GIAMBELLI'S FORMULA AND THE BEST CONSTANT OF SOBOLEV INEQUALITY IN ONE DIMENSIONAL EUCLIDEAN SPACE

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Abstract. The best constant of Sobolev inequality associated with $2N$-th order Hurwitz-type differential operator is computed. Giambelli's formula which appears in representation theory of finite groups plays an important role.

1 Conclusion

For $N = 1, 2, 3, \cdots$, we introduce the following characteristic polynomial with real coefficients.

$$Q(z) \,=\, \prod_{j=0}^{N-1}(z+a_j) \,=\, \sum_{j=0}^{N} q_j\, z^{N-j} \tag{1.1}$$

We impose the following three equivalent assumptions.

Assumption 1.1 *$Q(z)$ is Hurwitz polynomial with distinct characteristic roots.*

Assumption 1.2 *Suppose that $N = L + 2M$ $(L, M = 0, 1, 2, \cdots)$*

$$\begin{array}{ll} a_i \,\neq\, a_j \quad (i \neq j), \qquad a_j \,>\, 0 \quad (0 \leq j \leq L-1) \\ a_{L+j} \,=\, \overline{a}_{L+M+j}, \quad \operatorname{Re} a_{L+j} \,>\, 0, \quad \operatorname{Im} a_{L+j} \,>\, 0 \qquad (0 \leq j \leq M-1) \end{array}$$

Assumption 1.3

$$\text{G.C.D.}\,(\,Q(z), Q'(z)\,) \,=\, 1, \qquad \left| \; q_{-i+2j+1} \; \right|_{0 \leq i,j \leq k-1} > 0 \qquad (k = 1, 2, \cdots, N)$$

where $q_k = 0$ $(k < 0$ or $k > N)$.

In relation to $Q(z)$, we introduce another polynomial $P(z)$ defined by

$$P(z) \,=\, \prod_{j=0}^{N-1}(z+a_j^2) \,=\, \sum_{j=0}^{N} p_j\, z^{N-j} \tag{1.2}$$

which satisfies $P(-z^2) = Q(-z)Q(z)$. The following relations hold

$$p_k \,=\, \sum_{j=0}^{2k}(-1)^{k+j} q_j\, q_{2k-j} \qquad (0 \leq k \leq N)$$

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where $q_j = 0$ $(N+1 \le j < \infty)$. From Assumption 1.3 we have $q_j > 0$ $(0 \le j \le N-1)$. We also have $p_0 = 1$, $p_N = \prod_{j=0}^{N-1} a_j^2 > 0$ from (1.2).

We introduce Sobolev space

$$H \;=\; H(N) \;=\; \left\{ u(x) \;\middle|\; u^{(i)}(x) \,\in\, L^2(-\infty,\infty) \quad (0 \le i \le N) \right\} \tag{1.3}$$

equipped with Sobolev inner product

$$(u,v)_H \;=\; \int_{-\infty}^{\infty} \Big(Q(D)\, u(x) \Big) \overline{\Big(Q(D)\, v(x) \Big)} dx \tag{1.4}$$

In section 4, $(\cdot,\cdot)_H$ is shown to be an inner product of H and rewritten as

$$(u,v)_H \;=\; \int_{-\infty}^{\infty} \sum_{j=0}^{N} p_j\, u^{(N-j)}(x)\, \overline{v}^{(N-j)}(x)\, dx \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\xi^2)\, \widehat{u}(\xi)\, \overline{\widehat{v}}(\xi)\, d\xi, \tag{1.5}$$

where $D = d/dx$ and $\widehat{u}(\xi)$ is Fourier transform of $u(x)$,

$$\widehat{u}(\xi) \;=\; \int_{-\infty}^{\infty} e^{-\sqrt{-1}\xi x} u(x)\, dx \qquad (-\infty < \xi < \infty).$$

We also introduce Sobolev energy

$$\| \, u \, \|_H^2 \;=\; (u,u)_H \;=\; \int_{-\infty}^{\infty} \Big| \, Q(D)\, u(x) \Big|^2 dx \;=$$

$$\int_{-\infty}^{\infty} \sum_{j=0}^{N} p_j \, \Big| \, u^{(N-j)}(x) \Big|^2 dx \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\xi^2)\, \left| \, \widehat{u}(\xi) \, \right|^2 d\xi. \tag{1.6}$$

