_H)_{*,0}$ are onto homomorphisms under the assumption that H is α-open and α-closed in (X, τ) (cf. Theorem 5.7 (i-2)).

Let $X = U_1 \cup U_2$ for some subsets U_1 and U_2 and $f_1 : (U_1, \tau|U_1) \to (Y, \sigma)$ and $f_2 : (U_2, \tau|U_2) \to (Y, \sigma)$ be two maps satisfying a property $f_1(x) = f_2(x)$ for every point $x \in U_1 \cap U_2$. Then, a map $f_1 \nabla f_2$ is well defined as follows: $(f_1 \nabla f_2)(x) = f_1(x)$ for every $x \in U_1$ and $(f_1 \nabla f_2)(x) = f_2(x)$ for every $x \in U_2$; we call this map the *combination* of f_1 and f_2.

Lemma 5.6 (*A pasting lemma for preirresolute maps*) *For a topological space (X, τ), we assume that $X = U_1 \cup U_2$, where U_1 and U_2 are subsets of X and $f_1 : (U_1, \tau|U_1) \to (Y, \sigma)$ and $f_2 : (U_2, \tau|U_2) \to (Y, \sigma)$ are two maps satisfying a property $f_1(x) = f_2(x)$ for every point $x \in U_1 \cap U_2$.*

(i) *If $U_i \in PO(X, \tau)$ for each $i \in \{1, 2\}$ and f_1 and f_2 are preirresolute, then its combination $f_1 \nabla f_2 : (X, \tau) \to (Y, \sigma)$ is preirresolute.*

(ii) *Assume that $PO(X, \tau) \subseteq \tau^\alpha$ holds for a topological space (X, τ). If $U_i \in PC(X, \tau)$ for each $i \in \{1, 2\}$ and f_1 and f_2 are preirresolute, then its combination $f_1 \nabla f_2 : (X, \tau) \to (Y, \sigma)$ is preirresolute.*

Proof. **(i)** Let $V \in PO(Y, \sigma)$. By Theorem 5.3 (i) and (iii-2), it is proved that $(f_1 \nabla f_2)^{-1}(V) \in PO(X, \tau)$, because $f_i^{-1}(V) \in PO(U_i, \tau|U_i), f_i^{-1}(V) \in PO(X, \tau)$ for each $i \in \{1, 2\}$ and $(f_1 \nabla f_2)^{-1}(V) = f_1^{-1}(V) \cup f_2^{-1}(V)$ hold. **(ii)** Let $V \in PC(Y, \sigma)$. For each $i \in \{1, 2\}$, it follows from assumptions that $f_i^{-1}(V) \in PC(U_i, \tau|U_i)$ and so $f_i^{-1}(V) \in PC(X, \tau)$ (cf. Theorem 5.3 (iv)). Using the assumption of (ii), we may consider that $f_1^{-1}(V) \in PC(X, \tau)$ and $f_2^{-1}(V)$ is α-closed. By Theorem 5.3 (i), $f_1^{-1}(V) \cup f_2^{-1}(V)$ is preclosed in (X, τ), i.e., $(f_1 \nabla f_2)^{-1}(V)$ is preclosed in (X, τ). Therefore, $f_1 \nabla f_2 : (X, \tau) \to (Y, \sigma)$ is preirresolute. $\square$

Theorem 5.7 *Let H be a subset of a topological space (X, τ).*

(i) (i-1) *If H is α-open in (X, τ), then the maps $(r_H)_* : pch(X, X \setminus H; \tau) \to pch(H; \tau|H)$ and $(r_H)_{*,0} : pch_0(X, X \setminus H; \tau) \to pch(H; \tau|H)$ are homomorphisms of groups* (cf. Definition 5.5). *Moreover, $(r_H)_*|pch_0(X, X \setminus H; \tau) = (r_H)_{*,0}$ holds.*

(i-2) *If H is α-open and α-closed in (X, τ), then the maps $(r_H)_* : pch(X, X \setminus H; \tau) \to pch(H; \tau|H)$ and $(r_H)_{*,0} : pch_0(X, X \setminus H; \tau) \to pch(H; \tau|H)$ are onto homomorphisms of groups.*

(ii) *For an α-open subset H of (X, τ), we have the following isomorphisms of groups:*

(ii-1) *$pch(X, X \setminus H; \tau)/Ker(r_H)_* \cong Im(r_H)_*$;*

(ii-2) *$pch_0(X, X \setminus H; \tau) \cong Im(r_H)_{*,0}$,*

where $Ker(r_H)_* := \{a \in pch(X, X \setminus H; \tau) | (r_H)_*(a) = 1_X\}$ is a normal subgroup of $pch(X, X \setminus H; \tau)$; $Im(r_H)_* := \{(r_H)_*(a) | a \in pch(X, X \setminus H; \tau)\}$ and $Im(r_H)_{*,0} := \{(r_H)_{*,0}(b) | b \in pch_0(X, X \setminus H; \tau)\}$ are subgroups of $pch(H; \tau)$.

(iii) *For an α-open and α-closed subset H of (X, τ), we have the following isomorphisms of groups:*

(iii-1) *$pch(H; \tau|H) \cong pch(X, X \setminus H; \tau)/Ker(r_H)_*$;*

(iii-2) *$pch(H; \tau|H) \cong pch_0(X, X \setminus H; \tau)$.*

Proof. **(i) (i-1)** Let $a, b \in pch(X, X \setminus H; \tau)$. Since H is α-open in (X, τ), the maps $(r_H)_*$ and $(r_H)_{*,0}$ are well defined (cf. Definition 5.5). Then, we have that $(r_H)_*(\omega_X(a, b)) = (r_H)_*(b \circ a) = r_{H,H}(b \circ a) = (r_{H,H}(b)) \circ (r_{H,H}(a)) = \omega_H((r_H)_*(a), (r_H)_*(b))$ hold, where ω_H is the binary operation of $pch(H; \tau|H)$ (cf. Theorem 4.3 (ii)). Thus, $(r_H)_*$ is a homomorphism of groups. For the map $(r_H)_{*,0} : pch_0(X, X \setminus H; \tau) \to pch(H; \tau|H)$, we have that $(r_H)_{*,0}(\omega_{X,0}(a, b)) = (r_H)_{*,0}(b \circ a) = r_{H,H}(b \circ a) = (r_{H,H}(b)) \circ (r_{H,H}(a)) = \omega_H((r_H)_*(a), (r_H)_*(b))$ hold, where ω_H is the binary operation of $pch(H, \tau|H)$ (cf. Theorem4.3 (ii)). Thus, $(r_H)_{*,0}$ is also a homomorphism of groups. It is obviously shown that $(r_H)_*|pch_0(X, X \setminus H; \tau) = (r_H)_{*,0}$ holds (cf. Definition 5.1, Definition 5.5). **(i-2)** We first recall that, in general, $\tau^\alpha = PO(X, \tau) \cap SO(X, \tau)$ holds. In order to prove that $(r_H)_*$ and $(r_H)_{*,0}$ are onto, let $h \in pch(H; \tau|H)$. Let $j_H : (H; \tau|H) \to (X; \tau)$ and $j_{X \setminus H} : (X \setminus H, \tau|(X \setminus H)) \to (X, \tau)$ be the inclusions defined by $j_H(x) = x$ for every $x \in H$ and $j_{X \setminus H}(x) = x$ for every $x \in X \setminus H$. We consider the combination $h_1 := (j_H \circ h) \nabla (j_{X \setminus H} \circ 1_{X \setminus H}) : (X, \tau) \to (X, \tau)$. By Proposition 5.4 (ii-1), under the assumption of semi-openness on H, it is shown that the two maps $j_H \circ h : (H, \tau|H) \to (X, \tau)$

and $j_H \circ h^{-1} : (H, \tau|H) \to (X, \tau)$ are preirresolute; moreover, under the assumption of semi-openness on $X \setminus H$, $j_{X\setminus H} \circ 1_{X\setminus H} : (X \setminus H, \tau|(X \setminus H)) \to (X, \tau)$ is preirresolute. Using Lemma 5.6 (i) for a preopen cover $\{H, X \setminus H\}$ of X, the combination above $h_1 : (X, \tau) \to (X, \tau)$ is preirresolute. Since h_1 is bijective, its inverse map $h_1^{-1} = (j_H \circ h^{-1})\nabla(j_{X\setminus H} \circ 1_{X\setminus H})$ is also preirresolute. Thus, under the assumption that both H and $X \setminus H$ are semi-open and preopen in (X, τ), we have that $h_1 \in pch(X, \tau)$. Since $h_1(x) = x$ for every point $x \in X \setminus H$, we conclude that $h_1 \in pch_0(X, X \setminus H; \tau)$ and so $h_1 \in pch(X, X \setminus H; \tau)$. Moreover, $(r_H)_{*,0}(h_1) = (r_H)_*(h_1) = r_{H,H}(h_1) = h$; hence $(r_H)_{*,0}$ and $(r_H)_*$ are onto, under the assumption that H is α-open and α-closed in (X, τ). **(ii)** By (i-1) above and the first isomorphism theorem of group theory, it is shown that there are group isomorphisms below, under the assumption that H is α-open in (X, τ):

$(*)$ $pch(X, X \setminus H; \tau)/Ker(r_H)_* \cong Im(r_H)_*$ and

$(**)$ $pch_0(X, X \setminus H; \tau)/Ker(r_H)_{*,0} \cong Im(r_H)_{*,0}$,

where $ker(r_H)_{*,0} := \{a \in pch_0(X, X \setminus H; \tau) | (r_H)_{*,0}(a) = 1_X\}$. Moreover, under the assumption of α-openness on H, it is shown that $ker(r_H)_{*,0} = \{1_H\}$. Therefore, using $(**)$ above, we have the isomorphism (ii-2). **(iii)** By (i-2) above, it is shown that $(r_H)_*$ and $(r_H)_{*,0}$ are onto homomorphisms of groups, under the assumption that H is α-open and α-closed in (X, τ). Therefore, by (ii) above, the isomorphisms (iii-1) and (iii-2) are obtained. □

Example 5.8 **(i)** (cf. Proposition 6.1 below) The following topological space (X, τ) and a subset H show that the α-closedness of H in Theorem 5.7 (iii-2) can not be removed. Let $X := \mathbb{Z}$ and $\tau := \kappa$ (the Khalimsky topology). Namely, let (X, τ) be the digital line $(\mathbb{Z}, \kappa)$ (cf. the end of Section 2) and $H := U(0) = \{-1, 0, +1\}$ be the smallest open set containing $0 \in \mathbb{Z}$. Since $H \in \kappa$ and $\kappa = PO(\mathbb{Z}, \kappa) = \kappa^\alpha$ (e.g., [17, Theorem 2.1 (a), (b)]), the subset H is α-open in $(\mathbb{Z}, \kappa)$. In Section 6 below, it is shown that $pch_0(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa) \ncong pch(H; \kappa|H)$ and H is not α-closed in $(\mathbb{Z}, \kappa)$ (cf. Proposition 6.1 (ii), (iii)).

