Scientiae Mathematicae Japonicae Online, **e-2009**, 637–648

ON π GENERALIZED β-CLOSED SETS IN TOPOLOGICAL SPACES II

S.C. Arora, Sanjay Tahiliani and H. Maki

Received June 4, 2009

Abstract. The concept of $\pi g\beta$-closed sets is introduced and investigated by Tahiliani [12] earlier. In the present paper we investigate some more properties of $\pi g\beta$-closed sets. Their relations in group theory and digital line are investigated.

1 Introduction Throughout the present paper, $(X, \tau), (Y, \sigma)$ and (Z, η) (or X, Y and Z) represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. The closure and interior of a subset $A \subseteq X$ will be denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a topological space (X, τ) is called *β-open* [1] or *semi-preopen* [2] if $A \subseteq Cl(Int(Cl(A)))$. The compliment of a β-open set is called *β-closed* [1]. The intersection of all β-closed sets containing A is called *β-closure* of A and it is denoted by $\beta Cl(A)$. A subset A of (X, τ) is called *regular open* (resp. *regular closed*) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A finite union of regular open set is said to be *π-open*. The complement of π-open set is said to be *π-closed* [3]. A subset A of (X, τ) is said to be $g\beta$-closed [5] (resp. $\pi g\beta$-closed [12]) if $\beta Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open (resp. π-open). It's complement is said to be $g\beta$-open (resp. $\pi g\beta$-open). Using the concept of β-closed sets, classes of some functions (e.g., β-irresoluteness [11], pre-β-closedness [11], *gsp*-irresolute [5] (or $g\beta$-irresoluteness), $\pi g\beta$-irresoluteness [12] and contra $g\beta$-irresoluteness and contra $\pi g\beta$-irresoluteness) are introduced (cf. Definition 2.1, Definition 4.1 below).

The present paper is a continuation of [12] due to one of the present authors; we investigate more properties of functions preserving $\pi g\beta$-closed sets, some groups of such functions and properties on digital line (so called the Khalimsky line) [7],[8],[9],[10], e.g., [6]. In Section 2, we recall some definitions on functions and we need some properties on functions (cf. Lemma 2.2 and Theorem 2.3). In Section 3, for a topological space (X, τ), we introduce and investigate goups of functions, say $\pi g\beta ch(X, \tau), g\beta ch(X, \tau), \beta ch(X, \tau)$, preserving $\pi g\beta$-closed sets, $g\beta$-closed sets and β-closed sets, respecticely; they contain the homeomorphism group $h(X, \tau)$ as a subgroup (cf. Theorem 3.3). Morever, these groups have an importante property that they are one of topological invariants (Theorem 3.4). Using the concept of contra-β-irresoluteness (resp. contra-$g\beta$-irresoluteness, contra-$\pi g\beta$-irresoluteness), in Section 4, we construct more groups of functions, say $\beta ch(X, \tau) \cup$ *con*-$\beta ch(X, \tau), g\beta ch(X, \tau) \cup$ *con-g*$\beta ch(X, \tau)$ and $\pi g\beta ch(X, \tau) \cup$ *con*-$\pi g\beta ch(X, \tau)$ for a topological space (X, τ); they contain the homeomorpism group $h(X, \tau)$ as a subgroup (cf. Theorem 4.4). They are also examples of topological invariants (cf. Theorem 4.5). Some examples on the digital line $(\mathbb{Z}, \kappa)$ are given in Section 5. If A is a β-open set of $(\mathbb{Z}, \kappa)$, the inverse image by a digital translation $f_{2m+1} : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa)$, say $f_{2m+1}^{-1}(A)$, is expressible to the union of any β-closed sets. Namely, $f_{2m+1} \in$ *con-st*-$\beta h(\mathbb{Z}, \kappa)$ (cf. Theorem 5.10 (iii)).

2000 *Mathematics Subject Classification.* Primary:54C10,54D10.

Key words and phrases. $\pi g\beta$-closed sets, $g\beta$-closed sets, preopen sets, preirresolute maps, preclosed maps, β-irresolute maps, gβ-irresolute maps, πgβ-irresolute, $\pi\beta$-sets, digital lines, homeomorpisms group.

2 Preliminalies We need the following definition, lemma and theorem:

Definition 2.1 For topological spaces (X,τ) and (Y,σ), a function $f:(X,\tau)\to(Y,\sigma)$ is said to be:

(i) *π-irresolute* [3] (resp. *β-irresolute* [11]), if $f^{-1}(V)$ is π-closed (resp. β-closed) in (X,τ) for every π-closed set (resp. β-closed set) V of (Y,σ);

(ii) *pre-β-closed* [11], if $f(V)$ is β-closed in (Y,σ) for every β-closed set V of (X,τ);

(iii) *gsp-irresolute* [5] or *$g\beta$-irresolute*, if $f^{-1}(F)$ is $g\beta$-closed (X,τ) for every $g\beta$-closed set F of (Y,σ);

(iv) *$\pi g\beta$-irresolute* [12], if $f^{-1}(V)$ is $\pi g\beta$-closed in (X,τ) for every $\pi g\beta$-open set V of (Y,σ).

Lemma 2.2 *Let $f:(X,\tau)\to(Y,\sigma)$ and $g:(Y,\sigma)\to(Z,\zeta)$ be two functions between topological spaces.*

(i-1) *If f and g are $\pi g\beta$-irresolute* (cf. [12]), *then the composition $g\circ f$ is also $\pi g\beta$-irresolute.*

(i-2) *The identity function $1_X:(X,\tau)\to(X,\tau)$ is $\pi g\beta$-irresolute.*

(ii-1) *If f and g are $g\beta$-irresolute, then the composition $g\circ f$ is also $g\beta$-irresolute.*

(ii-2) *The identity function $1_X:(X,\tau)\to(X,\tau)$ is $g\beta$-irresolute.*

(iii)([11, Theorem 2.7 (i)]) *If f and g are β-irresolute, then the composition $g\circ f$ is also β-irresolute. The identity function $1_X:(X,\tau)\to(X,\tau)$ is β-irresolute.*

Proof. The proofs are obvious from definitions. □

Theorem 2.3 *Let $f:(X,\tau)\to(Y,\sigma)$ be a function.*

(i) *If f is a homeomorphism, then f is π-irresolute.*

(ii) *If f is a homeomorphism, then f is pre-β-closed* (i.e., *f^{-1} is β-irresolute*).

(iii) *If f is a homeomorphism, then $f(A)$ is $\pi g\beta$-closed in (Y,σ) for every $\pi g\beta$-closed set A of (X,τ)* (i.e., *f^{-1} is $\pi g\beta$-irresolute* [12, Definition 4.2, Theorem 4.2]).

(iv) *Every homeomorphism is $\pi g\beta$-irresolute, $g\beta$-irresolute and β-irresolute.*

Proof. **(i)** Let A be a π-open set of (Y,σ), say $A=\bigcup\{V_i|i\in\{1,2,...,m\}\}$, where m is a positive integer and V_i is regular open in (Y,σ) for each i with $1\le i\le m$. Since f is a homeomorphism, $f^{-1}(V_i)=f^{-1}(Int(Cl(V_i)))=Int(Cl(f^{-1}(V_i)))$ holds for each $i(1\le i\le m)$ and so $f^{-1}(A)=\bigcup\{f^{-1}(V_i)|i\in\{1,2,...,m\}\}$ holds. Namely, by definition, $f^{-1}(A)$ is π-open in (X,τ). Thus, we have that f is π-irresolute. Indeed, in general, a function is π-irresolute if and only if an inverse image of every π-open set is π-open.

(ii) Let V be a β-closed set of (X,τ), i.e., $Int(Cl(Int(V)))\subseteq V$ holds. Because of the homeomorphism on f, it is shown that $f(Int(Cl(Int(V))))=Cl(Int(Cl(f(V))))\subseteq f(V)$ and so $f(V)$ is β-closed in (Y,σ).

(iii) By (i) and (ii), f is π-irresolute and pre-β-closed. It follows from [12, Theorem 4.2] that if A is $\pi g\beta$-closed in (X,τ) then $f(A)$ is $\pi g\beta$-closed in (Y,σ).