The purpose of this paper is to find the supremum of Sobolev functional given by

$$S(u) \;=\; \left(\sup_{-\infty<y<\infty} |\, u(y) \,| \right)^2 \Big/ \| \, u \, \|_H^2. \tag{1.7}$$

In our previous paper [1], we have obtained the supremum of $S(u)$ under the assumption that $P(z) = \sum_{j=0}^{N} p_j z^{N-j}$ is factorized as follows.

$$P(z) = \prod_{j=0}^{N-1} (z + a_j), \quad 0 < a_0 < a_1 < \cdots < a_{N-1}$$

We extend the above result in the case $P(z)$ is given by (1.2), where not all the coefficients p_j $(0 \le j \le N-1)$ are positive.

We introduce a function $G(x,y) = G(x-y)$ defined by

$$G(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\sqrt{-1}\, x\xi}\, \widehat{G}(\xi)\, d\xi \qquad (-\infty < x < \infty) \tag{1.8}$$

where

$$\widehat{G}(\xi) = \frac{1}{P(\xi^2)} \qquad (-\infty < \xi < \infty) \tag{1.9}$$

As is shown later in section 2, the above function $G(x-y)$ is Green function of the boundary value problem for $2N$-th order differential operator $P(-D^2)$. We remark that the inequality

$$\delta\left(\xi^{2N}+1\right)^{-1} \le \widehat{G}(\xi) \le \delta^{-1}\left(\xi^{2N}+1\right)^{-1} \tag{1.10}$$

holds for suitable number $\delta > 0$, which follows from

$$P(\xi^2) = \left|\,Q\big(\sqrt{-1}\xi\big)\,\right|^2 = \prod_{j=0}^{L-1}(\xi^2+a_j^2)\prod_{j=0}^{M-1}\Big[\,(\xi+\mathrm{Im}\,a_{L+j})^2+(\mathrm{Re}\,a_{L+j})^2\,\Big]^2$$
$$(-\infty < \xi < \infty)$$

Our conclusion is as follows.

Theorem 1.1 (1) $C(N) = \sup_{u\in H,\ u\not\equiv 0} S(u)$ *is given by* $C(N) = G(0)$*. For any real number* y *and complex number* c*, we have* $S(c\,G(x-y)) = C(N)$*.*

(2) $$\inf_{u\in H,\ u\not\equiv 0} S(u) = 0 \tag{1.11}$$

The above theorem (1) is equivalently rewritten as follows.

Theorem 1.2 *For any function* $u(x) \in H$*, there exists a positive constant* C *which is independent of* $u(x)$ *such that the following Sobolev inequality holds.*

$$\left(\sup_{-\infty<y<\infty}|\,u(y)\,|\right)^2 \le C\int_{-\infty}^{\infty}\sum_{j=0}^{N} p_j\left|\,u^{(N-j)}(x)\,\right|^2 dx \tag{1.12}$$

Among such C *the best constant* $C(N)$ *is the same as that in Theorem* 1.1(1)*. If we replace* C *by* $C(N)$ *in* (1.12)*, the equality holds for*

$$u(x) = c\,G(x-y) \qquad (-\infty < x < \infty) \tag{1.13}$$

where y *is an arbitrary real number and* c *is an arbitrary complex number.*

The engineering meaning of Sobolev inequality is that the square of the maximum bending of a string [2] ($N=1$) or a beam ($N=2$) is estimated from above by the constant multiple of the potential energy.

Theorem 1.3 *The best constant* $C(N)$ *of Sobolev inequality is expressed in the following two ways.*

(1) $$C(1) = \frac{1}{2a_0}$$

$$C(N) = \frac{(-1)^{N+1}}{2a_0\cdots a_{N-1}}\left|\begin{array}{c} a_j^{2i+1} \\ \hline \cdots \quad 1 \quad \cdots \end{array}\right| \Big/ \left|\begin{array}{c} a_j^{2i} \end{array}\right| \qquad (N = 2,3,4,\cdots) \tag{1.14}$$

In the numerator of the right hand side of (1.14), *we have* $0 \le i \le N-2,\ 0 \le j \le N-1$ *and in the denominator* $0 \le i, j \le N-1$.

(2)
$$C(1) = \frac{1}{2q_1}$$

$$C(2) = \frac{1}{2q_1q_2}$$

$$C(N) = \frac{1}{2\,q_N} \left| \, q_{N-2-2i+j} \, \right| \Big/ \left| \, q_{N-1-2i+j} \, \right| \qquad (N = 3, 4, 5, \cdots) \tag{1.15}$$

In the numerator of the right hand side of (1.15), *we have* $0 \le i, j \le N-3$ *and in the denominator* $0 \le i, j \le N-2$.