(ii) Under the assumption that H is α-open and α-closed in (X, τ), Theorem 5.7 (iii) is proved. Let (X, τ) be a three point topological space, where $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a\}, \{b, c\}, X\}$, and $(H, \tau|H)$ a subspace of (X, τ), where $H := \{a\}$. Then, $PO(X, \tau) = P(X)$(=the power set of X) and $\tau = \tau^\alpha$ and so H is α-open and also α-closed in (X, τ). We apply Theorem 5.7 (iv) to the present case; we have the group isomorphisms. Directly, we obtain the following date on groups: $pch(X; \tau) \cong S_3$ (=the symmetric group of degree 3), $pch(X, X\setminus H; \tau) = \{1_X, h_a\}$, $Ker(r_H)_* = \{1_X, h_a\}$, $pch(H; \tau|H) = \{1_H\}$ and so $pch_0(X, X\setminus H; \tau) = \{1_X\}$, where $h_a : (X, \tau) \to (X, \tau)$ is a map defined by $h_a(a) = a, h_a(b) = c$ and $h_a(c) = b$. therefore, for this example, we have $pch(H; \tau|H) \cong pch(X, X \setminus H; \tau)/Ker(r_H)_*$; $pch(H; \tau|H) \cong pch_0(X, X \setminus H; \tau)$. Moreover, we have $h(X; \tau) = \{1_X, h_a\}$.

(iii) Even if a subset H of a topological space (X, τ) is not α-closed and it is α-open, we have possibilities to investigate isomorphisms of groups corresponding to a subspace $(H, \tau|H)$ and $(r_H)^*$ using Theorem 5.7 (ii). For example, let (X, τ) be a three point topological space, where $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a, b\}, X\}$, and $(H, \tau|H)$ a subspace of (X, τ), where $H := \{a, b\}$. Then, $PO(X, \tau) = P(X) \setminus \{\{c\}\}$ and $\tau^\alpha = \tau$. The subset H is α-open and it is not α-closed in (X, τ). By Theorem 5.7 (i)(i-1), the maps $(r_H)_* : pch(X, X \setminus H; \kappa) \to pch(H; \tau|H)$ and $(r_H)_{*,0} : pch_0(X, X \setminus H; \kappa) \to pch(H; \tau|H)$ are homomorphisms of groups and by Theorem 5.7 (ii) two isomorphisms of groups are obtained:

$(*$-1) $pch(X, X \setminus H; \tau)/Ker(r_H)_* \cong Im(r_H)_*$; $(*$-2) $pch_0(X, X \setminus H; \tau) \cong Im(r_H)_{*,0}$.

We need notation on maps as follows: let $h_c : (X, \tau) \to (X, \tau)$ and $t_{a,b} : (H, \tau|H) \to (H, \tau|H)$ are maps defined by $h_c(a) = b, h_c(b) = a, h_c(c) = c$ and $t_{a,b}(a) = b, t_{a,b}(b) = a$, respectively. Then, it is shown directly that $pch(X, X \setminus H; \tau) = \{1_X, h_c\} \cong \mathbb{Z}_2, (h_c)^2 = 1_X$, and $Ker(r_H)_*$=$\{a \in pch(X, X \setminus H; \tau) \mid (r_H)_*(a)$=$1_H\} = \{a \in \{1_X, h_c\}|(r_H)_*(a) = 1_H\}$ =$\{1_X\}$, because $(r_H)_*(1_X) = 1_H$ and $(r_H)_*(h_c) = t_{a,b} \neq 1_H$. By using (∗-1) above, $Im(r_H)_* \cong pch(X, X \setminus H{:}\tau)$=$\{1_X, h_c\}$ and so $Im(r_H)_* = \{1_H, r_{H,H}(h_c)\} = \{1_H, t_{a,b}\}$. Since $Im(r_H)_* \subseteq pch(H; \tau|H) \subseteq \{1_H, t_{a,b}\}$, we have that $Im(r_H)_*$=$pch(H; \tau|H)$ =$\{1_H, t_{a,b}\}$ and hence $(r_H)_*$ is onto. Namely, we have an isomorphism $(r_H)_* : pch(X, X \setminus H; \tau) \cong pch(H; \tau|H) \cong \mathbb{Z}_2$. Moreover, it is shown that $pch_0(X, X \setminus H; \tau) = \{a \in pch(X, X \setminus H; \tau)|a(x) = x$ for any $x \in \{c\}\}$ =$\{1_X, h_c\} = pch(X, X \setminus H; \tau)$ hold and so $(r_H)_* = (r_H)_{*,0}$ holds.

We conclude this section with an open question and a related definition:
Question. Study analogous theorems to Theorem 5.2 and Theorem 5.7 or new theorems for families using Definition 5.9 below, i.e., $pch(X, X \setminus H; \tau) \cup contpch(X, X \setminus H; \tau)$ and $pch_0(X, X \setminus H; \tau) \cup contpch_0(X, H; \tau)$.

Definition 5.9 For a topological space (X, τ) and a subset H of X, we define the following families of maps:
(i) $contpch(X, X \setminus H; \tau) := \{b|b \in contpch(X; \tau)$ and $b(X \setminus H) = X \setminus H\}$;
(ii) $contpch_0(X, X \setminus H; \tau) := \{b|b \in contpch(X, X \setminus H; \tau)$ and $b(x) = x$ for every $x \in X \setminus H\}$.

6 Examples on digital lines and digital planes We first investigate an example concerning the digital line $(\mathbb{Z}, \kappa)$ (cf. the end of Section 2). We recall the concept of the *smallest open set* $U(x)$ containing a point $x \in \mathbb{Z}$, i.e., $U(x) := \{2m-1, 2m, 2m+1\}$ for $x = 2m$ $(m \in \mathbb{Z})$ and $U(x) = \{2s+1\}$ for $x = 2s+1$ $(s \in \mathbb{Z})$. Then, for any open subset G containing a point $x \in \mathbb{Z}$, $U(x) \subseteq G$ holds in $(\mathbb{Z}, \kappa)$. We have $Cl(\{2m+1\}) = \{2m, 2m+1, 2m+2\}$, where $m \in \mathbb{Z}$. Let $H := U(0)$ be the smallest open set containing $0 \in \mathbb{Z}$ and $(H, \kappa|H)$ be a subspace of $(\mathbb{Z}, \kappa)$. Since $\kappa = PO(\mathbb{Z}, \kappa) = \kappa^\alpha$ (e.g., [17, Theorem 2.1 (a), (b)]), H is α-open in $(\mathbb{Z}, \kappa)$. Because of $Cl(Int(Cl(H))) = Cl(Int(\{-2, -1, 0, 1, 2\})) = Cl(\{-1, 0, 1\}) = \{-2, -1, 0, 1, 2\}$, we have $Cl(Int(Cl(H))) \not\subseteq H$. Namely, the subset H is not α-closed in $(\mathbb{Z}, \kappa)$. The following proposition shows that, for the above subspace $(H, \kappa|H)$,
(6-a) $pch_0(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa) \ncong pch(H; \kappa|H)$ (cf. Definition 5.1).
Then, the α-closedness of H in Theorem 5.7 (iii-2) can not be removed (cf. Example 5.8 (i) above, Proposition 6.1 (ii), (iii) below). Let $t_0 : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa)$ be a map defined by $t_0(x) := -x$ for every $x \in \mathbb{Z}$.

Proposition 6.1 (cf. Remark 5.8 (i)) *Let $(H, \kappa|H)$ be a subspace of $(\mathbb{Z}, \kappa)$, where $H := U(0) = \{-1, 0, +1\}$ be the smallest open set containing $0 \in \mathbb{Z}$. Then, the following properties hold.*
(i) $pch(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa) = \{1_\mathbb{Z}, t_0\}$.
(ii) $pch_0(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa) = \{1_\mathbb{Z}\}$.
(iii) $pch(H; \kappa|H) = \{1_H, t_0|H\}$.
(iv) $Im(r_H)_* = \{1_H, t_0|H\}$ *and* $(r_H)_* : pch(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa) \to pch(H, \kappa|H)$ *is onto.*
(v) $Ker(r_H)_* = \{1_\mathbb{Z}\}$.

Proof. To prove Proposition 6.1, we need the following notation (6-b) related on the subset H:
(6-b) $H_1 := H = U(0), H_2 := H_1 \cup U(2) \cup U(-2), \ldots$
(∗) $H_i := H_{i-1} \cup U(2i-2) \cup U(-(2i-2))$ for each integer $i \geq 2$.
It is easily shown that $H_i = \bigcup\{U(2j-2) \cup U(-(2j-2))|j \in \mathbb{Z}$ with $1 \leq j \leq i\}$ for each integer $i \geq 2$; if $i < j$, then $H_i \subseteq H_j$ and $\bigcup\{H_j|j \in \mathbb{Z}$ with $j \geq 1\} = \mathbb{Z}$. We claim the following property:

Claim. *Let* $f \in pch(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa)$ *and* $\{H_j\}_{j \in \mathbb{Z}}$*, where* $j \geq 1$*, a family of subsets defined by* (6-b) *above.*

(i) *If* $f|H = t_0|H$*, then* $f|H_m = t_0|H_m$ *for any* $m \in \mathbb{Z}$ *with* $m \geq 2$*.*

(ii) *If* $f|H = 1_H$*, then* $f|H_m = 1_{H_m}$ *for any* $m \in \mathbb{Z}$ *with* $m \geq 2$*.*

Proof of Claim. We prove the above properties by induction on m as follows.

(i) *For case* $m = 2$: for this case, we prove $f|H_2 = t_0|H_2$. We recall that $H_2 = H_1 \cup U(2) \cup U(-2)$ and $U(2), U(-2), \{1\}$ and $\{-1\}$ are preopen in $(\mathbb{Z}, \kappa)$. Since f^{-1} is preirresolute, we have that $\{-1\} = f(\{1\}) \in f(U(2)), f(\{2\})$ is preclosed, $f(\{3\})$ is an open singleton and $f(U(2)) \in PO(\mathbb{Z}, \kappa)$. Thus, we can assume that $f(2) = 2s$ for some integer s and $f(3) = 2u + 1$ for some integer u, because $\{2\}$ is preclosed and $\{3\}$ is preopen in $(\mathbb{Z}, \kappa)$. Since $U(2s)$ is the smallest open set containing $2s$ and $2 \in f^{-1}(\{2s\})$, we have that $f^{-1}(U(2s)) = f^{-1}(\{2s-1, 2s, 2s+1\})$ is the smallest open set containing 2. Thus, we have $1 \in \{1, 2, 3\} = U(2) = f^{-1}(\{2s-1, 2s, 2s+1\})$ and $3 \in f^{-1}(\{2s-1, 2s, 2s+1\})$. Then, $-1 = f(1) \in \{2s-1, 2s, 2s+1\}$ and $f(3) = 2u+1 \in \{2s-1, 2s, 2s+1\}$ hold. Namely, we have that

(6-c) $-1 = 2s - 1$ or $-1 = 2s + 1$ and $2u + 1 = 2s - 1$ or $2u + 1 = 2s + 1$.

If $s = 0$, then $f(2) = 0$ and $f(0) = t_0(0) = 0$ and so $0 = 2$ because f is bijective. Thus, it follows from (6-c) that $s = -1$ and so $f(2) = 2s = -2$. For the integer $f(3) = 2u+1, 2u+1 = -3$ or $2u+1 = -1$. If $2u+1 = -1$, then $f(3) = -1 = t_0(1) = f(1)$ and so $3 = 1$, because f is bijective. Thus, we have $f(3) = 2u + 1 = -3$. We conclude that $f(2) = -2$ and $f(3) = -3$; i.e., $f|H_1 \cup U(2) = t_0|H_1 \cup U(2)$. It is shown similarly that $f|H_1 \cup U(-2) = t_0|H_1 \cup U(-2)$. Thus, we prove that $f|H_2 = t_0|H_2$ holds.