(iv) Let f be a homeomorphism. Then, $f^{-1}:(Y,\sigma)\to(X,\tau)$ is also a homeomorphism. By (iii) for the homeomorphism f^{-1}, it is shown that $f=(f^{-1})^{-1}$ is $\pi g\beta$-irresolute. Let F be a $g\beta$-closed set (Y,σ). Let U be an open subset of (X,τ) such that $f^{-1}(F)\subseteq U$. Then, $F=f(f^{-1}(F))\subseteq f(U)$ and $f(U)$ is open in (Y,σ). It follows from the $g\beta$-closedness of F that $\beta Cl(F)\subseteq f(U)$ and so $f^{-1}(\beta Cl(F))=f^{-1}(F)\cup Int(Cl(Int(f^{-1}(F))))=\beta Cl(f^{-1}(F))\subseteq U$. Thus we have that $f^{-1}(F)$ is $g\beta$-closed in (X,τ). Hence, f is $g\beta$-irresolute. It is similarly proved that f is β-irresolute. □

3 More on functions preserving $\pi g\beta$-closed sets, $g\beta$-closed sets, β-closed sets

Definition 3.1 (i) A function $f : (X,\tau) \to (Y,\sigma)$ is called a *$\pi g\beta c$-homeomorphism* (resp. *$g\beta c$-homeomorphism*) if f is a $\pi g\beta$-irresolute (resp. $g\beta$-irresolute) bijection and f^{-1} is πgβ-irresolute (resp. $g\beta$-irresolute).

(ii) A function $f : (X,\tau) \to (Y,\sigma)$ is called a *βc-homeomorphism* if f is a β-irresolute bijection and f^{-1} is β-irresolute.

For a topological space (X,τ), we introduce the following:
(1) $\pi g\beta ch(X;\tau) := \{f |\ f : (X,\tau) \to (X,\tau)$ is a $\pi g\beta c$-homeomorphism$\}$;
(2) $g\beta ch(X;\tau) := \{f |\ f : (X,\tau) \to (X,\tau)$ is a $g\beta c$-homeomorphism$\}$;
(3) $\beta ch(X;\tau) := \{f |\ f : (X,\tau) \to (X,\tau)$ is a βc-homeomorphism$\}$;
(4) $h(X;\tau) := \{f |\ f : (X,\tau) \to (X,\tau)$ is a homeomorphism$\}$.

Theorem 3.2 *For a topological space (X,τ), the following properties hold.*
(i) $h(X;\tau) \subseteq \pi g\beta ch(X;\tau)$.
(ii) $h(X;\tau) \subseteq g\beta ch(X;\tau)$.
(iii) $h(X;\tau) \subseteq \beta ch(X;\tau)$.

Proof. Let $f \in h(X;\tau)$. Then, by Theorem 2.3 (iii) (iv) (resp. (v), (ii)) and Definition 3.1 (i) (resp. (i), (ii)), it is shown that f and f^{-1} are $\pi g\beta$-irresolute (resp. $g\beta$-irresolute, β-irresolute) and so f is $\pi g\beta$c-homeomorphism (resp. $g\beta c$-homeomorphism, βc-homeomorphism), i.e., $f \in \pi g\beta ch(X;\tau)$ (resp. $f \in g\beta ch(X;\tau)$, $f \in \beta ch(X;\tau)$) . □

Theorem 3.3 *Let (X,τ) be a topological space. Then, we have the following properties.*
(i) *The collection $\pi g\beta ch(X;\tau)$ forms a group under the composition of functions.*
(ii) *The collection $g\beta ch(X;\tau)$ forms a group under the composition of functions.*
(iii) *The collection $\beta ch(X;\tau)$ forms a group under the composition of functions.*
(iv) *The homeomorphism group $h(X;\tau)$ is a subgroup of the group $\pi g\beta ch(X;\tau)$.*
(v) *The homeomorphism group $h(X;\tau)$ is a subgroup of the group $g\beta ch(X;\tau)$.*
(vi) *The homeomorphism group $h(X;\tau)$ is a subgroup of the group $\beta ch(X;\tau)$.*

Proof. **(i-1)** A binary operation $\eta_X : \pi g\beta ch(X;\tau) \times \pi g\beta ch(X;\tau) \to \pi g\beta ch(X;\tau)$ is well defined by $\eta_X(a,b) := b \circ a$, where $b \circ a : X \to X$ is the composite function of the functions a and b such that $(b \circ a)(x) := b(a(x))$ for every point $x \in X$. Indeed, by Lemma 2.2 (i), it is shown that, for every $\pi g\beta c$-homeomorphisms a and b, the composition $b \circ a$ is also $\pi g\beta c$-homeomorphism. Namely, for every pair $(a,b) \in \pi g\beta ch(X,\tau), \eta_X(a,b) = b \circ a \in \pi g\beta ch(X;\tau)$. Then, it is claimed that the binary operation $\eta_X : \pi g\beta ch(X;\tau) \times \pi g\beta ch(X;\tau) \to \pi g\beta ch(X;\tau)$ satisfies the axiom of group. Namely, putting $a \cdot b := \eta_X(a,b)$, the following properties hold $\pi g\beta ch(X;tau)$.

(1) $((a \cdot b) \cdot c) = (a \cdot (b \cdot c))$ holds for every elements $a, b, c \in \pi g\beta ch(X;\tau)$;

(2) for all element $a \in \pi g\beta ch(X;\tau)$, there exists an element $e \in \pi g\beta ch(X;\tau)$ such that $a \cdot e = e \cdot a = a$ hold in $\pi g\beta ch(X;\tau)$;

(3) for each element $a \in \pi g\beta ch(X;\tau)$, there exists an element $a_1 \in \pi g\beta ch(X;\tau)$ such that $a \cdot a_1 = a_1 \cdot a = e$ hold in $\pi g\beta ch(X;\tau)$.

Indeed, the proof of (1) is obvious; the proof of (2) is obtained by taking $e := 1_X$, where 1_X is the identity function on X and using Lemma 2.2(i-2); the proof of (3) is obtained by taking $a_1 := a^{-1}$ for each $a \in \pi g\beta ch(X;\tau)$ and Definition 3.1, where a^{-1} is the inverse function of a. Therefore, by definition of groups, the pair $(\pi g\beta ch(X;\tau), \eta_X)$ forms a group under the composition of functions, i.e., $\pi g\beta ch(X;\tau)$ is a group.

(ii) Let $\eta'_X : g\beta ch(X;\tau) \times g\beta ch(X;\tau) \to g\beta ch(X;\tau)$ be a binary operation defined by $\eta'_X(a,b) := b \circ a$ (the composition) for every pair $(a,b) \in g\beta ch(X;\tau) \times g\beta ch(X;\tau)$. Then, by using Lemma 2.2 (ii-1), (ii-2) and the same argument as that in the proof of (i) above, it is shown that the collection $g\beta ch(X;\tau)$ forms a group under the composition of functions.

(iii) Let $\eta''_X : \beta ch(X;\tau) \times \beta ch(X;\tau) \to \beta ch(X;\tau)$ be a binary operation defined by $\eta''_X(a,b) := b \circ a$ (the composition) for every pair $(a,b) \in \beta ch(X;\tau) \times \beta ch(X;\tau)$. Then, by using Lemma 2.2 (iii) and the same argument as that in the proof of (i) above, it is shown that the collection $\beta ch(X;\tau)$ forms a group under the composition of functions.

(iv) It is obvious that $1_X : (X,\tau) \to (X,\tau)$ is a homeomorphism and so $h(X;\tau) \neq \emptyset$. It follows from Theorem 3.2(i) that $h(X;\tau) \subseteq \pi g\beta ch(X;\tau)$. Let $a,b \in h(X;\tau)$. Then we have that $\eta_X(a,b^{-1}) = b^{-1} \circ a \in h(X;\tau)$, here $\eta_X : \pi g\beta ch(X;\tau) \times \pi g\beta ch(X;\tau) \to \pi g\beta ch(X;\tau)$ is the binary operation (cf. Proof of Theorem 3.3(i)). Therefore, the group $h(X;\tau)$ is a subgroup of $\pi g\beta ch(X;\tau)$.

(v) Let $a,b \in h(X;\tau)$. Then we have that $\eta'_X(a,b^{-1}) = b^{-1} \circ a \in h(X;\tau)$, where $\eta'_X : g\beta ch(X;\tau) \times g\beta ch(X;\tau) \to g\beta ch(X;\tau)$ is the binary operation (cf. Proof of Theorem 3.3(ii)). By this binary operation, the group $h(X;\tau)$ is a subgroup of $g\beta ch(X;\tau)$ (cf. Theorem 3.2(ii)).