We here list explicit forms of $C(N)$ $(N = 1, 2, 3, 4, 5)$.

$$C(1) = \frac{1}{2a_0} = \frac{1}{2q_1}$$

$$C(2) = -\frac{1}{2a_0a_1} \begin{vmatrix} a_0 & a_1 \\ 1 & 1 \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 \\ a_0^2 & a_1^2 \end{vmatrix} = \frac{1}{2a_0a_1(a_0+a_1)} = \frac{1}{2q_1q_2}$$

$$C(3) = \frac{1}{2a_0a_1a_2} \begin{vmatrix} a_0 & a_1 & a_2 \\ a_0^3 & a_1^3 & a_2^3 \\ 1 & 1 & 1 \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 & 1 \\ a_0^2 & a_1^2 & a_2^2 \\ a_0^4 & a_1^4 & a_2^4 \end{vmatrix} =$$

$$\frac{a_0+a_1+a_2}{2a_0a_1a_2(a_0+a_1)(a_0+a_2)(a_1+a_2)} = \frac{1}{2q_3}\, q_1 \Big/ \begin{vmatrix} q_2 & q_3 \\ q_0 & q_1 \end{vmatrix} = \frac{q_1}{2q_3(q_1q_2-q_3)}$$

$$C(4) = -\frac{1}{2a_0a_1a_2a_3} \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 \\ a_0^5 & a_1^5 & a_2^5 & a_3^5 \\ 1 & 1 & 1 & 1 \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 & 1 & 1 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^4 & a_1^4 & a_2^4 & a_3^4 \\ a_0^6 & a_1^6 & a_2^6 & a_3^6 \end{vmatrix} =$$

$$\frac{1}{2q_4} \begin{vmatrix} q_2 & q_3 \\ q_0 & q_1 \end{vmatrix} \Big/ \begin{vmatrix} q_3 & q_4 & 0 \\ q_1 & q_2 & q_3 \\ 0 & q_0 & q_1 \end{vmatrix} = \frac{q_1q_2-q_3}{2q_4(q_1q_2q_3-q_3^2-q_1^2q_4)}$$

$$C(5) =$$

$$\frac{1}{2a_0a_1a_2a_3a_4} \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 & a_4^3 \\ a_0^5 & a_1^5 & a_2^5 & a_3^5 & a_4^5 \\ a_0^7 & a_1^7 & a_2^7 & a_3^7 & a_4^7 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} \Big/ \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_0^4 & a_1^4 & a_2^4 & a_3^4 & a_4^4 \\ a_0^6 & a_1^6 & a_2^6 & a_3^6 & a_4^6 \\ a_0^8 & a_1^8 & a_2^8 & a_3^8 & a_4^8 \end{vmatrix} =$$

$$\frac{1}{2q_5} \begin{vmatrix} q_3 & q_4 & q_5 \\ q_1 & q_2 & q_3 \\ 0 & q_0 & q_1 \end{vmatrix} \Big/ \begin{vmatrix} q_4 & q_5 & 0 & 0 \\ q_2 & q_3 & q_4 & q_5 \\ q_0 & q_1 & q_2 & q_3 \\ 0 & 0 & q_0 & q_1 \end{vmatrix} =$$

$$\frac{q_1q_2q_3-q_3^2-q_1^2q_4+q_1q_5}{2q_5(q_1q_2q_3q_4-q_3^2q_4-q_1^2q_4^2-q_1q_2^2q_5+q_2q_3q_5+2q_1q_4q_5-q_5^2)}$$

This paper is organized as follows. In section 2, we consider the $2N$-th order boundary value problem and find its Green function $G(x)$. In section 3, we give expressions of $G(0)$, where Giambelli's formula [3, 4] plays an important role. In section 4, it is shown that Green function is a reproducing kernel for H and $(\cdot,\cdot)_H$. The section 5 and 6 are devoted to proofs of the main Theorems 1.2 and 1.1(2), respectively. Finally, in section 7, we consider the special case of Theorem 1.2($N=1,2,3$).

2 Green function

We consider the following boundary value problem for a $2N$-th order linear ordinary differential operator $P(-D^2)=Q(-D)Q(D)$.