Assuming that (i) *is true for an arbitrary integer* m *with* $m \geq 2$*, we prove that* (i) *is true for an integer* $m + 1$ *as follows.* We recall that $H_{m+1} = H_m \cup U(2(m+1) - 2) \cup U(-2(m+1) + 2) = H_m \cup U(2m) \cup U(-2m)$, where $U(2m) = \{2m-1, 2m, 2m+1\}$. It follows from assumption that $f(2m-1) = t_0(2m-1) = -(2m-1)$ holds, because $2m - 1 \in U(2m-2) \subset H_m$. We can assume that $f(2m) = 2s$ for some integer s, because $\{2m\}$ is preclosed, and $f(2m+1) = 2u+1$ for some integer u, because $\{2m+1\} \in PO(\mathbb{Z}, \kappa)$. Since $U(2s)$ is the smallest open set containing $2s$ and $\{2m\} = f^{-1}(\{2s\}), f^{-1}(U(2s)) = f^{-1}(\{2s-1, 2s, 2s+1\})$ is the smallest open set $U(2m)$ containing $2m$. Thus, we have $2m-1 \in f^{-1}(\{2s-1, 2s, 2s+1\})$ and $2m+1 \in f^{-1}(\{2s-1, 2s, 2s+1\})$. Then, $f(2m-1) = -(2m-1) \in \{2s-1, 2s, 2s+1\}$ holds because of $2m - 1 \in H_m$, and $f(2m+1) = 2u+1 \in \{2s-1, 2s, 2s+1\}$ hold. Namely, we have that

(6-d) $-2m + 1 = 2s - 1$ or $-2m + 1 = 2s + 1$ (i.e., $2s = -2(m-1)$ or $2s = 2m$), and $2u + 1 = 2s - 1$ or $2u + 1 = 2s + 1$.

If $2s = -2(m-1)$, then $f(2m) = 2s = -2(m-1) = t_0(2(m-1)) = f(2(m-1))$ and so $2m = 2(m-1)$, i.e., $0 = -2$. Thus, it follows from (6-d) that $2s = -2m$ and hence $f(2m) = -2m$. For the integer $f(2m+1) = 2u+1$, if $2u+1 = 2s+1$, then $f(2m+1) = 2u+1 = 2s+1 = -2m+1 = -(2m-1) = t_0(2m-1) = f(2m-1)$, because of $2m-1 \in H_m$, and so $2m + 1 = 2m - 1$ by the bijectivity of f. This conclude that $2u + 1 = 2s - 1$ holds. Thus, $f(2m+1) = 2u+1 = 2s-1 = -2m-1 = -(2m+1)$. Therefore, we have that $f(2m) = -2m = t_0(2m)$ and $f(2m+1) = -(2m+1) = t_0(2m+1)$ hold. Namely, we have that $f|H_m \cup U(2m) = t_0|H_m \cup U(2m)$ holds. It is shown similarly that $f|H_m \cup U(-2m) = t_0|H_m \cup U(-2m)$. Consequently, we have that $f|H_{m+1} = t_0|H_{m+1}$ holds for the integer $m + 1$. By induction on m, (i) is proved. **(ii)** The proof of (ii) is the same as that of (i) except for obvious modifications.

We conclude the proof of Claim above.

Proof of Proposition 6.1 (i). First we shall prove that $\{1_{\mathbb{Z}}, t_0\} \subseteq pch(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa)$ holds. We have that $t_0(U(2m)) = \{-(2m-1), -2m, -(2m+1)\} = U(-2m)$, where $U(2m) = \{2m-1, 2m, 2m+1\}$ and $m \in \mathbb{Z}$, and $t_0(U(2s+1)) = \{-(2s+1)\} = U(-(2s+1))$, where

$U(2s+1) = \{2s+1\}$ and $s \in \mathbb{Z}$. Thus, $t_0(U(y)) = U(-y) \in \kappa = PO(\mathbb{Z}, \kappa)$ for any $y \in \mathbb{Z}$. A preopen set V of $(\mathbb{Z}, \kappa)$ is expressible as $V = \bigcup\{U(y)|y \in V\}$, because $PO(\mathbb{Z}, \kappa) = \kappa$ holds. Then, ${t_0}^{-1}(V) = t_0(V) = \bigcup\{t_0(U(y))|y \in V\} = \bigcup\{U(-y)|y \in V\} \in \kappa = PO(\mathbb{Z}, \kappa)$ and so ${t_0}^{-1}(V) = t_0(V)$ is preopen in $(\mathbb{Z}, \kappa)$. Thus, ${t_0}^{-1}$ and t_0 are preirresolute. Since t_0 is bijective, we have $\{1_{\mathbb{Z}}, t_0\} \subseteq pch(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa)$.

Finally we shall prove that $pch(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa) \subseteq \{1_{\mathbb{Z}}, t_0\}$ holds. Let $f \in pch(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa)$. Since f^{-1} is preirresolute and $f^{-1}(H) = H, f(V) \in PO(\mathbb{Z}, \kappa) = \kappa$ for any $V \in PO(\mathbb{Z}, \kappa) = \kappa$ with $V \subseteq H$. For example, $\{-1\}, \{1\}$ and $\{-1, 1\}$ are open and so preopen. Then, $f(\{-1\}) = \{-1\}$ or $\{1\}, f(\{1\}) = \{1\}$ or $\{-1\}$ and $f(\{-1, 1\}) = \{-1, 1\}$. Since $\{0\}$ is a unique preclosed singleton in $(\mathbb{Z}, \kappa)$ such that $\{0\} \subseteq H$ and $f(H) = H$, we have $f(\{0\}) = \{0\}$. Thus, we obtain

(6-e) $f|H = t_0|H$ or $f|H = 1_H$.

Using Claim above, we now prove the following inclusion: $pch(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa) \subseteq \{1_{\mathbb{Z}}, t_0\}$ as follows. Indeed, let $f \in pch(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa)$ and $x \in \mathbb{Z}$. There exists an integer m such that $x \in H_m$. When $f|H = t_0|H$, by (6-e), it is obtained that, by Claim (i) above, $f(x) = (f|H_m)(x) = (t_0|H_m)(x) = t_0(x)$ for the point x, i.e., we have $f = t_0$. When $f|H = 1_H$, by (6-e), it is obtained that, by Claim (ii) above, $f(x) = (f|H_m)(x) = (1_{\mathbb{Z}}|H_m)(x) = x = 1_{\mathbb{Z}}(x)$ for the point x, i.e., we have $f = 1_{\mathbb{Z}}$. Consequently, $f = t_0$ or $f = 1_{\mathbb{Z}}$ for any $f \in pch(\mathbb{Z}; \mathbb{Z} \setminus H; \kappa)$, i.e., $pch(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa) \subseteq \{1_{\mathbb{Z}}, t_0\}$. Therefore, we conclude the proof of (i): $pch(\mathbb{Z}; \mathbb{Z} \setminus H; \kappa) = \{1_{\mathbb{Z}}, t_0\}$ and $pch(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa) \cong \mathbb{Z}_2$, because $(t_0)^2 = 1_{\mathbb{Z}}$.

Proof of Proposition 6.1 (ii). It is obviously shown that $t_0 \notin pch_0(\mathbb{Z}; \mathbb{Z} \setminus H; \kappa)$. Indeed, for a point $2 \notin H, t_0(2) = -2 \neq 2$ and hence $t_0|(\mathbb{Z} \setminus H) \neq 1_{\mathbb{Z}}$.

Proof of Proposition 6.1 (iii). We first prove that $\{1_H, t_0|H\} \subseteq pch(H; \kappa|H)$ holds. It is obvious that $1_H \in pch(H; \kappa|H)$. We show that $t_0|H \in pch(H, \kappa|H)$. For a subset $V \subseteq H$, under the assumption that $H \in PO(H, \kappa|H)$, $V \in PO(H, \kappa|H)$ if and only if $V \in PO(\mathbb{Z}, \kappa)$ (cf. Theorem 5.3 (v)). Consequently, we have that $PO(H; \kappa|H) = \{\{-1\}, \{+1\}, \{-1, 1\}, H, \emptyset\}$, because $PO(\mathbb{Z}, \kappa) = \kappa$. Then, it is easily shown that $(t_0|H)^{-1}(V) = (t_0|H)(V) \in PO(H, \kappa|H)$ for any $V \in PO(H, \kappa|H)$. Thus, we conclude $\{1_H, t_0|H\} \subseteq pch(H; \kappa|H)$. Conversely, let $f \in pch(H; \kappa|H)$. Then, $f(\{0\}) = \{0\}$ holds, because the singleton $\{0\}$ is a unique preclosed singleton of $(H, \kappa|H)$; $f(\{-1\}) = \{1\}$ or $\{-1\}$ and $f(\{1\}) = \{-1\}$ or $\{1\}$. Thus, we have that $f = 1_H$ or $t_0|H$ and hence $pch(H, \kappa|H) \subseteq \{1_H, t_0|H\}$. Therefore, (iii) is proved.

Proof of Proposition 6.1 (iv). It follows from (i), (iii) and definition that $Im(r_H)_* = \{(r_H)_*(1_{\mathbb{Z}}), (r_H)_*(t_0)\} = \{1_H, r_{H,H}(t_0)\} = \{1_H, t_0|H\}$. This property shows directly that $(r_H)_*$ is onto. (We note that the α-openness and α-closedness of H are assumed in order to show the onto homomorphism of $(r_H)_*$ in general (cf. Theorem 5.7 (ii), (ii-2)) ; this property (iv) shows that this homomorphism $(r_H)_*$ is onto even if H is not α-closed in $(\mathbb{Z}, \kappa)$).

Proof of Proposition 6.1 (v). It follows from (i), (ii) and definition that $Ker(r_H)_* = \{a \in pch(\mathbb{Z}, \mathbb{Z} \setminus H; \kappa)|(r_H)_*(a) = 1_H\} = \{a \in \{1_{\mathbb{Z}}, t_0\}|r_{H,H}(a) = 1_H\} = \{1_{\mathbb{Z}}\}$, because $r_{H,H}(t_0) = t_0|H \neq 1_H$. □

In the end of this section, we investigate characterizations of preirresolute functions and preirresolute homeomorphisms on digital planes (cf. Theorem 6.8) and some examples (cf. Theorem 6.12, Proposition 6.14). Let $(\mathbb{Z}^2, \kappa^2)$ be the digital plane throughout this section. First we recall some definitions and properties on digital planes (e.g., [7], [17], [18], [42], [9]). The topological product of two copies of the digital line $(\mathbb{Z}, \kappa)$ is called *the digital plane*, it is denoted by $(\mathbb{Z}^2, \kappa^2)$ (cf. in [24, Definition 4], it is called *digital 2-space*).

Let $x = (x_1, x_2)$ be a point of $(\mathbb{Z}^2, \kappa^2)$, where $x_1, x_2 \in \mathbb{Z}$.

($*^1$) When the integers x_1, x_2 are all odd, then a singleton $\{x\} = \{(x_1, x_2)\}$ is open in $(\mathbb{Z}^2, \kappa^2)$. We use the following notation for the point x (i.e., $\{x\}$ is open): $U(x) := \{x\}$; this set $U(x)$ is the smallest open set containing the point x.