(vi) We have that $\eta''_X(a,b^{-1}) = b^{-1} \circ a \in h(X;\tau)$ for every $a,b \in h(X;\tau)$, where $\eta''_X : \beta ch(X;\tau) \times \beta ch(X;\tau) \to \beta ch(X;\tau)$ is the binary operation (cf. Theorem 3.2(iii)). It is shown that $h(X;\tau)$ is a subgroup of $\beta ch(X;\tau)$. □

Theorem 3.4 *Let (X,τ) and (Y,σ) be topological spaces.*
If (X,τ) and (Y,σ) are homeomorphic, then there exist isomorphisms:
(i) $\pi g\beta ch(X,\tau) \cong \pi g\beta ch(Y,\sigma)$;
(ii) $g\beta ch(X,\tau) \cong g\beta ch(Y,\sigma)$;
(iii) $\beta ch(X,\tau) \cong \beta ch(Y,\sigma)$.

Proof. It follows from assumption that there exsts a homeomorphism, say $f : (X,\tau) \to (Y,\sigma)$. We define a function $f_* : \pi g\beta ch(X,\tau) \to \pi g\beta ch(Y,\sigma)$ by $f_*(a) := f \circ a \circ f^{-1}$ for every element $a \in \pi g\beta ch(X,\tau)$; by Theorem 2.3 (iv) (or Theorem 3.2) and Lemma 2.2 (i-1), the bijections $f \circ a \circ f^{-1}$ and $(f \circ a \circ f^{-1})^{-1}$ are $\pi g\beta$-irresolute and so f_* is well defined. The induced function f_* is a homomorphism. Indeed, $f_*(\eta_X(a,b)) = f \circ b \circ f^{-1} \circ f \circ a \circ f^{-1} = (f_*(b)) \circ (f_*(a)) = \eta_X(f_*(a), f_*(a))$ hold. Obviously, f_* is bijective. Thus, we have (i), i.e., f_* is an isomorphism. By using Theorem 2.3 (iv) (or Theorem 3.2) and Lemma 2.2 (ii-1) (resp. Lemma 2.2 (iii)), (ii) (resp. (iii)) is obtained with similar argument above. □

In Theorem 3.4, by using Lemma 2.2 (i-2), (ii-2), (iii), it is obtained that $f_*(1_X) = 1_Y$ holds.

4 More on the groups including the homeomorphism group $h(X;\tau)$ as subgroup

Definition 4.1 For a topological spaces (X,τ) and (Y,σ), we define the following functions. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be *contra β-irresolute* [4] (resp. *contra $g\beta$-irresolute, contra $\pi g\beta$-irresolute*) if $f^{-1}(V)$ is β-closed (resp. $g\beta$-closed, $\pi g\beta$-closed) in (X,τ) for every β-open (resp. $g\beta$-open, $\pi g\beta$-open) set V of (Y,σ).

For these, we can immediately see the following lemma:

Lemma 4.2 *Let $f : (X,\tau) \to (Y,\sigma)$ and $g : (Y,\sigma) \to (Z,\zeta)$ be two functions between topological spaces.*
(i-1) *If f and g are contra-β-irresolute, then the composition $g \circ f$ is also β-irresolute.*

(i-2) *If f is β-irresolute* (resp. *contra-β-irresolute*) *and g are contra-β-irresolute* (resp. *β-irresolute*), *then the composition $g \circ f$ is contra-β-irresolute.*

(ii-1) *If f and g are contra-$g\beta$-irresolute, then the composition $g \circ f$ is also $g\beta$-irresolute.*

(ii-2) *If f is $g\beta$-irresolute* (resp. *contra-$g\beta$-irresolute*) *and g are contra-$g\beta$-irresolute* (resp. *$g\beta$-irresolute*), *then the composition $g \circ f$ is contra-$g\beta$-irresolute.*

(iii-1) *If f and g are contra-$\pi g\beta$-irresolute, then the composition $g \circ f$ is also $\pi g\beta$-irresolute.*

(iii-2) *If f is $\pi g\beta$-irresolute* (resp. *contra-$\pi g\beta$-irresolute*) *and g are contra-$\pi g\beta$-irresolute* (resp. *$\pi g\beta$-irresolute*), *then the composition $g \circ f$ is contra-$g\beta$-irresolute.* □

Definition 4.3 For a topological space (X, τ), we define the following collection of functions:

(1) *con-$\beta ch(X;\tau)$* $:= \{f| \ f : (X,\tau) \to (X,\tau)$ is a contra-β-irresolute bijection and f^{-1} is contra-β-irresolute $\}$;

(2) *con-$g\beta ch(X;\tau)$* $:= \{f| \ f : (X,\tau) \to (X,\tau)$ is a contra-$g\beta$-irresolute bijection and f^{-1} is contra-$g\beta$-irresolute $\}$;

(3) *con-$\pi g\beta ch(X;\tau)$* $:= \{f| \ f : (X,\tau) \to (X,\tau)$ is a contra-$\pi g\beta$-irresolute bijection and f^{-1} is contra-$\pi g\beta$-irresolute$\}$.

For a topological space (X, τ), we construct alternative groups, say $\beta ch(X;\tau) \cup$*con-$\beta ch(X;\tau)$*, $g\beta ch(X;\tau) \cup$ *con-$g\beta ch(X;\tau)$* and $\pi g\beta ch(X;\tau) \cup$ *con-$\pi g\beta ch(X;\tau)$*.

Theorem 4.4 *Let (X, τ) be a topological space. Then, we have the following properties.*

(i) *The union of two collections, $\beta ch(X;\tau) \cup$ con-$\beta ch(X;\tau)$, forms a group under the composition of functions.*

(ii) *The union of two collections, $g\beta ch(X;\tau) \cup$ con-$g\beta ch(X;\tau)$, forms a group under the composition of functions.*

(iii) *The union of two collections, $\pi g\beta ch(X;\tau) \cup$ con-$\pi g\beta ch(X;\tau)$, forms a group under the composition of functions.*

(iv) *The group $\beta ch(X;\tau)$* (resp. *$g\beta ch(X;\tau)$, $\pi g\beta ch(X;\tau)$*) *is a subgroup of $\beta ch(X;\tau) \cup$ con-$\beta ch(X;\tau)$* (resp. *$g\beta ch(X;\tau) \cup$ con-$g\beta ch(X;\tau)$, $\pi g\beta ch(X;\tau) \cup$ con-$\pi g\beta ch(X;\tau)$*).

(v) *The homeomorphism group $h(X;\tau)$ is a subgroup of $\beta ch(X;\tau) \cup$ con-$\beta ch(X;\tau)$* (resp. *$g\beta ch(X;\tau) \cup$ con-$g\beta ch(X;\tau)$, $\pi g\beta ch(X;\tau) \cup$ con-$\pi g\beta ch(X;\tau)$*).

Proof. **(i)** Let $B_X := \beta ch(X;\tau) \cup$*con-$\beta ch(X;\tau)$*. A binary operation $w_X : B_X \times B_X \to B_X$ is well defined by $W_X(a,b) := b \circ a$, where $b \circ a : X \to X$ is the composite function of the functions a and b. Indeed, let $(a,b) \in B_X$; if $a \in \beta ch(X;\tau)$ and $b \in$ *con-$\beta ch(X;\tau)$*, then $b \circ a : (X,\tau) \to (X,\tau)$ a contra-β-irresolute bijection and $(b \circ a)^{-1}$ is also contra-β-irresolute and so $w_X(a,b) = b \circ a \in$ *con-$\beta ch(X;\tau)$* $\subset B_X$ (cf. Lemma 4.2 (i-2)); if $a \in \beta ch(X;\tau)$ and $b \in \beta ch(X;\tau)$, then $b \circ a : (X,\tau) \to (X,\tau)$ is a β-irresolute bijection and so $w_X(a,b) = b \circ a \in \beta ch(X,\tau) \subseteq B_X$ (cf. Lemma 2.2 (iii)); if $a \in$ *con-$\beta ch(X;\tau)$* and $b \in$ *con-$\beta ch(X;\tau)$*, then $b \circ a : (X,\tau) \to (X,\tau)$ is a β-irresolute bijection and $(b \circ a)^{-1}$ is also β-irresolute and so $w_X(a,b) = b \circ a \in \beta ch(X;\tau) \subseteq B_X$ (cf. Lemma 4.2 (i-1)); if $a \in$ *con-$\beta ch(X;\tau)$* and $b \in \beta ch(X;\tau)$, then $b \circ a : (X,\tau) \to (X,\tau)$ is a contra-β-irresolute bijection and $(b \circ a)^{-1}$ is also contra-β-irresolute and so $w_X(a,b) = b \circ a \in$ *con-$\beta ch(X;\tau)$* $\subseteq B_X$ (cf. Lemma 4.2 (i-2)). By the similar arguments of Theorem 3.3, it is claimed that the binary operation $w_X : B_X \times B_X \to B_X$ satisfies the axiom of group; for the identity element e of $B_X, e := 1_X : (X,\tau) \to (X,\tau)$ (the identity function). Thus, the pair (B_X, w_X) forms a group under the composition of functions, i.e., $\beta ch(X;\tau) \cup$ *con-$\beta ch(X;\tau)$* is a group.