BVP(N)

$$\begin{cases} P(-D^2)\,u \,=\, f(x) & (-\infty<x<\infty) \quad (2.1)\\ u^{(i)}(x) \,\in\, L^2(-\infty,\infty) & (0\le i\le 2N) \quad (2.2)\end{cases}$$

Concerning the uniqueness and existence of the solution to BVP(N), we have the following theorem.

Theorem 2.1 *For any function $f(x)\in L^2(-\infty,\infty)$,* BVP($N$) *has a unique solution $u(x)$ expressed as*

$$u(x) \,=\, \int_{-\infty}^{\infty} G(x,y)\,f(y)\,dy \qquad (-\infty<x<\infty) \tag{2.3}$$

where $G(x,y)=G(x-y)$ $(-\infty<x,y<\infty)$ is Green function given by (1.8)*. It also has the following equivalent expressions.*

(1) $$G(x) \,=\, \sum_{j=0}^{N-1} \frac{1}{P'(-a_j^2)}\,G_j(x) \tag{2.4}$$

$$G_j(x) \,=\, \frac{1}{2a_j}\,e^{-a_j|x|} \qquad (0\le j\le N-1, \quad -\infty<x<\infty)$$

(2) $$G(x) \,=\, (-1)^{N+1} \begin{vmatrix} a_j^{2i} \\ \hline G_j(x) \end{vmatrix} \Bigg/ \begin{vmatrix} a_j^{2i} \end{vmatrix} \tag{2.5}$$

In the numerator of the right hand side of (2.5)*, we have $0\le i\le N-2,\ 0\le j\le N-1$ and in the denominator $0\le i,j\le N-1$.*

(3) $$G(x) \,=\, (\,G_0*\cdots*G_{N-1}\,)(x) \tag{2.6}$$

where $*$ denotes the convolution operator defined by

$$(f*g)(x) \,=\, \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy \qquad (-\infty<x<\infty). \tag{2.7}$$

From (3)*, in the case of $a_j>0$ $(0\le j\le N-1)$, we have $G(x)>0$.*

In order to prove Theorem 2.1(2), we prepare the following well-known fact.

Lemma 2.1 *For any $N \times N$ regular matrix $\boldsymbol{A}$ and $N \times 1$ matrices $\boldsymbol{b}$ and $\boldsymbol{c}$, we have the following equality.*

$$ {}^t\boldsymbol{b}\boldsymbol{A}^{-1}\boldsymbol{c} = - \begin{vmatrix} \boldsymbol{A} & \boldsymbol{c} \\ \hline {}^t\boldsymbol{b} & 0 \end{vmatrix} \Bigg/ \begin{vmatrix} \boldsymbol{A} \end{vmatrix} $$

Proof of Theorem 2.1 (2.3) is obtained by considering Fourier transform of (2.1). We here show (1), (2) and (3). From the partial fraction expansion

$$ \frac{1}{P(z)} = \sum_{j=0}^{N-1} \frac{1}{P'(-a_j^2)} (z + a_j^2)^{-1}, $$

we have

$$ \widehat{G}(\xi) = \frac{1}{P(\xi^2)} = \sum_{j=0}^{N-1} \frac{1}{P'(-a_j^2)} \widehat{G}_j(\xi) \qquad (-\infty < \xi < \infty) $$

where $\widehat{G}_j(\xi) = (\xi^2 + a_j^2)^{-1}$ $(-\infty < \xi < \infty,\ 0 \le j \le N-1)$. This shows (1). Using well-known fact

$$ \begin{pmatrix} \\ \dfrac{1}{P'(-a_i^2)} \\ \\ \end{pmatrix} = \begin{pmatrix} \\ (-a_j^2)^i \\ \\ \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} $$

we have

$$ \widehat{G}(\xi) = \begin{pmatrix} \quad \widehat{G}_j(\xi) \quad \end{pmatrix} \begin{pmatrix} \\ (-a_j^2)^i \\ \\ \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = $$

$$ - \begin{vmatrix} (-a_j^2)^i & \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \end{matrix} \\ \hline \widehat{G}_j(\xi) & 0 \end{vmatrix} \Bigg/ \begin{vmatrix} (-a_j^2)^i \end{vmatrix} = $$

$$ (-1)^{N+1} \begin{vmatrix} a_j^{2i} \\ \hline \widehat{G}_j(\xi) \end{vmatrix} \Bigg/ \begin{vmatrix} a_j^{2i} \end{vmatrix} $$