($*^2$) When the integers x_1, x_2 are all even, then a singleton $\{x\} = \{(x_1, x_2)\}$ is closed in $(\mathbb{Z}^2, \kappa^2)$. For the point x above, we put the four points $p_x^{(1)} := (x_1 + 1, x_2 + 1), p_x^{(2)} := (x_1 - 1, x_2 + 1), p_x^{(3)} := (x_1 - 1, x_2 - 1), p_x^{(4)} := (x_1 + 1, x_2 - 1)$. Each singleton $\{p_x^{(i)}\}$ is open in $(\mathbb{Z}^2, \kappa^2)$ for $i \in \{1, 2, 3, 4\}$, because all components of the points are odd (cf. ($*^1$)). For the point $x = (x_1, x_2)$ above (i.e., $\{x\}$ is closed), put $U(x) := \{x_1 - 1, x_1, x_1 + 1\} \times \{x_2 - 1, x_2, x_2 + 1\}$. This set $U(x)$ is the smallest open set containing x and $\{x, p_x^{(1)}, p_x^{(2)}, p_x^{(3)}, p_x^{(4)}\} \subseteq U(x)$.

($*^3$) Let $x = (x_1, x_2)$ be a point of $(\mathbb{Z}^2, \kappa^2)$ such that x_i is odd and x_j is even for distinct integers i and j with $i, j \in \{1, 2\}$. Such point x is called a mixed point of $(\mathbb{Z}^2, \kappa^2)$. When $(x_1, x_2) := (2s, 2m + 1)$, where $s, m \in \mathbb{Z}$, let $x^+ := (2s + 1, 2m + 1), x^- := (2s - 1, 2m + 1)$; when $(x_1, x_2) := (2s + 1, 2m)$, where $s, m \in \mathbb{Z}$, let $x^+ := (2s + 1, 2m + 1), x^- := (2s + 1, 2m - 1)$. By using above notations, we define $U(x) := \{x^-, x, x^+\}$ for this point x; this set $U(x)$ is also the smallest open set containing x. The set $U(x)$ has the following property: for $x = (2s, 2m + 1), U(x) := \{2s - 1, 2s, 2s + 1\} \times \{2m + 1\}$; for $x = (2s + 1, 2m), U(x) := \{2s + 1\} \times \{2m - 1, 2m, 2m + 1\}$.

It is well known that for any open subset G containing a point $x \in \mathbb{Z}^2$, $U(x) \subseteq G$ holds in $(\mathbb{Z}^2, \kappa^2)$.

We use the following notation (e.g., [17], [18], [7], [42], [9]).

Definition 6.2 (i) $(\mathbb{Z}^2)_{\kappa^2} := \{x \in \mathbb{Z}^2 | \{x\}$ is open in $(\mathbb{Z}^2, \kappa^2)\}$;
$(\mathbb{Z}^2)_{\mathcal{F}^2} := \{x \in \mathbb{Z}^2 | \{x\}$ is closed in $(\mathbb{Z}^2, \kappa^2)\}$;
$(\mathbb{Z}^2)_{mix} := \mathbb{Z}^2 \setminus ((\mathbb{Z}^2)_{\kappa^2} \cup (\mathbb{Z}^2)_{\mathcal{F}^2})$.

(ii) For a subset A of $(\mathbb{Z}^2, \kappa^2)$,
$A_{\kappa^2} := (\mathbb{Z}^2)_{\kappa^2} \cap A$ and so $A_{\kappa^2} = \{x \in A | \{x\}$ is open in $(\mathbb{Z}^2, \kappa^2)\}$ holds;
$A_{\mathcal{F}^2} := (\mathbb{Z}^2)_{\mathcal{F}^2} \cap A$ and so $A_{\mathcal{F}^2} = \{x \in A | \{x\}$ is closed in $(\mathbb{Z}^2, \kappa^2)\}$ holds;
$A_{mix} := A \setminus (A_{\kappa^2} \cup A_{\mathcal{F}^2})$.

Example 6.3 (i) It is obviously shown that, in $(\mathbb{Z}^2, \kappa^2)$,

(i-1) $(\mathbb{Z}^2)_{\kappa^2} = \{(2s + 1, 2m + 1) \in \mathbb{Z}^2 | s, m \in \mathbb{Z}\}$,

(i-2) $(\mathbb{Z}^2)_{\mathcal{F}^2} = \{(2s, 2m) \in \mathbb{Z}^2 | s, m \in \mathbb{Z}\}$,

(i-3) $(\mathbb{Z}^2)_{mix} = \{(2s + 1, 2m) \in \mathbb{Z}^2 | s, m \in \mathbb{Z}\} \cup \{(2s, 2m + 1) \in \mathbb{Z}^2 | s, m \in \mathbb{Z}\}$,

(i-4) $\mathbb{Z}^2 = (\mathbb{Z}^2)_{\kappa^2} \cup (\mathbb{Z}^2)_{\mathcal{F}^2} \cup (\mathbb{Z}^2)_{mix}$ (disjoint union) and
$A = A_{\kappa^2} \cup A_{\mathcal{F}^2} \cup A_{mix}$ (disjoint union) for a subset A of $(\mathbb{Z}^2, \kappa^2)$.

(ii) (ii-1) For a point $x \in (\mathbb{Z}^2)_{\kappa^2}$, $(U(x))_{\mathcal{F}^2} = \emptyset, (U(x))_{\kappa^2} = \{x\}$ and $(U(x))_{mix} = \emptyset$.

(ii-2) For a point $x := (x_1, x_2) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, $(U(x))_{\mathcal{F}^2} = \{x\}, (U(x))_{\kappa^2} = \{p_x^{(i)} | i \in \{1, 2, 3, 4\}\}$ (cf. ($*$-2) above) and $(U(x))_{mix} = \{(x_1, x_2 + 1), (x_1, x_2 - 1), (x_1 + 1, x_2), (x_1 - 1, x_2)\}$.

(ii-3) For a point $x \in (\mathbb{Z}^2)_{mix}$, $(U(x))_{\mathcal{F}^2} = \emptyset, (U(x))_{\kappa^2} = \{x^+, x^-\}$ and $(U(x))_{mix} = \{x\}$ (cf. ($*$-3)).

(iii) Let $s, m \in \mathbb{Z}$. (iii-1) For a point $(2s + 1, 2m + 1) \in (\mathbb{Z}^2)_{\kappa^2}$, $Cl(\{(2s + 1, 2m + 1)\}) = \{2s, 2s + 1, 2s + 2\} \times \{2m, 2m + 1, 2m + 2\}$.

(iii-2) For a point $(2s, 2m) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, $Cl(\{(2s, 2m)\}) = \{(2s, 2m)\}$.

(iii-3) For a point $(2s, 2m + 1) \in (\mathbb{Z}^2)_{mix}$, $Cl(\{(2s, 2m + 1)\}) = \{2s\} \times \{2m, 2m + 1, 2m + 2\}$; for a point $(2s + 1, 2m) \in (\mathbb{Z}^2)_{mix}$, $Cl(\{(2s + 1, 2m)\}) = \{2s, 2s + 1, 2s + 2\} \times \{2m\}$.

Moreover, we recall the following well known definitions and examples.

Definition 6.4 (i) For a topological space (X, τ) and a subset A of X,
(i-1) $Ker(A) := \bigcap\{V | A \subseteq V, V \in \tau\}$;
(i-2) $Ker_p(A) := \bigcap\{V | A \subseteq V, V \in PO(X, \tau)\}$.
(ii) For a subspace $(H, \tau|H)$ of (X, τ) and a subset B of H,
(ii-1) $(\tau|H)$-$Ker(B) := \bigcap\{W | B \subseteq W, W \in \tau|H\}$;
(ii-2) $(\tau|H)$-$Ker_p(B) := \bigcap\{V | B \subseteq W, W \in PO(H, \tau|H)\}$.

Example 6.5 For $(X, \tau) = (\mathbb{Z}^2, \kappa^2)$ in Definition 6.4, we have the following properties: for any $s, m \in \mathbb{Z}$,
$Ker(\{(2s+1, 2m+1)\}) = \{(2s+1, 2m+1)\} = U((2s+1, 2m+1))$,
$Ker(\{(2s, 2m)\}) = \{2s-1, 2s, 2s+1\} \times \{2m-1, 2m, 2m+1\} = U((2s, 2m))$,
$Ker(\{(2s, 2m+1)\}) = \{2s-1, 2s, 2s+1\} \times \{2m+1\} = U((2s, 2m+1))$,
$Ker(\{(2s+1, 2m)\}) = \{2s+1\} \times \{2m-1, 2m, 2m+1\} = U((2s+1, 2m))$.

We need the following properties in order to characterize the group $pch(\mathbb{Z}^2; \kappa^2)$ (cf. Theorem 6.8).

Proposition 6.6 (i) (cf. [9, Lemma 2.2 (ii)]) $\{x\} \cup (U(x))_{\kappa^2} \in PO(\mathbb{Z}^2, \kappa^2)$ *holds for every* $x \in \mathbb{Z}^2$.
(ii) $Ker_p(\{x\}) = \{x\} \cup (U(x))_{\kappa^2}$ *holds in* $(\mathbb{Z}^2, \kappa^2)$ *for every* $x \in \mathbb{Z}^2$.
(iii) *Let* $x \in \mathbb{Z}^2$, $x_0 := (0, 0)$ *and* $H := U(x_0)$, *i.e.,* $H = \{-1, 0, 1\} \times \{-1, 0, 1\}$.
(iii-1) $\{x_0\} \cup (U(x_0))_{\kappa^2} \in PO(H, \kappa^2|H)$ *holds.*
(iii-2) $(\kappa^2|H)$-$Ker_p(\{x_0\}) = \{x_0\} \cup (U(x_0))_{\kappa^2}$ *holds.*

Proof. **(i) Case 1**. $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$: in [9, Lemma 2.2 (ii)], recently one of the present authors proved this property (i) for a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$; let $x := (2s, 2m)$, where $s, m \in \mathbb{Z}$; for this case, it is shown that $Int(Cl(\{x\} \cup (U(x))_{\kappa^2})) = Int(\{2s-2, 2s-1, 2s, 2s+1, 2s+2\} \times \{2m-2, 2m-1, 2m, 2m+1, 2m+2\}) = \{2s-1, 2s, 2s+1\} \times \{2m-1, 2m, 2m+1\} = U((2s, 2m)) \supseteq \{x\} \cup (U(x))_{\kappa^2}$ hold. Thus the set $\{x\} \cup (U(x))_{\kappa^2}$ is preopen in $(\mathbb{Z}^2, \kappa^2)$. **Case 2**. $x \in (\mathbb{Z}^2)_{\kappa^2}$: for this case, we have that $U(x) = \{x\}$ (cf. $(*^1)$) and so $\{x\} \cup (U(x))_{\kappa^2} = \{x\} \in \kappa^2$ and so it is preopen. **Case 3**. $x \in (\mathbb{Z}^2)_{mix}$: for this case, by Example 6.3 (ii-3), it is shown that $\{x\} \cup (U(x))_{\kappa^2} = \{x^+, x, x^-\} = U(x) \in \kappa^2 \subseteq PO(\mathbb{Z}^2, \kappa^2)$. **(ii)** Using (i), we have that $Ker_p(\{x\}) \subseteq \{x\} \cup (U(x))_{\kappa^2}$. We should prove $\{x\} \cup (U(x))_{\kappa^2} \subseteq Ker_p(\{x\})$. By using [7, Lemma 6.3 (i)] and [18, Lemma 4.7], it is shown that

$(*)$ every preopen set V such that $x \in V$ includes the preopen set $\{x\} \cup (U(x))_{\kappa^2}$.