(ii) (resp. (iii)) The proof is obtained by similar arguments of (i) above using Lemma 4.2 (ii-1), (ii-2)) (resp. (iii-1), (iii-2)) and Lemma 2.2 (ii-1), (ii-2)) (resp. (i-1), (i-2)) in the place of Lemma 4.2 (i-1), (i-2) and Lemma 2.2 (iii).

(iv) The group $\beta ch(X;\tau)$ (resp. $g\beta ch(X;\tau)$, $\pi g\beta ch(X;\tau)$) is not empty (cf. Lemma 2.2 (iii) (resp. (ii-2), (i-2)). Using the binary operation in the proof (i) above, it is shown that $w_X(a,b^{-1}) = b^{-1}\circ a \in \beta ch(X;\tau)$ for any $a,b\in\beta ch(X;\tau)$ and so $\beta ch(X;\tau)$ is a subgroup of $\beta ch(X;\tau)\cup con\text{-}\beta ch(X;\tau)$. For the other cases, they are similarly proved (cf. Proof of (ii),(iii) above).

(v) By Theorem 3.3 (vi) above, it is shown that $h(X;\tau)$ is a subgroup of $\beta ch(X;\tau)\cup con\text{-}\beta ch(X;\tau)$ (resp. $g\beta ch(X;\tau)\cup con\text{-}g\beta ch(X;\tau)$, $\pi g\beta ch(X;\tau)\cup con\text{-}\pi g\beta ch(X;\tau)$). □

The groups of Theorem 4.4 are also invariant concepts under homeomorphisms between topological spaces (cf. Theorem 3.4).

Theorem 4.5 *Let (X,τ) and (Y,σ) be topological spaces.*
If (X,τ) and (Y,σ) are homeomorphic, then there exist isomorphisms:
(i) $\beta ch(X;\tau)\cup con\text{-}\beta ch(X;\tau)\cong\beta ch(Y;\sigma)\cup con\text{-}\beta ch(Y;\sigma)$;
(ii) $g\beta ch(X;\tau)\cup con\text{-}g\beta ch(X;\tau)\cong g\beta ch(Y;\sigma)\cup con\text{-}g\beta ch(Y;\sigma)$;
(iii) $\pi g\beta ch(X;\tau)\cup con\text{-}\pi g\beta ch(X;\tau)\cong\pi g\beta ch(Y;\sigma)\cup con\text{-}\pi g\beta ch(Y;\sigma)$.

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be a homeomorphism. We put $B_X := \beta ch(X;\tau)\cup con\text{-}\beta ch(X;\tau)$ (resp. $B_Y := \beta ch(Y;\sigma)\cup con\text{-}\beta ch(Y;\sigma)$) for a topological space (X,τ) (resp. (Y,σ)). First we have a well defined function $f_*: B_X\to B_Y$ by $f_*(a) := f\circ a\circ f^{-1}$ for every element $a\in B_X$. Indeed, by Theorem 2.3 (iv) (or Theorem 3.2), f and f^{-1} are β-irresolute; by Lemma 2.2 (iii) and Lemma 4.2 (i-2), the bijections $f\circ a\circ f^{-1}$ and $(f\circ a\circ f^{-1})^{-1}$ are β-irresolute or contra-β-irresolute and so f_* is well defined. The induced function f_* is a homomorphism. Indeed, $f_*(w_X(a,b)) = f\circ b\circ f^{-1}\circ f\circ a\circ f^{-1} = (f_*(b))\circ(f_*(a)) = w_Y(f_*(a),f_*(a))$ hold, $w_X: B_X\times B_X\to B_X$ and $w_Y: B_Y\times B_Y\to B_Y$ are the binary operations defined in Proof of Theorem 4.4 (i). Obviously, f_* is bijective. Thus, we have the isomorphism of (i). By using Theorem 2.3 (iv) (or Theorem 3.2), Lemma 2.2 (ii-1) (resp. Lemma 2.2 (iii-1)) and Lemma 4.2 (ii-2) (resp. Lemma 4.2 (iii-2)), the isomorphism of (ii) (resp. (iii)) is obtained with similar argument above. □

In Theorem 4.5, by using Lemma 2.2 (i-2), (ii-2), (iii), it is obtained that $f_*(1_X) = 1_Y$ holds.

5 Examples on digital line $(\mathbb{Z},\kappa)$

Definition 5.1 The *digital line* ([7], [8], [9], [10], e.g.,[6]) or so called the *Khalimsky line* is the set of all integers $\mathbb{Z}$, equipped with the topology κ having $\{\{2m-1,2m,2m+1\}|m\in\mathbb{Z}\}$ as a subbase; the digital line is denotedd $(\mathbb{Z},\kappa)$.

A subset V is open in $(\mathbb{Z},\kappa)$ if and only if whenever $x\in V$ and x is an even integer, then $x-1,x+1\in V$ (cf. [10, page 175]). It is clear that a singleton $\{2s+1\}$ is open, a singleton $\{2m\}$ is closed and a subset $\{2k-1,2k,2k+1\}$ is the smallest open set containing $2k$, where s,m and k are any integers. In the present paper (cf. [6]), we use the following notation:
$U(2s+1) := \{2s+1\}$ and $U(2s) := \{2s-1,2s,2s+1\}$ for each $s\in\mathbb{Z}$,
$\mathbb{Z}_\kappa := \{x\in\mathbb{Z}|\{x\}$ is open in $(\mathbb{Z},\kappa)\}$,
$\mathbb{Z}_{\mathcal{F}} := \{x\in\mathbb{Z}|\{x\}$ is closed in $(\mathbb{Z},\kappa)\}$,
for a subset A of $(\mathbb{Z},\kappa)$, $A_\kappa := A\cap\mathbb{Z}_\kappa$ and $A_{\mathcal{F}} := A\cap\mathbb{Z}_{\mathcal{F}}$.
Obviously, we have that $\mathbb{Z}=\mathbb{Z}_\kappa\cup\mathbb{Z}_{\mathcal{F}}$ (disjoint union) and, for a subset A, $A = A_\kappa\cup A_{\mathcal{F}}$ (disjoint union).

Example 5.2 For a fixed integer m, we define the functions $f_{2m}:(\mathbb{Z},\kappa)\to(\mathbb{Z},\kappa)$ and $f_{2m+1}:(\mathbb{Z},\kappa)\to(\mathbb{Z},\kappa)$, respectively:

$f_{2m}(x) := x + 2m$ for every point $x \in \mathbb{Z}$;
$f_{2m+1}(x) := x + (2m + 1)$ for every point $x \in \mathbb{Z}$.
We claim that:
(a) f_{2m+1} is not continuous and so $f_{2m+1} \notin h(\mathbb{Z};\kappa)$ (cf. Theorem 5.10 (i));
(b) $f_{2m+1} \notin \beta ch(\mathbb{Z};\kappa)$ (cf. Theorem 5.10 (i));
(c) there exists a β-open set A such that $f^{-1}_{2m+1}(A)$ is β-closed (cf. in general, Theorem 5.10 (i), (ii) below).
(d) $f_{2m} \in h(\mathbb{Z};\kappa)$ (cf. Theorem 5.10 (i), (ii-3)).