This shows (2). (3) follows immediately from

$$ \widehat{G}(\xi) = \prod_{j=0}^{N-1} \widehat{G}_j(\xi) \qquad (-\infty < \xi < \infty) $$

which completes the proof of Theorem 2.1. ■

Theorem 2.2 *Green function $G(x,y)=G(N;x,y)$ satisfies the following properties.*

(1) $P(-\partial_x^2)\,G(x,y) \;=\; Q(-\partial_x)\,Q(\partial_x)\,G(x,y) \;=\; 0 \qquad (-\infty<x,y<\infty, \quad x\neq y)$ (2.8)

(2) $\xi^i\widehat{G}(\xi) \,\in\, L^\infty(-\infty,\infty) \qquad (0\le i\le 2N)$ (2.9)

(3)
$$\partial_x^i G(x,y)\Big|_{y=x-0} - \partial_x^i G(x,y)\Big|_{y=x+0} = \begin{cases} 0 & (0\le i\le 2N-2) \\ (-1)^N & (i=2N-1) \quad (-\infty<x<\infty) \end{cases} \tag{2.10}$$

(4)
$$\partial_x^i G(x,y)\Big|_{x=y+0} - \partial_x^i G(x,y)\Big|_{x=y-0} = \begin{cases} 0 & (0\le i\le 2N-2) \\ (-1)^N & (i=2N-1) \quad (-\infty<y<\infty) \end{cases} \tag{2.11}$$

The condition (2) assures that for every $f(x)\in L^2(-\infty,\infty)$ we have $\partial_x^i(G*f)(x)\in L^2(-\infty,\infty)$ $(0\le i\le 2N)$.

Proof of Theorem 2.2 If $x\neq y$, we have

$$P(-\partial_x^2)\,G(x,y) \;=\; \prod_{k=0}^{N-1}\left(-\partial_x^2+a_k^2\right)\sum_{j=0}^{N-1}\frac{1}{P'(-a_j^2)}\,G_j(x-y) \;=$$
$$\sum_{j=0}^{N-1}\frac{1}{P'(-a_j^2)}\prod_{k=0}^{N-1}\left(-\partial_x^2+a_k^2\right)G_j(x-y) \;=$$
$$\sum_{j=0}^{N-1}\frac{1}{2a_jP'(-a_j^2)}\prod_{k=0}^{N-1}\left(-a_j^2+a_k^2\right)e^{-a_j|x-y|} \;=\; 0,$$

which proves (1). (2) is obvious from (1.9). Next we show (3), the left-hand side of which is written as

$$\partial_x^k G(x,y)\Big|_{y=x-0} - \partial_x^k G(x,y)\Big|_{y=x+0} =$$
$$(-1)^{N+1}\left|\begin{array}{c} a_j^{2i} \\ \hline \partial_x^k G_j(x-y)\Big|_{y=x-0} - \partial_x^k G_j(x-y)\Big|_{y=x+0} \end{array}\right| \Bigg/ \left|\quad a_j^{2i} \quad\right|$$
$$(0\le k\le 2N-1, \quad -\infty<x<\infty).$$

Employing the fact

$$\partial_x^k G_j(x-y)\Big|_{y=x-0} - \partial_x^k G_j(x-y)\Big|_{y=x+0} \;=\; -\frac{1}{2}\left(1-(-1)^k\right)a_j^{k-1} \;=$$
$$\begin{cases} 0 & (k=2l) \\ -\,a_j^{2l} & (k=2l+1) \qquad (0\le l\le N-1) \qquad (-\infty<x<\infty), \end{cases}$$

we have (3). (4) follows from (3). This completes the proof of Theorem 2.2. ■

3 The best constant of Sobolev inequality

In this section, we prove the equivalence between (1.14) and (1.15) in Theorem 1.3.