Using this property $(*)$, we have $\{x\} \cup (U(x))_{\kappa^2} \subseteq \bigcap\{V | \{x\} \subseteq V, V \in PO(\mathbb{Z}^2, \kappa^2)\} = Ker_p(\{x\})$. Therefore, $Ker_p(\{x\}) = \{x\} \cup (U(x))_{\kappa^2}$ holds. **(iii) (iii-1)** Using (i) above and Theorem 5.3 (ii), $\{x_0\} \cup (U(x_0))_{\kappa^2}$ is preopen in $(H, \kappa^2|H)$. **(iii-2)** Since $H := U(x_0)$ and $x_0 = (0, 0)$, we have that H is α-open and so $PO(\mathbb{Z}^2, \kappa^2)|H = PO(H, \kappa^2|H)$ holds (cf. Theorem 5.3 (vi-3)). By the definition of p-kernels (cf. Definition 6.4) and (iii-1), it is shown that $(\kappa^2|H)$-$Ker_p(\{x_0\}) = \bigcap\{W | \{x_0\} \subset W, W \in PO(\mathbb{Z}^2, \kappa^2)|H\} = \bigcap\{V \cap H | \{x_0\} \subset V \cap H, V \in PO(\mathbb{Z}^2, \kappa^2)\} = (\bigcap\{V | \{x_0\} \subset V, V \in PO(\mathbb{Z}^2, \kappa^2)\}) \cap H = (Ker_p(\{x_0\})) \cap H = (\{x_0\} \cup ((U(x_0))_{\kappa^2}) \cap H = \{x_0\} \cup ((U(x_0))_{\kappa^2}$ hold. $\square$

In Theorem 6.7 below, we use the following notation:
$(\mathbb{Z}^2)_{\mathcal{PO}} := \{x | \{x\} \in PO(\mathbb{Z}^2; \kappa^2)\}$,
$(\mathbb{Z}^2)_{\mathcal{PC}} := \{x | \{x\}$ is preclosed in $(\mathbb{Z}^2, \kappa^2)\}$.
Then, it is shown that $(\mathbb{Z}^2)_{\mathcal{PO}} = (\mathbb{Z}^2)_{\kappa^2}$, $(\mathbb{Z}^2)_{\mathcal{PC}} = (\mathbb{Z}^2)_{\mathcal{F}^2} \cup (\mathbb{Z}^2)_{mix}$ (disjoint union) and $\mathbb{Z}^2 = (\mathbb{Z}^2)_{\mathcal{PO}} \cup (\mathbb{Z}^2)_{\mathcal{PC}}$ (disjoint union) hold (cf. Definition 6.2, Example 6.3).
For a finite set K, we denote the cardinal number of K by $^{\#}(K)$.

Theorem 6.7 *Assume $f \in pch(\mathbb{Z}^2;\kappa^2)$. Then, the following properties are verified. Let $x \in \mathbb{Z}^2$.*

(i) *$x \in (\mathbb{Z}^2)_{\kappa^2}$ if and only if $f(x) \in (\mathbb{Z}^2)_{\kappa^2}$.*

(ii) *$x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ if and only if $f(x) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$.*

(iii) *$x \in (\mathbb{Z}^2)_{mix}$ if and only if $f(x) \in (\mathbb{Z}^2)_{mix}$.*

(iv) *For a point $x \in \mathbb{Z}^2, f(Ker_p(\{x\})) = Ker_p(\{f(x)\})$ holds. Explicitly, if $x \in (\mathbb{Z}^2)_{\kappa^2}$, then $f(U(x)) = U(f(x))$ holds; if $x \in (\mathbb{Z}^2)_{mix}$, then $f(U(x)) = U(f(x))$ holds; if $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, then $f(\{x\} \cup (U(x))_{\kappa^2}) = \{f(x)\} \cup (U(f(x)))_{\kappa^2}$ holds (i.e., $f((U(x))_{\kappa^2}) = (U(f(x)))_{\kappa^2}$).*

(v) (v-1) *If $z \in (\mathbb{Z}^2)_{\kappa^2}$, then $f(U(z)) = \{f(z)\} = U(f(z))$ hold.*

(v-2) *If $y \in (\mathbb{Z}^2)_{mix}$, then $f(U(y)) = U(f(y))$ holds.*

(v-3) *If $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, then $f((U(x))_{\kappa^2}) = (U(f(x)))_{\kappa^2}, f((U(x))_{\mathcal{F}^2}) = \{f(x)\} = (U(f(x)))_{\mathcal{F}^2}$ and $f((U(x))_{mix}) = (U(f(x)))_{mix}$ hold.*

(vi) *For a point $x \in \mathbb{Z}^2, f(U(x)) = U(f(x))$ holds.*

Proof. **(i)** Since $f \in pch(\mathbb{Z}^2,\kappa^2)$, it is shown that $x \in (\mathbb{Z}^2)_{\mathcal{PO}}$ if and only if $f(x) \in (\mathbb{Z}^2)_{\mathcal{PO}}$. Thus, (i) is obtained by using the property of $(\mathbb{Z}^2)_{\mathcal{PO}} = (\mathbb{Z}^2)_{\kappa^2}$.

(ii) (**Necessity**) Suppose $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$. Since $(\mathbb{Z}^2)_{\mathcal{F}^2} \subseteq (\mathbb{Z}^2)_{\mathcal{PC}}$, we have $f(x) \in (\mathbb{Z}^2)_{\mathcal{PC}} = (\mathbb{Z}^2)_{mix} \cup (\mathbb{Z}^2)_{\mathcal{F}^2}$ (disjoint union). Assume $f(x) \in (\mathbb{Z}^2)_{mix}$. Then, first let $f(x) = (2s+1, 2m)$, where $s, m \in \mathbb{Z}$. Take two open singletons $\{y^+\} := \{(2s+1, 2m+1)\}$ and $\{y^-\} := \{(2s+1, 2m-1)\}$. Since $U(f(x)) = \{y^+, f(x), y^-\} \in \kappa^2 \subseteq PO(\mathbb{Z}^2,\kappa^2)$ and $f \in pch(\mathbb{Z}^2;\kappa^2)$, we have that $f^{-1}(U(f(x)))=\{f^{-1}(y^+), x, f^{-1}(y^-)\}$ and so $f^{-1}(U(f(x))) \in PO(\mathbb{Z}^2,\kappa^2)$. Thus, we have that $Ker_p(\{x\}) \subseteq \{f^{-1}(y^+), x, f^{-1}(y^-)\}$ holds and so $^{\#}(Ker_p(\{x\})) \leq {}^{\#}(\{f^{-1}(y^+), x, f^{-1}(y^-)\}) = 3$ holds. However, by Proposition 6.6 (i) and (ii), it is shown that $^{\#}(Ker_p(\{x\})) = 5$ for the point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$; we have a contradiction for this case. Finally, we have similarly a contradiction for the case where $f(x) = (2s, 2m+1)$, where $s, m \in \mathbb{Z}$. Therefore, we claim $f(x) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$.

(**Sufficiency**) Suppose $f(x) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$. Since $f(x) \in (\mathbb{Z}^2)_{\mathcal{PC}}$ and $f^{-1} \in pch(\mathbb{Z}^2;\kappa^2)$, we have $x = f^{-1}(f(x)) \in (\mathbb{Z}^2)_{\mathcal{PC}} = (\mathbb{Z}^2)_{mix} \cup (\mathbb{Z}^2)_{\mathcal{F}^2}$ (disjoint union). Assume that $x \in (\mathbb{Z}^2)_{mix}$; we have a contradiction using similar arguments in the proof of the necessity. Thus, we conclude that $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ holds.

(iii) We recall that $\mathbb{Z}^2$ is the disjoint union of $(\mathbb{Z}^2)_{\mathcal{F}^2}, (\mathbb{Z}^2)_{\kappa^2}$ and $(\mathbb{Z}^2)_{mix}$. Then, $x \in (\mathbb{Z}^2)_{mix}$ if and only if $x \notin (\mathbb{Z}^2)_{\mathcal{F}^2}$ and $x \notin (\mathbb{Z}^2)_{\kappa^2}$; by (i) and (ii) above, if and only if $f(x) \notin (\mathbb{Z}^2)_{\mathcal{F}^2}$ and $f(x) \notin (\mathbb{Z}^2)_{\kappa^2}$; i.e., $f(x) \in (\mathbb{Z}^2)_{mix}$ holds.

(iv) Case 1. $x \in (\mathbb{Z}^2)_{\kappa^2}$: for this case, we have $Ker_p(\{x\}) = \{x\}$ and $Ker_p(f(\{x\})) = \{f(x)\} = f(Ker_p(\{x\}))$ (cf. (i) above).

Case 2. $x \in (\mathbb{Z}^2)_{mix}$: for this case, there exist two points, say x^+ and x^-, such that $U(x) = \{x^+, x, x^-\}$ is the smallest open set containing the point x, where $\{x^+, x^-\} \subseteq (\mathbb{Z}^2)_{\kappa^2}$ and $x^+ \neq x^-$. Using Proposition 6.6 (ii), it is shown that $Ker_p(\{x\}) = U(x) = \{x^+, x, x^-\} \in PO(\mathbb{Z}^2,\kappa^2)$. Thus, we have $f(Ker_p(\{x\})) = f(U(x))= \{f(x^+), f(x), f(x^-)\}$, $f(Ker_p(\{x\})) \in PO(\mathbb{Z}^2,\kappa^2), f(x^+) \in (\mathbb{Z}^2)_{\kappa^2}$, $f(x^-) \in (\mathbb{Z}^2)_{\kappa^2}$ and $f(x) \in (\mathbb{Z}^2)_{mix}$ (cf. (i) and (iii) above). We claim that for the mixed point $f(x), U(f(x)) =\{f(x^+), f(x), f(x^-)\}$, i.e., $U(f(x)) = f(U(x))$ holds. Indeed, $U(f(x)) = Ker_p(\{f(x)\})$ is the smallest preopen set containing $f(x)$ (cf. Proposition 6.6 (ii) for the mixed point $f(x)$), say $U(f(x)) := \{p^{(1)}, f(x), p^{(2)}\}$, where $\{p^{(1)}\}$ and $\{p^{(2)}\}$ are the open singletons determined by the point $f(x)$. Then, $U(f(x)) = Ker_p(\{f(x)\}) =\{p^{(1)}, f(x), p^{(2)}\} \subseteq \{f(x^+), f(x), f(x^-)\}$ and so $3 = {}^{\#}(U(f(x)))= {}^{\#}(\{f(x^+), f(x), f(x^-)\} \cap \{p^{(1)}, f(x), p^{(2)}\})$. Thus, we have $\{f(x^+), f(x), f(x^-)\} =\{p^{(1)}, f(x), p^{(2)}\}$; we claimed $U(f(x)) = f(U(x))$. Hence, we have that $f(Ker_p(\{x\})) = f(U(x)) = U(f(x)) = Ker_p(\{f(x)\})$ for a mixed point x. (cf. Proposition 6.6 (ii)).