Proof of (a). Because $f^{-1}_{2m+1}(\{1\}) = \{1 - (2m + 1)\} = \{-2m\} \notin \kappa$ for a subset $\{1\} \in \kappa$. Thus, $f_{2m+1} : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa)$ is not continuous.

Proof of (b) and (c). The function f_{2m+1} is not β-irresolute. Indeed, a subset $U(2u+1) := \{2u+1\}$ is a β-open, where $u \in \mathbb{Z}$, because $\{2u+1\} \in \kappa$. We have that $f^{-1}_{2m+1}(U(2u+1)) = \{2u+1-(2m+1)\} = \{2(u-m)\}$ and $Cl(Int(Cl(f^{-1}_{2m+1}(U(2u+1))))) = Cl(Int(Cl(\{2(u-m)\}))) = Cl(Int(\{2(u-m)\})) = \emptyset \not\supseteq f^{-1}_{2m+1}(U(2u+1))$. Thus, we have that $f^{-1}_{2m+1}(U(2u+1))$ is not β-open for a β-open set $U(2u + 1)$. Namely, f_{2m+1} is not β-irresolute. Put $A := U(2u + 1) = \{2u + 1\}$. Then, $Int(Cl(Int(f^{-1}_{2m+1}(A)))) = \emptyset \subseteq f^{-1}_{2m+1}(A)$ holds and so $f^{-1}_{2m+1}(A)$ is β-closed.

Proof of (d). By the definition of the topology κ, an open subset A is expressible as $A = \bigcup\{U(x)|x \in A\}$, where $U(x)$ is the smallest open set containing x (i.e., $U(2s) := \{2s-1, 2s, 2s+1\}$ and $U(2u+1) := \{2u+1\}(s, u \in \mathbb{Z})$). Then we have that $f^{-1}_{2m}(U(2u+1)) = \{2u+1-2m\} \in \kappa$ and $f^{-1}_{2m}(U(2s)) = \{2s-1-2m, 2s-2m, 2s+1-2m\} \in \kappa$. Therefore, we have that $f^{-1}_{2m}(A) = \bigcup\{f^{-1}_{2m}(U(x))|x \in A\} \in \kappa$ and hence f_{2m} is continuous and bijective. Similarly, it is shown that $f^{-1}_{2m} : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa)$ is continuous. Therefore, $f_{2m} \in h(\mathbb{Z};\kappa)$. □

We characterlize β-open sets of $(\mathbb{Z}, \kappa)$ (cf. Theorem 5.7 below). First we need the following definition:

Definition 5.3 For a nonempty subset A of $(\mathbb{Z}, \kappa)$, we introduce the following subsets of $(\mathbb{Z}, \kappa)$.
(i) ([6]) $A_{\mathcal{F}} := \{x \in A|\{x\}$ is closed in $(\mathbb{Z}, \kappa)\}$.
(ii) For a point $x \in \mathbb{Z}$ and a subset $A \subseteq \mathbb{Z}$, $V_A(x) := \{x, x+1\}$ if $x+1 \in A$ (sometimes it is denoted by $V^+(x)$); $V_A(x) := \{x-1, x\}$ if $x+1 \notin A$ (sometimes, it is denoted by $V^-(x)$). We note that the concept of $V_A(x)$ is uniquely well determined for each point $x \in \mathbb{Z}$ and A.
(iii) $V_A := \bigcup\{V_A(x)|x \in A_{\mathcal{F}}\}$, where $A_{\mathcal{F}} \neq \emptyset$.

Example 5.4 In order to understand the concept of the set V_A for a subset A, we see some examples.
(i) Let $A := \{0, 4, 7\}$. The set A is not β-open in $(\mathbb{Z}, \kappa)$. Indeed, by definition, $Cl(Int(Cl(A))) = Cl(Int(\{0, 4, 6, 7, 8\})) = Cl(\{7\}) = \{6, 7, 8\} \not\supseteq A$ hold. We note that $A_{\mathcal{F}} := \{x \in A|\{x\}$ is closed (i.e., x is even)$\}$; $A_\kappa := \{x \in A|\{x\}$ is open (i.e., x is odd)$\}$. Then, $A_{\mathcal{F}} = \{0, 4\}$ and $A_\kappa = \{7\}$. For this set $A_{\mathcal{F}}$, we have $V_A := \bigcup\{V_A(x)|x \in A_{\mathcal{F}}\} = V^-(0) \cup V^-(4) = \{-1, 0\} \cup \{3, 4\}$ and we have the set $V_A \cup A_\kappa$ as follow: $V_A \cup A_\kappa = \{-1, 0\} \cup \{3, 4\} \cup \{7\} \neq A$. Using Theorem 5.7 below, it is concluded also that A is not β-open, because $A \neq V_A \cup A_\kappa$.
(ii) Let $A := \{0, 1, 3, 4, 9, 11\}$. Then, we have $V_A \cup A_\kappa = V^+(0) \cup V^-(4) \cup A_\kappa = \{0, 1\} \cup \{3, 4\} \cup \{1, 3, 9, 11\} = A$. We have that A has an expression of the following form: $A = V_A \cup A_\kappa$. By Theorem 5.7 below, the set A is β-open in $(\mathbb{Z}, \kappa)$. We have directly that $Cl(Int(Cl(A))) = Cl(Int(\{0, 1, 2, 3, 4, 8, 9, 10, 11, 12\})) = Cl(U(2) \cup U(10)) = Cl(\{1, 2, 3, 9, 10, 11\}) = \{0, 1, 2, 3, 4\} \cup \{8, 9, 10, 11, 12\} \supseteq A$ and so A is β-open in $(\mathbb{Z}, \kappa)$.

Example 5.5 Let $2s, 2u \in \mathbb{Z}_{\mathcal{F}}$ and $2m+1 \in \mathbb{Z}_\kappa$, where $s, u, m \in \mathbb{Z}$.

(i) A subset $V^+(2s)$ is β-open and β-closed.

Indeed, $Cl(Int(Cl(V^+(2s)))) = Cl(Int(Cl(\{2s, 2s+1\}))) = Cl(Int(\{2s, 2s+1, 2s+2\})) = Cl(\{2s+1\}) = \{2s, 2s+1, 2s+2\} \supset \{2s, 2s+1\} = V^+(2s)$ and so $V^+(2s)$ is β-open; $Int(Cl(Int(V^+(2s)))) = Int(Cl(\{2s+1\})) = Int(\{2s, 2s+1, 2s+2\}) = \{2s+1\} \subset V^+(2s)$ and so $V^+(2s)$ is β-closed.

(ii) A subset $V^-(2s)$ is β-open and β-closed. Indeed, we have that $Cl(Int(Cl(V^-(2s)))) = Cl(Int(Cl(\{2s-1, 2s\}))) = Cl(Int(\{2s-2, 2s-1, 2s\})) = Cl(\{2s-1\}) = \{2s-2, 2s-1, 2s\} \supset \{2s-1, 2s\} = V^-(2s)$ and so $V^-(2s)$ is β-open; $Int(Cl(Int(V^-(2s)))) = Int(Cl(\{2s-1\})) = Int(\{2s-2, 2s-1, 2s\}) = \{2s-1\} \subset \{2s-1, 2s\} = V^-(2s)$ and so $V^-(2s)$ is β-closed.

(iii) A subset $V^-(2s) \cup V^+(2s+2)$ is β-open and β-closed.
Indeed, by (i) and (ii), the union $V^-(2s) \cup V^+(2s+2)$ is β-open. Since $Int(Cl(Int(V^-(2s) \cup V^+(2s+2)))) = Int(Cl(Int(\{2s-1, 2s, 2s+2, 2s+3\}))) = Int(Cl(\{2s-1, 2s+3\})) = Int(\{2s-2, 2s-1, 2s, 2s+2, 2s+3, 2s+4\}) = \{2s-1, 2s+3\} \subset V^-(2s) \cup V^+(2s+2)$, we have that $V^-(2s) \cup V^+(2s+2)$ is β-closed.

(iv) A subset $V^+(2u) \cup V^-(2u+4)$ is β-open; it is not β-closed; by (i) and (ii), $V^+(2u)$ and $V^-(2u+4)$ are β-closed.
Indeed, $Int(Cl(Int(V^+(2u) \cup V^-(2u+4)))) = Int(Cl(Int(\{2u, 2u+1, 2u+3, 2u+4\}))) = Int(Cl(\{2u+1, 2u+3\})) = Int(\{2u, 2u+1, 2u+2, 2u+3, 2u+4\}) = \{2u+1, 2u+2, 2u+3\} \not\subseteq V^+(2u) \cup V^-(2u+4))$ hold and so $V^+(2u) \cup V^-(2u+4)$ is not β-closed.