Putting $x = 0$ in (2.5) and employing $G_j(0) = (2a_j)^{-1}$ $(0 \leq j \leq N-1)$, we have

$$G(0) = (-1)^{N+1} \left| \begin{array}{c} a_j^{2i} \\ \hline (2a_j)^{-1} \end{array} \right| \Bigg/ \left| \begin{array}{c} a_j^{2i} \end{array} \right| =$$

$$(-1)^{N+1} \frac{1}{2a_0 \cdots a_{N-1}} \left| \begin{array}{c} a_j^{2i+1} \\ \hline 1 \end{array} \right| \Bigg/ \left| \begin{array}{c} a_j^{2i} \end{array} \right|$$

from which we obtain (1.14). Changing the row of the above determinant, we have

$$G(0) = \frac{1}{2q_N} \left| \begin{array}{c} a_j^{2N-3-2i} \\ \hline 1 \end{array} \right| \Bigg/ \left| \begin{array}{c} a_j^{2(N-1-i)} \end{array} \right| \tag{3.1}$$

where $q_N = a_0 \cdots a_{N-1}$.

Here we introduce two partitions of natural numbers

$$\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_{N-1}) \quad \text{and} \quad \mu = (\mu_0, \mu_1, \cdots, \mu_{N-1})$$

where λ_i and μ_i are given as follows.

$$\lambda_i = N - 1 - i \qquad (0 \leq i \leq N-1)$$

$$\mu_i = \begin{cases} \lambda_i - 1 & (0 \leq i \leq N-2) \\ 0 & (i = N-1) \end{cases}$$

By using the above λ and μ, (3.1) is rewritten as follows.

$$G(0) = \frac{1}{2q_N} \left| a_j^{N-1-i+\mu_i} \right| \Bigg/ \left| a_j^{N-1-i+\lambda_i} \right| = \frac{1}{2q_N} S_\mu(a) \Big/ S_\lambda(a) \tag{3.2}$$

In the above expression, $S_Y(a)$ denotes Schur polynomial associated with a partition $Y = (Y_0, Y_1, \cdots, Y_{N-1})$ $(Y_0 \geq Y_1 \geq \cdots \geq Y_{N-1} \geq 0)$, which is defined by

$$S_Y(a) = S_Y(a_0, \cdots, a_{N-1}) = \left| a_j^{N-1-i+Y_i} \right| \Big/ \left| a_j^{N-1-i} \right|.$$

The following statement is the most important lemma in this paper.

Lemma 3.1 (Giambelli [4]) *For a partition*

$$Y = (Y_0, Y_1, \cdots, Y_{N-1}) \qquad (Y_0 \geq Y_1 \geq \cdots \geq Y_{N-1} \geq 0) \tag{3.3}$$

of a natural number, let $\widehat{Y}$ *be a conjugate of* Y *defined by*

$$\widehat{Y} = (\widehat{Y}_0, \widehat{Y}_1, \cdots, \widehat{Y}_{N-1}) \qquad \widehat{Y}_i = \#\{\, j \mid Y_j \geq i+1 \,\} \qquad (0 \leq i \leq N-1) \tag{3.4}$$

Then we have

$$S_Y(a) = \left| q_{j-i+\widehat{Y}_i} \right|_{0 \leq i,j \leq N-1} \tag{3.5}$$

where q_j $(1 \leq j \leq N)$ *is the* j*-th fundamental symmetric polynomial of* $a = (a_0, \cdots, a_{N-1})$*. We also assume that* $q_0 = 1$ *and* $q_j = 0$ *for* $j < 0$ *or* $j > N$*.*

Applying Giambelli's formula to (3.2) and considering that $\widehat{\lambda}_i = \lambda_i$ and $\widehat{\mu}_i = \mu_i$, we have the following equality.

$$G(0) = \frac{1}{2q_N} \left| \, q_{j-i+\widehat{\mu}_i} \, \right|_{0\le i,j\le N-1} \Big/ \left| \, q_{j-i+\widehat{\lambda}_i} \, \right|_{0\le i,j\le N-1} =$$

$$\frac{1}{2q_N} \begin{vmatrix} q_{N-2} & q_{N-1} & \cdots & q_{2N-4} & q_{2N-3} \\ q_{N-4} & q_{N-3} & \cdots & q_{2N-6} & q_{2N-5} \\ \vdots & \vdots & & \vdots & \vdots \\ q_{-N+2} & q_{-N+3} & \cdots & q_0 & q_1 \\ q_{-N+1} & q_{-N+2} & \cdots & q_{-1} & q_0 \end{vmatrix} \Big/$$

$$\begin{vmatrix} q_{N-1} & q_N & \cdots & q_{2N-3} & q_{2N-2} \\ q_{N-3} & q_{N-2} & \cdots & q_{2N-5} & q_{2N-4} \\ \vdots & \vdots & & \vdots & \vdots \\ q_{-N+3} & q_{-N+4} & \cdots & q_1 & q_2 \\ q_{-N+1} & q_{-N+2} & \cdots & q_{-1} & q_0 \end{vmatrix} =$$