Case 3. $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$: for this point x, $\{x\}$ is closed, $^{\#}(U(x)) = 9$, $^{\#}((U(x))_{\kappa^2}) = 4$ and $^{\#}(f((U(x))_{\kappa^2})) = 4$. Then, by Proposition 6.6 (ii) and (i), (ii) above, it is shown that $f(Ker_p(\{x\})) = f(\{x\}) \cup f((U(x))_{\kappa^2})$. And so we have
$(**)^{(1)}$ $(f(Ker_p(\{x\})))_{\kappa^2} = f((U(x))_{\kappa^2})$ and $^{\#}((f(Ker_p(\{x\})))_{\kappa^2}) = 4$.
Since $f(x) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, we have $(Ker_p(\{f(x)\}))_{\kappa^2} = (U(f(x)))_{\kappa^2}$ and so
$(**)^{(2)}$ $^{\#}(Ker_p(\{f(x)\}))_{\kappa^2}) = 4$.
Now, $f(Ker_p(\{x\}))$ is a preopen set containing $f(x)$, because $f^{-1} \in pch(\mathbb{Z}^2;\kappa^2)$ and $Ker_p(\{x\}) \in PO(\mathbb{Z}^2,\kappa^2)$ (cf. Proposition 6.6 (i) (ii)). On the other hand, $Ker_p(\{f(x)\})$ is the smallest preopen set containing $f(x)$ and $f(x) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$. Thus, we have that $f(x) \in Ker_p(\{f(x)\}) \subseteq f(Ker_p(\{x\}))$ and so
$(**)^{(3)}$ $(Ker_p(\{f(x)\}))_{\kappa^2} \subseteq (f(Ker_p(\{x\})))_{\kappa^2}$ holds.

By using $(**)^{(1)}$, $(**)^{(2)}$ and $(**)^{(3)}$, it is shown that $(Ker_p(\{f(x)\}))_{\kappa^2} = (f(Ker_p(\{x\})))_{\kappa^2} = f((U(x))_{\kappa^2})$ holds, i.e., $(U(f(x)))_{\kappa^2} = f((U(x))_{\kappa^2})$ holds. Therefore, we have $\{f(x)\} \cup (U(f(x)))_{\kappa^2} = f(\{x\}) \cup f((U(x))_{\kappa^2}) = f(\{x\} \cup (U(x))_{\kappa^2})$. Namely, we show that $Ker_p(\{f(x)\}) = f(Ker_p(\{x\}))$ holds for this closed singleton $\{x\}$ (cf. Proposition 6.6 (ii)).

(v) (v-1) (v-2) These are obtained by (iv).

(v-3) Since $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}, \{x\}$ is closed and $U(x) = (U(x))_{\kappa^2} \cup (U(x))_{\mathcal{F}^2} \cup (U(x))_{mix}$ (disjoint union) and $(U(x))_{\mathcal{F}^2} = \{x\}$ hold. By (iv), it is first shown that $f((U(x))_{\kappa^2}) = (U(f(x)))_{\kappa^2}$ holds for the point x; by definition and (ii) above, $f((U(x))_{\mathcal{F}^2}) = (U(f(x)))_{\mathcal{F}^2}$ holds. Finally, we claim that $f((U(x))_{mix}) = (U(f(x)))_{mix}$ hold. [Indeed, let $f(y) \in f((U(x))_{mix})$, where $y \in (U(x))_{mix}$. We set $U(y) = \{y^-, y, y^+\}$, where $y^-, y^+ \in (U(y))_{\kappa^2} \subseteq (U(x))_{\kappa^2}$. Since $y \in \mathbb{Z}^2, y \in U(x)$ and $U(y) \subseteq U(x)$, by using (iv), it is shown that $f(U(y)) = U(f(y))$, $f(y) \in U(f(y))$ and so $(U(f(y)))_{\kappa^2} = f((U(y))_{\kappa^2}) \subseteq f(U(x)_{\kappa^2})$. Thus we have $f(U(y))_{\kappa^2}) \subseteq (U(f(x)))_{\kappa^2}$ and so $\{f(y^+), f(y^-)\} = (U(f(x)))_{\kappa^2} \cap (U(f(y)))_{\kappa^2} = (U(f(y)) \cap U(f(x)))_{\kappa^2}$. We note that $f(x) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$. By using [41, Lemma 2.3] for the points $f(y) \in U(f(y))$ and $f(x) \in U(f(x))$, it is shown that $f(y) \in U(f(x))$. Indeed, $f(y) \in (\mathbb{Z}^2)_{mix(1)}$ and $f(x) \in (\mathbb{Z}^2)_{mix(2)}(a' = 1, a = 2, n = 2)$ and $U(f(y)) \cap U(f(x))$ contains the 2-open singletons $\{f(y^+)\}$ and $\{f(y^-)\}$ (Note: $2^{a'} = 2$). Since $f(y) \in (\mathbb{Z}^2)_{mix}$, we conclude that $f(y) \in (U(f(x))_{mix}$ and hence $f((U(x))_{mix}) \subseteq (U(f(x)))_{mix}$. We can prove the converse implication using the case where $f^{-1} \in pch(\mathbb{Z}^2,\kappa^2)$ and the point $f(x) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$. Then, we have that $f^{-1}((U(f(x)))_{mix}) \subseteq (U(f^{-1}(f(x))))_{mix}$, i.e., $(U(f(x)))_{mix} \subseteq f((U(x))_{mix})$.] Therefore, we proved that $(U(f(x)))_{mix} = f((U(x))_{mix})$ holds.

(vi) For a point $x \in (\mathbb{Z}^2)_{\kappa^2} \cup (\mathbb{Z}^2)_{mix}$, the proof is obtained by (v-1) or (v-2). For a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, the proof is as follows. Since $U(x) = (U(x))_{\kappa^2} \cup (U(x))_{mix} \cup \{x\}$ (cf. Example 6.3 (i-4)), we have that $f(U(x)) = f((U(x))_{\kappa^2}) \cup f((U(x))_{mix}) \cup \{f(x)\} = (U(f(x)))_{\kappa^2}) \cup ((U(f(x)))_{mix}) \cup \{f(x)\} = U(f(x))$ hold (cf. (v) above). □

Theorem 6.8 *Let $f : (\mathbb{Z}^2,\kappa^2) \to (\mathbb{Z}^2,\kappa^2)$ be a function.*

(i) *If f is a function satisfying the following property that $f(Ker_p(\{x\})) = Ker_p(\{f(x)\})$ holds for every $x \in \mathbb{Z}^2$, then f is preirresolute.*

(ii) *For a bijection $f : (\mathbb{Z}^2,\kappa^2) \to (\mathbb{Z}^2,\kappa^2)$, the following properties are equivalent:*

(1) *$f \in pch(\mathbb{Z}^2;\kappa^2)$ holds;*

(2) *$f(Ker_p(\{x\})) = Ker_p(\{f(x)\})$ and $f^{-1}(Ker_p(\{x\})) = Ker_p(\{f^{-1}(x)\})$ hold for every $x \in \mathbb{Z}^2$;*

(3) *$f(\{x\} \cup (U(x))_{\kappa^2}) = \{f(x)\} \cup (U(f(x)))_{\kappa^2}$ and*
$f^{-1}(\{x\} \cup (U(x))_{\kappa^2}) = \{f^{-1}(x)\} \cup (U(f^{-1}(\{x\})))_{\kappa^2}$ hold for every $x \in \mathbb{Z}^2$.

Proof. **(i)** Let $V \in PO(\mathbb{Z}^2,\kappa^2)$. For each $x \in f^{-1}(V)$, by Definition 6.4 (i-2) and the assumption, it is shown that, for the point $f(x) \in V, Ker_p(\{f(x)\}) \subseteq V$ and so $Ker_p(\{f(x)\}) = f(Ker_p(\{x\})) \subseteq V$. Thus, we have that $f^{-1}(V) = \bigcup\{Ker_p(\{x\}) |\ x \in f^{-1}(V)\}$ and so $f^{-1}(V) \in PO(\mathbb{Z}^2,\kappa^2)$, because $Ker_p(\{x\}) \in PO(\mathbb{Z}^2,\kappa^2)$ for each $x \in V$

(cf. Proposition 6.6 (i), (ii)). Therefore, f is preirresolute. **(ii) (1)⇒(2)** It is proved in Proposition 6.7 (iv). **(2)⇒(1)** Using (i) above for the bijections f and f^{-1}, we have that f and f^{-1} are preirresolute. **(2)⇔(3)** It is proved by Proposition 6.6 (ii). □

Using Theorem 6.8, we have some examples of functions on $(\mathbb{Z}^2, \kappa^2)$.

Definition 6.9 Let $t_{c,d} : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ be a bijection defined as follow, where $c, d \in \mathbb{Z}$:
$t_{c,d}(x_1, x_2) := (x_1 + c, x_2 + d)$ for every point (x_1, x_2) of $(\mathbb{Z}^2, \kappa^2)$.

Theorem 6.10 (i) *A collection $\{t_{2a,2b} | a, b \in \mathbb{Z}\}$ forms a subgroup of $pch(\mathbb{Z}^2; \kappa^2)$; explicitly, the collection is a subgroup of $h(\mathbb{Z}^2; \kappa^2)$.*

(ii) *Let a and b integers. The following properties are verified.*
(ii-1) $t_{2a+1,2b} \notin pch(\mathbb{Z}^2; \kappa^2)$; (ii-2) $t_{2a,2b+1} \notin pch(\mathbb{Z}^2; \kappa^2)$;
(ii-3) $t_{2a+1,2b+1} \notin pch(\mathbb{Z}^2; \kappa^2)$.

Proof. **(i)** We put $\mathbf{T} := \{t_{2a,2b} | a, b \in \mathbb{Z}\}$. The collection $\mathbf{T}$ is nonempty, because $t_{0,0}$ is the identity function and $t_{0,0} \in \mathbf{T}$. We claim $\mathbf{T} \subseteq pch(\mathbb{Z}^2; \kappa^2)$ (cf. Theorem 6.8 (i)). For $t_{2a,2b} \in \mathbf{T}$ and a point $x \in \mathbb{Z}^2$, it is shown that
$t_{2a,2b}(\{x\} \cup (U(x))_{\kappa^2}) = \{t_{2a,2b}(x)\} \cup (U(t_{2a,2b}(x)))_{\kappa^2}$ and
$(t_{2a,2b})^{-1}(\{x\} \cup (U(x))_{\kappa^2}) = \{(t_{2a,2b})^{-1}(x)\} \cup (U((t_{2a,2b})^{-1}(x)))_{\kappa^2}$.
By using Theorem 6.8 (i), it is obtained that $t_{2a,2b} \in pch(\mathbb{Z}^2; \kappa^2)$ and so $\mathbf{T} \subseteq pch(\mathbb{Z}^2; \kappa^2)$. For two elements $t_{2a,2b}, t_{2h,2k} \in \mathbf{T}$, we have $\omega(t_{2a,2b}, (t_{2h,2k})^{-1}) = (t_{2h,2k})^{-1} \circ t_{2a,2b} = t_{2a-2h,2b-2k}$ and so $\omega(t_{2a,2b}, (t_{2h,2k})^{-1}) \in \mathbf{T}$, where ω is the binary operation defined in the proof of Theorem 4.3 (ii). Therefore, $\mathbf{T}$ is a subgroup of $pch(\mathbb{Z}^2; \kappa^2)$. We note that the above mentioned proof is one of using Theorem 6.8 and $t_{2a,2b} \in pch(\mathbb{Z}^2; \kappa^2)$, where $a, b \in \mathbb{Z}$. Using the above mentioned property, we claim that $\mathbf{T}$ is also a subgroup of $h(\mathbb{Z}^2; \kappa^2)$. Let $t_{2a,2b} \in \mathbf{T}$ and G an open set of $(\mathbb{Z}^2, \kappa^2)$. Then, using the concept of smallest open sets $U(x)$, the set G is expressible as $G = \bigcup\{U(x) | x \in G\}$ and so $(t_{2a,2b})^{-1}(G) = \bigcup\{(t_{2a,2b})^{-1}(U(x)) | x \in G\}$. Since $t_{2a,2b} \in pch(\mathbb{Z}^2; \kappa^2)$, using Theorem 6.7 (vi), we have $t_{2a,2b}(U(x)) = U(t_{2a,2b}(x))$ and also $(t_{2a,2b})^{-1}(U(x)) = U((t_{2a,2b})^{-1}(x))$ hold for any point $x \in \mathbb{Z}^2$ and $a, b \in \mathbb{Z}$. Thus, we conclude that $(t_{2a,2b})^{-1}(G) = \bigcup\{U((t_{2a,2b})^{-1}(x)) | x \in G\} \in \kappa^2$ and so $t_{2a,2b}$ is a homeomorphism, i.e., $t_{2a,2b} \in h(\mathbb{Z}^2; \kappa^2)$; by an argument similar to that above, we can prove $\mathbf{T}$ is a subgroup of $h(\mathbb{Z}^2; \kappa^2)$. **(ii) (ii-1)** (resp. **(ii-2), (ii-3)**) For a preopen set $V = \{(1, 1)\}$, $t^{-1}_{2a+1,2b}(V) = \{(-2a, 1-2b)\}$ (resp. $t^{-1}_{2a,2b+1}(V) = \{(1-2a, -2b)\}$, $t^{-1}_{2a+1,2b+1}(V) = \{(-2a, -2b)\}$) is not preopen. Thus, we have: $t_{2a+1,2b} \notin pch(\mathbb{Z}^2; \kappa^2)$ (resp. $t_{2a,2b+1} \notin pch(\mathbb{Z}^2; \kappa^2)$, $t_{2a+1,2b+1} \notin pch(\mathbb{Z}^2; \kappa^2)$). □