(v) A subset $\bigcup\{V_A(x) | x \in A_{\mathcal{F}}\}$, say V_A, is β-open, where $A_{\mathcal{F}} \neq \emptyset$. It is obtainrd by (i) and (ii) above and the well know fact that an arbitrary union of β-open sets is β-open in general (eg. [12]).

(vi) A subset $V^+(2m+1)$ is β-open and β-closed. Indeed, the proof is similar to one of (ii) above, because $V^+(2m+1) = \{2m+1, 2m+2\}$.

(vii) A subset $V^-(2m+1)$ is β-open and β-closed. Since $V^-(2m+1) = \{2m, 2m+1\}$, it is obtained by the proof in (i) above.

Definition 5.6 A subset F of a topological space (X, τ) is called:

(i) a *$\pi\beta$-set* of (X, τ), if F is expressible to the union of finitely β-closed sets;

(ii) a *stably $\pi\beta$-set* of (X, τ), if F is expressible to the union of any collection of β-closed sets.

We have a characterization on β-opennese of subsets in $(\mathbb{Z}, \kappa)$ as follows.

Theorem 5.7 *Let A be a subset of $(\mathbb{Z}, \kappa)$.*

(i) *Assume that $A_{\mathcal{F}} \neq \emptyset$.*

(i-1) *If A is β-open, then A is expressible as the union:$V_A \cup A_\kappa$, where $V_A := \bigcup\{V_A(x) | x \in A_{\mathcal{F}}\}$* (cf. Definition 5.3 (iii)).

(i-2) *If A satiesfies a property that $A = V_A \cup A_\kappa$, then A is β-open.*

(ii) *Assume that $A_{\mathcal{F}} = \emptyset$. Then, $V_A = \emptyset$ and $A = A_\kappa$ hold and A is open; it is β-open.*

Proof. **(i) (i-1)** We have that $A \subseteq V_A \cup A_\kappa$, because $A = A_\kappa \cup A_{\mathcal{F}}$ and $A_{\mathcal{F}} \subseteq V_A$ hold in general. Conversely, in order to prove that $V_A \cup A_\kappa \subseteq A$, let $y \in V_A \cup A_\kappa$.

Case 1. $y \in A_\kappa$: for this case, $y \in A$, because $A_\kappa \subseteq A$ in general.

Case 2. $y \in V_A$: for this case, there exists a point x such that $y \in V_A(x)$ and $x \in A_{\mathcal{F}}$. Then, $x = 2s$ for some integer $s \in \mathbb{Z}$ and $x \in A$. Because A is β-open, by [14], it is concluded that $A \subseteq Cl(A_\kappa)$ holds. Since $x = 2s \in A_{\mathcal{F}} \subseteq A$, we have that $U(x) \cap A_\kappa \neq \emptyset$, where $U(x) = \{x-1, x, x+1\}$ is the smallest open set containing the point $x = 2s$. If $y = x$, then $y \in A$. Hence, we suppose that $y \neq x$. We note that $y \in V_A(x) \subseteq U(x)$.
(Case 2-1). If $x+1 \in A$, then $V_A(x) = V^+(x) = \{x, x+1\}$ and so $y = x+1 \in A$ because

$y \neq x$.
(Case 2-2). If $x+1 \notin A$, then $V_A(x) = V^-(x) = \{x-1, x\}$ and so $y = x-1$ because $y \neq x$. Since $\{x-1, x, x+1\} \cap A_\kappa \neq \emptyset, x \notin A_\kappa$ and $x+1 \notin A_\kappa$, we have that $x-1 \in A_\kappa$ and hence $y = x-1 \in A$.
Thus we obtain that $y \in A$ for this point for Case 2.
Therefore, we prove that $V_A \cup A_\kappa \subseteq A$ and hence $V_A \cup A_\kappa = A$.

(i-2) Suppose that $A = V_A \cup A_\kappa$. We recall that $V_A := \bigcup\{V_A(x)|x \in A_\mathcal{F}\}$ and $V_A(x) = \{x, x+1\}$ or $V_A(x) = \{x-1, x\}$, where $x \in A_\mathcal{F}$. We first show that $\{x\} \subseteq Cl((V_A(x))_\kappa)$ for a point $x \in A_\mathcal{F}$. Indeed, if $V_A(x) = V^+(x)$, then $Cl((V_A(x))_\kappa) = Cl(V^+(x) \setminus \{x\}) = Cl(\{x+1\}) = \{x, x+1, x+2\}$; if $V_A(x) = V^-(x)$, then $Cl((V_A(x))_\kappa) = Cl(\{x-1\}) = \{x-2, x-1, x\}$; thus $x \in Cl((V_A(x))_\kappa)$. Secondly, by using the property above, it is shown that $Cl((V_A)_\kappa) = Cl((\bigcup\{V_A(x)|x \in A_\mathcal{F}\})_\kappa) \supseteq \bigcup\{Cl((V_A(x))_\kappa)|x \in A_\mathcal{F}\} \supseteq \bigcup\{\{x\}|x \in A_\mathcal{F}\} = A_\mathcal{F}$, i.e., $Cl((V_A)_\kappa) \supseteq A_\mathcal{F}$. Finally, using the assumption of (i-2), we show that $Cl(A_\kappa) = Cl((V_A \cup A_\kappa)_\kappa) = Cl((V_A)_\kappa \cup (A_\kappa)_\kappa) = Cl((V_A)_\kappa) \cup Cl(A_\kappa) \supseteq A_\mathcal{F} \cup A_\kappa = A$ and hence $Cl(A_\kappa) \supseteq A$. By [14], it is concluded that A is β-open in $(\mathbb{Z}, \kappa)$.

(ii) If $A_\mathcal{F} = \emptyset$, then $V_A = \emptyset$ and $A = A_\kappa$, because $A_\mathcal{F} = \emptyset$ and $A = A_\kappa \cup A_\mathcal{F}$ (disjoint union); A is open and hence A is β-open. □

We need the following notation:
$T^e(\mathbb{Z};\kappa) := \{f_{2m}|m \in \mathbb{Z}\}$, $T^o(\mathbb{Z};\kappa) := \{f_{2m+1}|m \in \mathbb{Z}\}$ and $T(\mathbb{Z};\kappa) := T^e(\mathbb{Z};\kappa) \cup T^o(\mathbb{Z};\kappa)$, where $f_{2m}(x) := x+2m$ and $f_{2m+1}(x) = x+2m+1$ for every $x \in \mathbb{Z}$ and for an integer m.

Lemma 5.8 *Let A and E be subsets of $\mathbb{Z}$. We have the following properties on the function $f_{2m+1} : \mathbb{Z} \to \mathbb{Z}$, where $m \in \mathbb{Z}$:*

(i) (i-1) $f^{-1}_{2m+1}(A_\mathcal{F}) = (f^{-1}_{2m+1}(A))_\kappa$ *and* $f_{2m+1}(E_\mathcal{F}) = (f_{2m+1}(E))_\kappa$ *hold;*

(i-2) $f^{-1}_{2m+1}(A_\kappa) = (f^{-1}_{2m+1}(A))_\mathcal{F}$ *and* $f_{2m+1}(E_\kappa) = (f_{2m+1}(E))_\mathcal{F}$ *hold.*

(ii) *For a point $x \in A_\mathcal{F}$, $f^{-1}_{2m+1}(V_A(x)) = V_B(f^{-1}_{2m+1}(x))$ holds, where $B := f^{-1}_{2m+1}(A)$.*

(iii) $f^{-1}_{2m+1}(V_A) = \bigcup\{V_B(y)|y \in (f^{-1}_{2m+1}(A))_\kappa\}$, *where $B := f^{-1}_{2m+1}(A)$ and V_A is defined by* Definition 5.3 (iii).

(iv) (Example 5.5 (v)) *V_A is β-open.*

(v) *$f^{-1}_{2m+1}(V_A)$ is β-open.*

(vi) *If A is a finite subset of $(\mathbb{Z}, \kappa)$ with $A_\mathcal{F} \neq \emptyset$, then V_A and $f^{-1}_{2m+1}(V_A)$ are the union of a finitely β-closed sets. Namely, they are $\pi\beta$-sets* (cf. Definition 5.6 (i)).

(vii) *If $A_\mathcal{F} \neq \emptyset$, then V_A and $f^{-1}_{2m+1}(V_A)$ are the union of any collection of β-closed sets. Namely, they are stably $\pi\beta$-sets* (cf. Definition 5.6 (ii)).

Proof. **(i) (i-1)** It is shown that $f^{-1}_{2m+1}(A_\mathcal{F}) = \{x-(2m+1)|x \in A_\mathcal{F}\} = \{2s-(2m+1)|2s \in A, s \in \mathbb{Z}\} = (\{x-(2m+1)|x \in A\})_\kappa$ hold, because $x-(2m+1) \in \mathbb{Z}_\kappa$ if and only if $x \in \mathbb{Z}_\mathcal{F}$. The later equality is obtained by similar argument. **(i-2)** They are proved by using (i-1).

(ii) For a point $x \in A_\mathcal{F}$, we have the following two cases:

(Case 1). $x+1 \in A$: for this case, we have that $f^{-1}_{2m+1}(x+1) \in f^{-1}_{2m+1}(A)$ and so $y+1 \in B$, where $y := f^{-1}_{2m+1}(x)$ and $B := f^{-1}_{2m+1}(A)$. Thus, for the subset B and the point $y, V_B(y)$ is well defined and $V_B(y) = V^+(y) = \{y, y+1\}$ (cf. Definition 5.3 (ii)). Then, since $V_A(x) = V^+(x)$ for this point x, it is shown that $f^{-1}_{2m+1}(V_A(x)) = f^{-1}_{2m+1}(\{x, x+1\}) = \{y, y+1\} = V_B(y) = V_B(f^{-1}_{2m+1}(x))$.

(Case 2). $x+1 \notin A$: for this case, we have that $y+1 \notin B$, where $y := f^{-1}_{2m+1}(x)$ and $B := f^{-1}_{2m+1}(A)$. Thus, $V_B(y)$ is well defined and $V_B(y) = V^-(y) = \{y-1, y\}$ (cf. Definition 5.3 (ii)). Then, since $V_A(x) = V^-(x)$ for this point x, it is shown that $f^{-1}_{2m+1}(V_A(x)) = f^{-1}_{2m+1}(\{x-1, x\}) = \{y-1, y\} = V_B(f^{-1}_{2m+1}(x))$.

(iii) Using (i) above, we note that $x \in A_{\mathcal{F}}$ if and only if $f_{2m+1}^{-1}(x) \in (f_{2m+1}^{-1}(A))_\kappa$. Then, we have that $f_{2m+1}^{-1}(V_A) = \bigcup\{f_{2m+1}^{-1}(V_A(x))|x \in A_{\mathcal{F}}\} = \bigcup\{V_B(z)|z \in (f_{2m+1}^{-1}(A))_\kappa\}$, where $B := f_{2m+1}^{-1}(A)$ (cf. (ii) above).

(v) The subset $f_{2m+1}^{-1}(V_A)$ is the union of a collection of β-open sets (cf. (iii) above and Example 5.5 (vi),(vii)). Thus, $f_{2m+1}^{-1}(V_A)$ is β-open.

(vi) The set V_A (resp. $f_{2m+1}^{-1}(V_A)$) is the union of finitely β-closed sets (cf. Definition 5.3 (iii) and Example 5.5 (i), (ii) (resp. (iii) above and Example 5.5 (vi), (vii))). Thus, V_A (resp. $f_{2m+1}^{-1}(V_A)$) is a $\pi\beta$-set (cf. Definition 5.6 (i)).

(vii) It is obvious from (iii) above and Example 5.5 (vi),(vii). □

The above Lemma 5.8 (vi) and (vii) suggest the following concepts:

Definition 5.9 (i-1) A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:

contra-stably-$\pi\beta$-continuous (briefly, *contra-st-$\pi\beta$-continuous*) if $f^{-1}(F)$ is a stably $\pi\beta$-set of (X, τ) for every β-open set F of (Y, σ).

(i-2) A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:
a *contra-stably-$\pi\beta$-homeomorphism* (briefly, *contra-st-$\pi\beta$-homeomorphism*) if f is a contra-stably $\pi\beta$-continuous bijection and f^{-1} is contra-stably $\pi\beta$-continuous.

(ii) For a topological space (X, τ), we denote a collection of all contra-st-$\pi\beta$-homeomorhisms from (X, τ) onto itself as follows:
con-st-$\pi\beta h(X;\tau)$ $:= \{f|f : (X, \tau) \to (X, \tau)$ is a contra-st-$\pi\beta$-homeomorphism$\}$.

Theorem 5.10 (i) *For a topological space (X, τ), we have the following implications:*
$h(X;\tau) \subseteq \beta ch(X;\tau) \cup$ *con-$\beta ch(X;\tau)$* $\subseteq \beta ch(X;\tau)\cup$ *con-st-$\pi\beta h(X;\tau)$.*

(ii) *For each integer m, the following properties hold:*

(ii-1) $f_{2m+1} \notin$ *con-$\beta ch(\mathbb{Z};\kappa)$*, $f_{2m+1} \notin \beta ch(\mathbb{Z};\kappa)$;

(ii-2) $f_{2m+1} \notin h(\mathbb{Z};\kappa)$;

(ii-3) (Example 5.2 (d)) $f_{2m} \in h(\mathbb{Z};\kappa)$.

(iii) *For each integer m, f_{2m+1} is contra-stably $\pi\beta$-continuous and so f_{2m+1} is a contra-stably $\pi\beta$-homeomorphism. Namely, $f_{2m+1} \in$ con-st-$\pi\beta h(\mathbb{Z};\kappa)$.*

(iv) *The collection $T^e(\mathbb{Z};\kappa)$ is a subgroup of $h(\mathbb{Z},\kappa)$.*

(v) *The collection $T^e(\mathbb{Z};\kappa)\cup T^o(\mathbb{Z};\kappa)$, say $T(\mathbb{Z};\kappa)$, forms a group under the compositions of functions; the group $T(\mathbb{Z};\kappa)$ is included in the family $h(\mathbb{Z};\kappa)\cup$ con-st-$\pi\beta h(\mathbb{Z};\kappa)$* (cf. (ii-3), (iii), (iv) *above*).

Proof. **(i)** By Theorem 4.4 (v), it was obtained that $h(X, \tau) \subseteq \beta ch(X, \tau) \cup$ *con-$\beta ch(X, \tau)$*. By definitions, it is shown that every contra-β-irresolute function, say f, is contra-stably $\pi\beta$-continuous. Indeed, for a β-open set A, $f^{-1}(A)$ is a β-closed set and so it is a stably $\pi\beta$-set (cf. Definition 4.1, Definition 5.9 (i-1)). Therefore, we have that *con-$\beta ch(X;\tau)$* $\subseteq$ *con-st-$\pi\beta h(X;\tau)$* and so the required implication.

(ii-1) (ii-2) Let $A := V^-(2s) \cup V^+(2s + 2)$, where $s \in \mathbb{Z}$. Then, the subset A is β-open (also it is β-closed) (cf. Example 5.5 (iii)) and $f_{2m+1}^{-1}(A) = V^+(2u) \cup V^-(2u + 4)$ is not β-closed and it is β-open (cf. Example 5.5 (iv)), where $u := s - m - 1$ and so $2u = 2s - (2m + 1) - 1, 2u + 4 = 2s + 2 - (2m + 1) + 1$. Therefore, f_{2m+1} is not contra-β-irresolute; f_{2m+1} is not β-irresolute. and hence $f_{2m+1} \notin$ *con-$\beta ch(\mathbb{Z};\kappa)$*; $f_{2m+1} \notin \beta ch(\mathbb{Z};\kappa)$.