To prove Theorem 6.12 below, we need the following lemma.

Lemma 6.11 (i) *For a subset F of $(\mathbb{Z}^2, \kappa^2)$, the following properties are equivalent:*
(1) *F is nowhere dense in $(\mathbb{Z}^2, \kappa^2)$ (i.e., $Int(Cl(F)) = \emptyset$);*
(2) *$F_{\kappa^2} = \emptyset$;*
(3) *$Int(F) = \emptyset$.*

(ii) *For a subset A of $(\mathbb{Z}^2, \kappa^2)$, $A_{mix} \cup A_{\mathcal{F}^2}$ is nowhere dense and so it is semi-closed in $(\mathbb{Z}^2, \kappa^2)$.*

(iii) *If A_i is nowhere dense in $(\mathbb{Z}^2, \kappa^2)$ for each $i \in \Lambda$, where Λ is an index set, then $\bigcup\{A_i | i \in \Lambda\}$ is nowhere dense in $(\mathbb{Z}^2, \kappa^2)$.*

Proof. **(i) (1)⇒(2)** Suppose $F_{\kappa^2} \neq \emptyset$. There exists an open singleton $\{x\}$ such that $x \in F$. Since $Int(Cl(\{x\})) = \{x\}$, we have $x \in Int(Cl(F))$ and so $Int(Cl(F)) \neq \emptyset$; this contradicts to (1). **(2)⇒(3)** Suppose that $Int(F) \neq \emptyset$. There exists a point $y \in Int(F)$ and so

$U(y) \subseteq F$. Thus, we have $(U(y))_{\kappa^2} \subseteq F_{\kappa^2}$ and $F_{\kappa^2} \neq \emptyset$; this contradicts to (2). **(3)⇒(1)** Suppose that $Int(Cl(F)) \neq \emptyset$. There exists a point $x \in Int(Cl(F))$ and so there the smallest open set $U(x)$ containing x such that $U(x) \subset Cl(F)$. Then, since $(U(x))_{\kappa^2} \neq \emptyset$, there exists an open singleton $\{z\}$ such that $z \in U(x)$ and so $\{z\} \subseteq F$, i.e., $z \in Int(F)$; this contradicts to the assumption (3). **(ii)** By definition, it is shown that $(A_{mix} \cup A_{\mathcal{F}^2})_{\kappa^2} = \emptyset$ and so $A_{mix} \cup A_{\mathcal{F}^2}$ is nowhere dense in $(\mathbb{Z}^2, \kappa^2)$ (cf. (i) above). It is clear that every nowhere dense set of a topological space is semi-closed. **(iii)** It follows from assumption and (i) above that $(A_i)_{\kappa^2} = \emptyset$ for each $i \in \Lambda$. Since $(\bigcup\{A_i | i \in \Lambda\})_{\kappa^2} = \bigcup\{(A_i)_{\kappa^2} | i \in \Lambda\}$ holds, we have that $(\bigcup\{A_i | i \in \Lambda\})_{\kappa^2} = \emptyset$ and $\bigcup\{A_i | i \in \Lambda\}$ is nowhere dense in $(\mathbb{Z}^2, \kappa^2)$ (cf. (i) above). □

Theorem 6.12 *For the function* $t_{2a+1,2b+1}$ (resp. $t_{2a,2b+1}, t_{2a+1,2b}$) $: (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$, *where* $a, b \in \mathbb{Z}$*, we have the following property:*
for every preopen subset V *of* $(\mathbb{Z}^2, \kappa^2)$, $(t_{2a+1,2b+1})^{-1}(V)$ (resp. $(t_{2a,2b+1})^{-1}(V)$, $(t_{2a+1,2b})^{-1}(V)$) *is the union of any collection of semi-closed sets of* $(\mathbb{Z}^2, \kappa^2)$.

Proof. Throughout this proof, we denote $W(x) := t^{-1}(\{x\} \cup (U(x))_{\kappa^2})$ for a point $x \in \mathbb{Z}^2$ and a function $t \in \{t_{2a+1,2b+1}, t_{2a,2b+1}, t_{2a+1,2b}\}$.

(1) Let $t_{2a+1,2b+1} : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ be the function. In (1), we first claim that: for any point $x \in \mathbb{Z}^2$,

$(*)$ $W(x)$ is semi-closed in $(\mathbb{Z}^2, \kappa^2)$.

Case 1-1. $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, say $x = (2s, 2m)$, where $s, m \in \mathbb{Z}$: for this case, we show that $(t_{2a+1,2b+1})^{-1}(\{(2s, 2m)\}) = \{(o_1, o_2)\}$, where $o_1 := 2s - 2a - 1$ and $o_2 := 2m - 2b - 1$, and $W(x) = \{(o_1, o_2)\} \cup \{(o_1+1, o_2+1), (o_1-1, o_2-1), (o_1+1, o_2-1), (o_1-1, o_2+1)\}$. Then, we have that $Int(Cl(W(x))) = Int(\{o_1-1, o_1, o_1+1\} \times \{o_2-1, o_2, o_2+1\}) = \{(o_1, o_2)\} \subseteq W(x)$ hold and hence $W(x)$ is semi-closed.

Case 1-2. $x \in (\mathbb{Z}^2)_{\kappa^2}$, say $x = (2s+1, 2m+1)$, where $s, m \in \mathbb{Z}$: for this case, we have that $(t_{2a+1,2b+1})^{-1}(\{(2s+1, 2m+1)\}) = (t_{2a+1,2b+1})^{-1}(U((2s+1, 2m+1))) = \{(e_1, e_2)\}$, where $e_1 := 2s - 2a$ and $e_2 := 2m - 2b$. Then, we have $W(x) = \{(e_1, e_2)\}$ hold and so $W(x)$ is closed; it is semi-closed.

Case 1-3. $x \in (\mathbb{Z}^2)_{mix}$, say $x = (2s, 2m+1)$ or $x = (2s+1, 2m)$, where $s, m \in \mathbb{Z}$: when $x = (2s, 2m+1)$, we have that $(t_{2a+1,2b+1})^{-1}(\{(2s, 2m+1)\}) = \{(o, e)\} \subseteq (\mathbb{Z}^2)_{mix}$, where $o := 2s - 2a - 1$ and $e := 2m - 2b$, and $(t_{2a+1,2b+1})^{-1}((U((2s, 2m+1)))_{\kappa^2}) = \{(o-1, e), (o+1, e)\} \subseteq (\mathbb{Z}^2)_{\mathcal{F}^2}$. Thus, we have $W(x) = W(x) \cap ((\mathbb{Z}^2)_{mix} \cup (\mathbb{Z}^2)_{\mathcal{F}^2}) = (W(x))_{mix} \cup (W(x))_{\mathcal{F}^2}$ and so $W(x)$ is semi-closed (cf. Lemma 6.11 (ii)). When $x = (2s+1, 2m)$, by similar argument above, we have $W(x) = (W(x))_{mix} \cup (W(x))_{\mathcal{F}^2}$ and $W(x)$ is semi-closed.

Thus we showed $(*)$ above. In (1), finally, let V be a preopen set of $(\mathbb{Z}^2, \kappa^2)$. For $(\mathbb{Z}^2, \kappa^2)$, it is proved by [40] that a subset V is preopen in $(\mathbb{Z}^2, \kappa^2)$ if and only if V is expressible as $V = \bigcup\{\{x\} \cup (U(x))_{\kappa^2} | x \in V\}$ ([40]);(cf. Proposition 6.6 (i)). Therefore, $(t_{2a+1,2b+1})^{-1}(V) = \bigcup\{(t_{2a+1,2b+1})^{-1}(\{x\} \cup (U(x))_{\kappa^2}) | x \in V\} = \bigcup\{W(x) | x \in V\}$ and so $(t_{2a+1,2b+1})^{-1}(V)$ is the union of the semi-closed sets $W(x)$, where $x \in V$ (cf. $(*)$ above).

(2) Let $t_{2a,2b+1} : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ be the function and $x \in \mathbb{Z}^2$.

Case 2-1. $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, say $x = (2s, 2m)$, where $s, m \in \mathbb{Z}$: for this case, we show that $(t_{2a,2b+1})^{-1}(\{(2s, 2m)\}) = \{(e, o)\} \subseteq (W(x))_{mix}$, where $e := 2s - 2a$ and $o := 2m - 2b - 1$, and $(t_{2a,2b+1})^{-1}((U((2s, 2m)))_{\kappa^2}) \subseteq (t_{2a,2b+1})^{-1}((\mathbb{Z}^2)_{\kappa^2}) = (\mathbb{Z}^2)_{mix}$ holds. Thus, we have $W(x) = (W(x))_{mix}$ and so $W(x)$ is semi-closed (cf. Lemma 6.11 (ii)).

Case 2-2. $x \in (\mathbb{Z}^2)_{\kappa^2}$, say $x = (2s+1, 2m+1)$, where $s, m \in \mathbb{Z}$: for this case, we have that $(t_{2a,2b+1})^{-1}(\{(2s+1, 2m+1)\}) = (t_{2a,2b+1})^{-1}(U((2s+1, 2m+1))) = \{(o, e)\} \subseteq (\mathbb{Z}^2)_{mix}$, where $o := 2s + 1 - 2a$ and $e := 2m - 2b$; thus, $W(x) = (W(x))_{mix}$ and it is semi-closed (cf. Lemma 6.11 (ii)).

Case 2-3. $x \in (\mathbb{Z}^2)_{mix}$, say $x = (2s, 2m+1)$ or $x = (2s+1, 2m)$, where $s, m \in \mathbb{Z}$: when $x = (2s, 2m+1)$, we have that $(t_{2a,2b+1})^{-1}(\{(2s, 2m+1)\}) = \{(e_1, e_2)\} \subseteq$

$(\mathbb{Z}^2)_{\mathcal{F}^2}$, where $e_1 := 2s-2a$ and $e_2 := 2m-2b$, and $(t_{2a,2b+1})^{-1}((U((2s,2m+1)))_{\kappa^2}) \subseteq (t_{2a,2b+1})^{-1}((\mathbb{Z}^2)_{\kappa^2}) = (\mathbb{Z}^2)_{mix}$ hold. Then, $W(x) = (W(x))_{\mathcal{F}^2} \cup (W(x))_{mix}$ holds and so it is semi-closed (cf. Lemma 6.11 (ii)). When $x = (2s+1, 2m)$, we have $(t_{2a,2b+1})^{-1}(\{(2s+1, 2m)\}) = \{(o_1, o_2)\} \subseteq W(x)$, where $o_1 := 2s-2a+1$ and $o_2 := 2m-2b-1$, and $(t_{2a,2b+1})^{-1}((U((2s+1,2m)))_{\kappa^2}) = \{(o_1, o_2-1), (o_1, o_2+1)\}$ hold. Then, we have that $Int(Cl(W(x)))$=$Int(\{o_1-1, o_1, o_1+1\} \times \{o_2-1, o_2, o_2+1\})$ =$\{(o_1,o_2)\} \subseteq W(x)$ and so $W(x)$ is semi-closed.

For every cases above in (2), $W(x)$ is semi-closed for any point $x \in \mathbb{Z}^2$. By the same argument in the end of the proof of (1) above, for a preopen set V, $(t_{2a,2b+1})^{-1}(V)$= $\bigcup\{(t_{2a,2b+1})^{-1}(\{x\} \cup (U(x))_{\kappa^2})|x \in V\}$ =$\bigcup\{W(x)|x \in V\}$ and so $(t_{2a,2b+1})^{-1}(V)$ is the union of the semi-closed sets $W(x)$, where $x \in V$.

(3) Let $t_{2a+1,2b} : (\mathbb{Z}^2, \kappa^2) \to (\mathbb{Z}^2, \kappa^2)$ be the functiom and $x \in \mathbb{Z}^2$.

Case 3-1. $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, say $x = (2s, 2m)$, where $s, m \in \mathbb{Z}$: for this case, we show that $(t_{2a+1,2b})^{-1}(\{(2s,2m)\}) \subseteq (t_{2a+1,2b})^{-1}((\mathbb{Z}^2)_{\mathcal{F}^2}) \subseteq (\mathbb{Z}^2)_{mix}$ and $(t_{2a+1,2b})^{-1}((U((2s,2m)))_{\kappa^2}) \subseteq (t_{2a+1,2b})^{-1}((\mathbb{Z}^2)_{\kappa^2}) \subseteq (\mathbb{Z}^2)_{mix}$ hold. Thus, we have $W(x) = (W(x))_{mix}$ and so $W(x)$ is semi-closed (cf. Lemma 6.11 (ii)).

Case 3-2. $x \in (\mathbb{Z}^2)_{\kappa^2}$, say $x = (2s+1, 2m+1)$, where $s, m \in \mathbb{Z}$: for this case, we have $W(x) = (t_{2a+1,2b})^{-1}(\{(2s+1, 2m+1)\}) \subseteq (t_{2a+1,2b})^{-1}((\mathbb{Z}^2)_{\kappa^2})$ =$(\mathbb{Z}^2)_{mix}$; by Lemma 6.11 (ii), $W(x)$ is semi-closed.

Case 3-3. $x \in (\mathbb{Z}^2)_{mix}$, say $x = (2s, 2m+1)$ or $x = (2s+1, 2m)$, where $s, m \in \mathbb{Z}$: when $x = (2s, 2m+1)$, we have that $(t_{2a+1,2b})^{-1}(\{(2s, 2m+1)\})$ =$\{(o_1, o_2)\}$, where $o_1 := 2s-2a-1$ and $o_2 := 2m+1-2b$, and $(t_{2a+1,2b})^{-1}(U((2s,2m+1))_{\kappa^2})$= $\{(o_1-1, o_2), (o_1+1, o_2)\}$. For this case, it is shown that $Int(Cl(W(x))) = \{(o_1, o_2)\} \subseteq W(x)$ holds and so $W(x)$ is semi-closed. When $x := (2s+1, 2m)$, it is shown that $W(x) = (W(x))_{\mathcal{F}^2} \cup (W(x))_{mix}$ holds and so it is semi-closed (cf. Lemma 6.11 (ii)).

For every cases above in (3), $W(x)$ is semi-closed for any point $x \in \mathbb{Z}^2$. By the same argument in the end of the proof of (1) above, for a preopen set V, $(t_{2a+1,2b})^{-1}(V)$= $\bigcup\{(t_{2a+1,2b})^{-1}(\{x\} \cup (U(x))_{\kappa^2})|x \in V\})$ =$\bigcup\{W(x)|x \in V\}$ and so $(t_{2a+1,2b})^{-1}(V)$ is the union of the semi-closed sets $W(x)$, where $x \in V$. □

Remark 6.13 **(i)** The functions $t_{2a,2b}, t_{2a+1,2b}, t_{2a,2b+1}, t_{2a+1,2b+1} : \mathbb{Z}^2 \to \mathbb{Z}^2$ (cf. Definition 6.9) are extended from $\mathbb{R}^2$ onto itself. It is obviously shown that $\{t_{2a,2b}, t_{2a+1,2b}, t_{2a,2b+1}, t_{2a+1,2b+1}\} \subseteq h(\mathbb{R}^2; \epsilon^2)$, as functions from the Euklidean plane $(\mathbb{R}^2, \epsilon^2)$ onto itself, where $a, b \in \mathbb{Z}$; however, as functions from a topological space $(\mathbb{Z}^2, \kappa^2)$ onto itself, by Theorem 6.10 (ii), it is shown that $t_{2a+1,2b} \notin h(\mathbb{Z}^2; \kappa^2), t_{2a,2b+1} \notin h(\mathbb{Z}^2; \kappa^2)$ and $t_{2a+1,2b+1} \notin h(\mathbb{Z}^2; \kappa^2)$. This is one of examples of property which shows differences between topological spaces $(\mathbb{R}^2, \epsilon^2)$ and $(\mathbb{Z}^2, \kappa^2)$. As alternative example, it is well known that $(\mathbb{R}^2, \epsilon^2)$ is a Hausdorff space and so a T_1-space; however $(\mathbb{Z}^2, \kappa^2)$ and $(\mathbb{Z}, \kappa)$ are not T_1 (cf. [22, page 7], [8, Example 4.6], e.g [18, Theorems 2.3, 4.8]). **(ii)** We are suggested to define a new kind of class of functions by observing Theorem 6.12.

Finally, we give more examples of functions belonging to $pch(\mathbb{Z}^2; \kappa^2)$ and a corollary. We recall the following known functions from $\mathbb{Z}^2$ onto itself. First we need the functions $\rho^{\mathbb{R}}, r_1^{\mathbb{R}}, r_2^{\mathbb{R}}, u^{\mathbb{R}}$ and $v^{\mathbb{R}}$ from $\mathbb{R}^2$ onto itself defined as follows: for every point $x \in \mathbb{R}^2$, let $\rho^{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation around $x_0 := (0,0)$ by angle $\pi/2$, i.e., $\rho^{\mathbb{R}}(x_1, x_2) = (-x_2, x_1)$; $r_1^{\mathbb{R}}(x_1, x_2) := (x_1, -x_2)$ and $r_2^{\mathbb{R}}(x_1, x_2) := (-x_1, x_2)$; $u^{\mathbb{R}}(x_1, x_2) := (-x_2, -x_1)$ and $v^{\mathbb{R}}(x_1, x_2) := (x_2, x_1)$.

Using the forms of the above functions, we have the following example.

Example 6.14 **(i)** Each function $f \in \{\rho, \rho^2, \rho^3, r_1, r_2, u, v\}$ is preirresolute (cf. Theorem 6.8) and $f \in pch(\mathbb{Z}^2; \kappa^2)$, where $\rho, \rho^2, \rho^3, r_1, r_2, u$ and v are defined as follows: for

$x =: (x_1, x_2) \in \mathbb{Z}^2$,
(1) $\rho(x) := \rho^{\mathbb{R}}(x) = (-x_2, x_1)$; (2) for each $i \in \{1, 2\}$, $r_i(x) := r_i^{\mathbb{R}}(x)$, i.e., $r_1(x) = (x_1, -x_2)$ and $r_2(x) = (-x_1, x_2)$; (3) $u(x) := u^{\mathbb{R}}(x) = (-x_2, -x_1)$; (4) $v(x) := v^{\mathbb{R}}(x) = (x_2, x_1)$. The proof of $f \in pch(\mathbb{Z}^2; \kappa^2)$ is done using Theorem 6.8 and they are omitted. **(ii)** By using (i) above and Theorem 6.7 (vi), it is obtained that each function f of (i) above is a homeomorphism from $(\mathbb{Z}^2, \kappa^2)$ onto itself, because $f(U(x)) = U(f(x))$ and $f^{-1}(U(x)) = U(f^{-1}(x))$ hold for every point $x \in \mathbb{Z}^2$ (cf. Proof of Theorem 6.10 (i) and Corollary 6.15 below).

We are inspired the following corollary by studying Example 6.14, Theorem 6.10 (i) and Theorem 6.7 (vi).

Corollary 6.15 *For $(\mathbb{Z}^2, \kappa^2)$, $h(\mathbb{Z}^2; \kappa^2) = pch(\mathbb{Z}^2; \kappa^2)$ holds.*

Proof. We have $h(\mathbb{Z}^2; \kappa^2) \subseteq pch(\mathbb{Z}^2; \kappa^2)$ (cf. Theorem 4.3 (iii)). We claim that $pch(\mathbb{Z}^2; \kappa^2) \subseteq h(\mathbb{Z}^2; \kappa^2))$ holds. Let $f \in pch(\mathbb{Z}^2; \kappa^2)$. Using Theorem 6.7 (vi) for the function f, we have the property:$f(U(x)) = U(f(x))$ for every point $x \in \mathbb{Z}^2$. Let $G \in \kappa^2$. Then, for each point $z \in G$ there exists the smallest open set $U(z)$ containing z such that $U(z) \subseteq G$ and so $f(U(z)) = U(f(z)) \subseteq f(G)$. We have that $f(G) = \bigcup\{U(f(z)) | z \in G\}$ and $f(G) \in \kappa^2$; thus, this shows that f^{-1} is continuous. By an argument similar to that above, it is shown that f is continuous; thus, $f \in h(\mathbb{Z}^2; \kappa^2)$. Therefore, we show $pch(\mathbb{Z}^2; \kappa^2) = h(\mathbb{Z}^2; \kappa^2)$. $\square$

Question Let $x_0 := (0, 0)$ and $H := U(x_0)$, i.e., $H = \{-1, 0, 1\} \times \{-1, 0, 1\}$. Study an analogous property to Proposition 6.1 on $pch(H; \kappa|H)$ and $pch(\mathbb{Z}^2, \mathbb{Z}^2 \setminus H; \kappa^2)$.

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