(iii) Let A be a β-open set of $(\mathbb{Z}, \kappa)$. First, suppose that $A_{\mathcal{F}} \neq \emptyset$. Using Theorem 5.7 (i-1), we put $A = V_A \cup A_\kappa$. For a point $x = 2s \in A_{\mathcal{F}}$, where $s \in \mathbb{Z}$, we set $B := f_{2m+1}^{-1}(A)$. Then, we have that if $x + 1 \in A$, then $f_{2m+1}^{-1}(V_A(x)) = \{x - (2m + 1), x + 1 - (2m + 1)\} = \{2(s - m) - 1, 2(s - m)\} = V^+(2(s - m) - 1) = V_B(f_{2m+1}^{-1}(x))$; if $x + 1 \notin A$, then $f_{2m+1}^{-1}(V_A(x)) = \{x - 1 - (2m + 1), x - (2m + 1)\} = \{2(s - m - 1), 2(s - m) - 1\} =$

$V^-(2(s-m)-1) = V_B(f_{2m+1}^{-1}(x))$. Using Theorem 5.7 (i-1) and Lemma 5.8 (i-2),(vii), we have that

($*$) $f_{2m+1}^{-1}(A) = (f_{2m+1}^{-1}(V_A)) \cup (f_{2m+1}^{-1}(A))_{\mathcal{F}}$ and

($**$) $f_{2m+1}^{-1}(V_A)$ is a stably $\pi\beta$-set.

Since the subset $(f_{2m+1}^{-1}(A))_{\mathcal{F}}$ is closed (cf. [13, Lemma 2.6 (ii)]), it is β-closed. Thus, it follows from ($*$) and ($**$) that the above subset $f_{2m+1}^{-1}(A)$ is a stably $\pi\beta$-set. Finally, suppose that $A_{\mathcal{F}} = \emptyset$. Then, $A = A_\kappa$ holds $(\mathbb{Z}, \kappa)$ (cf. Theorem 5.7 (ii)). We have that, by Lemma 5.8 (i-2), $f_{2m+1}^{-1}(A) = f_{2m+1}^{-1}(A_\kappa) = (f_{2m+1}^{-1}(A))_{\mathcal{F}}$ and so $f_{2m+1}^{-1}(A)$ is a closed set (cf. [13, Lemma 2.6 (ii)]); thus $f_{2m+1}^{-1}(A)$ is a stably $\pi\beta$-set.

For both cases above, $f_{2m+1}^{-1}(A)$ is a stably $\pi\beta$-set for every β-open set A. Namely, f_{2m+1} is stably $\pi\beta$-continuous. Since f_{2m+1} is bijective and $(f_{2m+1})^{-1} = f_{-(2m+1)}$ holds, $f_{2m+1} : (\mathbb{Z}, \kappa) \to (\mathbb{Z}, \kappa)$ is a contra-stably $\pi\beta$-homeomorphism.

(iv) Let $a, b \in T^e(\mathbb{Z}; \kappa)$. Then, there exist integers m and s such that $a = f_{2m}$ and $b = f_{2s}$. Since the binary operation $W_{\mathbb{Z}} : h(\mathbb{Z}; \kappa) \times h(\mathbb{Z}; \kappa) \to h(\mathbb{Z}; \kappa)$ is defined by $W_{\mathbb{Z}}(a', b') := b' \circ a'$ for every $a', b' \in h(\mathbb{Z}; \kappa)$ and $T^e(\mathbb{Z}; \kappa) \subset h(\mathbb{Z}; \kappa)$ (cf. (ii-3) above), we have that $W_{\mathbb{Z}}(a, b^{-1}) = (f_{2s})^{-1} \circ f_{2m} = f_{2(m-s)} \in T^e(\mathbb{Z}; \kappa)$. Moreover, $f_0 = 1_{\mathbb{Z}} \in T^e(\mathbb{Z}; \kappa) \neq \emptyset$ hold and so $T^e(\mathbb{Z}; \kappa)$ is a subgroup of $h(\mathbb{Z}; \kappa)$.

(v) Let $a, b \in T^e(\mathbb{Z}; \kappa) \cup T^o(\mathbb{Z}; \kappa)$. Then, if $a, b \in T^e(\mathbb{Z}; \kappa)$, then $b \circ a \in T^e(\mathbb{Z}; \kappa)$; if $a \in T^e(\mathbb{Z}; \kappa)$ and $b \in T^o(\mathbb{Z}; \kappa)$, then $b \circ a \in T^o(\mathbb{Z}; \kappa)$; if $a \in T^o(\mathbb{Z}; \kappa)$ and $b \in T^e(\mathbb{Z}; \kappa)$, then $b \circ a \in T^o(\mathbb{Z}; \kappa)$; if $a, b \in T^o(\mathbb{Z}; \kappa)$, then $b \circ a \in T^e(\mathbb{Z}; \kappa)$. Thus, a binary operation $W'_{\mathbb{Z}} : T(\mathbb{Z}; \kappa) \times T(\mathbb{Z}; \kappa) \to T(\mathbb{Z}; \kappa)$ is well defined by the composition of functions. It is obviously $T(\mathbb{Z}; \kappa)$ forms a group. Let $f \in T(\mathbb{Z}; \kappa)$; then if $f \in T^e(\mathbb{Z}; \kappa)$, then $f \in h(\mathbb{Z}; \kappa)$ (cf. (iv) above); if $f \in T^0(\mathbb{Z}; \kappa)$, then $f \in$ *con-st-*$\pi\beta h(\mathbb{Z}; \kappa)$ (cf. (iii) above). Therefore, we have $T(\mathbb{Z}; \kappa) \subseteq h(\mathbb{Z}; \kappa) \cup$ *con-st-*$\pi\beta h(\mathbb{Z}; \kappa)$ as subset. □

References

[1] M.E. Abd El-Monsef, S.N.El-Deeb and R.A. Mahmoud, *β-open sets and β-continuous mappings*, *Bull. Fac. Sci. Assiut Univ.*, **12**(1983), 77-90.

[2] D. Andrijević, *Semi-preopen sets*, Mat. Vesnik, **38**(1986), 24–32.

[3] G. Aslim, A. Cakshu Guler and T. Noiri, *On πgs-closed sets in topological spaces*, Acta. Math. Hungar., **112**(4) (2004), 275–283.

[4] M. Caldas and S. Jafari, *Weak and strong form of β-irresoluteness*, Arabian J. Sci. Engrg., **31** 1A(2006), 31-39.

[5] J.Dontchev, *On generalizing semi-preopen sets*, Mem. Fac. Sci. Kochi Univ. Ser.A, (Math.), **16**(1995), 35-48.

[6] M. Fujimoto, S. Takigawa, J. Dontchev, T. Noiri and H. Maki, *The topological structure and groups of digital n-spaces*, Kochi J. Math.,**1**(2006), 31-55.

[7] E.D.Khalimsky,*Applications of connected ordered topological spaces in topology*, Conference of Math. Department of Povolosia, (1970).

[8] E.D.Khalimsky, R.Kopperman and P.R.Meyer,*Computer graphics and connected topologies in finite ordered sets*, Topology Appl., **36**(1990), 1–17.

[9] T.Y.Kong, R.Kopperman and P.R.Meyer,*A topological approach to digital topology*, Amer. Math. Monthly, **98**(1991), 901–917.

[10] V. Kovalevsky and R.Kopperman,*Some topology-based image processing algorithms*, Ann. New York Acad. Sci., Papers on General Topology and Applications, **728**(1994), 174–182.

[11] R.A Mahmoud and M.E.Abd El-Monsef, *β-irresolute and β-topological invariant*, Proc. Pakistan Acad. Sci., **27**(1990), 285-296.

[12] S. Tahiliani, *On $\pi g\beta$-closed sets in topological spaces* (accepted for publication, Note di Math.)

[13] S. Takigawa and H. Maki,*Every nonempty open set of the digital n-space is expressible as the union of finitely many nonempty regular open sets*, Sci. Math. Jpn., **67**(3)(2008), 395–376;**e2007**, 601-612.

[14] S. Takigawa,*On characterizations on preopen sets and semi-open sets of digital n-spaces* (preparing).

[15] S. Takigawa, M. Ganster, H. Maki, T. Noiri and M. Fujimoto, *The digital n-space is quasi-submaximal*, Questions Answers Gen. Topology, **26**(2008), 45–52.

S.C. Arora:
Sanjay Tahiliani:
(Affiliation)
Department of Mathematics
Delhi University, Delhi-110007, India
(e-mail address):scrora@maths.du.ac.in
(e-mail address):sanjaytahiliani@yahoo.com

Haruo Maki:
(Postal address) Wakagi-dai 2-10-13, Fukutsu-shi
Fukuoka-ken, 811-3221 Japan
(e-mail address):makih@pop12.odn.ne.jp