$$\frac{1}{2q_N} \begin{vmatrix} q_{N-2} & q_{N-1} & \cdots & q_{2N-5} \\ q_{N-4} & q_{N-3} & \cdots & q_{2N-7} \\ \vdots & \vdots & & \vdots \\ q_{-N+4} & q_{-N+5} & \cdots & q_1 \end{vmatrix} \Big/ \begin{vmatrix} q_{N-1} & q_N & \cdots & q_{2N-3} \\ q_{N-3} & q_{N-2} & \cdots & q_{2N-5} \\ \vdots & \vdots & & \vdots \\ q_{-N+3} & q_{-N+4} & \cdots & q_1 \end{vmatrix}$$

where we have used the fact $q_0 = 1$ and $q_{-j} = 0$ $(j \ge 1)$. Thus we proved that (1.14) is equivalent to (1.15).

4 Reproducing kernel

In this section, we show that Green function $G(x,y)$ is a reproducing kernel for a set of Hilbert space H and its inner product $(\cdot,\cdot)_H$ introduced in section 1.

We first show that $(\cdot,\cdot)_H$ is positive definite. Applying Parseval equality to (1.4), we have

$$(u,v)_H = \int_{-\infty}^{\infty} \Big(Q(D)\,u(x)\Big)\overline{\Big(Q(D)\,v(x)\Big)}dx =$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Big(Q\big(\sqrt{-1}\,\xi\big)\,\widehat{u}(\xi)\Big)\overline{\Big(Q\big(\sqrt{-1}\,\xi\big)\,\widehat{v}(\xi)\Big)}d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\,Q\big(\sqrt{-1}\,\xi\big)\,|^2\widehat{u}(\xi)\,\overline{\widehat{v}}(\xi)\,d\xi\,. \tag{4.1}$$

Sobolev energy $(u,u)_H$ is calculated as

$$\|\,u\,\|_H^2 = (u,u)_H = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\,Q\big(\sqrt{-1}\,\xi\big)\,|^2\,|\,\widehat{u}(\xi)\,|^2\,d\xi$$

By the inequality (1.10) we have

$$\|\,u\,\|_H^2 \ge \frac{\delta}{2\pi}\int_{-\infty}^{\infty} \big(\xi^{2N}+1\big)\,|\,\widehat{u}(\xi)\,|^2\,d\xi$$

from which it is concluded that $(\cdot,\cdot)_H$ is positive definite.

Moreover, the right-hand side of (4.1) is expanded as follows.

$$(u,v)_H = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\xi^2)\,\widehat{u}(\xi)\,\overline{\widehat{v}}(\xi)\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=0}^{N} p_j\,\xi^{2(N-j)}\,\widehat{u}(\xi)\,\overline{\widehat{v}}(\xi)\,d\xi =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=0}^{N} p_j\left(\left(\sqrt{-1}\xi\right)^{N-j}\widehat{u}(\xi)\right)\overline{\left(\left(\sqrt{-1}\xi\right)^{N-j}\widehat{v}(\xi)\right)}d\xi =$$

$$\int_{-\infty}^{\infty}\sum_{j=0}^{N} p_j\,u^{(N-j)}(x)\,\overline{v}^{(N-j)}(x)\,dx$$

Theorem 4.1 (1) *For any $u(x) \in H$, we have the following reproducing relation.*

$$u(y) = (\,u(x), G(x,y)\,)_H \qquad (-\infty < y < \infty) \tag{4.2}$$

This means that Green function $G(x,y) = G(x-y)$ is a reproducing kernel for H with the inner product $(\cdot,\cdot)_H$.

(2) $$G(0) = G(y,y) = (\,G(x,y), G(x,y)\,)_H \qquad (-\infty < y < \infty) \tag{4.3}$$

Proof of Theorem 4.1 We note that

$$G(x-y) \quad \overset{\frown}{\longrightarrow} \quad e^{-\sqrt{-1}\,y\xi}\widehat{G}(\xi)$$

Using Parseval equality, we have

$$(\,u(x), G(x,y)\,)_H = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\xi^2)\,\widehat{u}(\xi)\,\overline{e^{-\sqrt{-1}y\xi}\widehat{G}(\xi)}d\xi =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sqrt{-1}y\xi}\widehat{u}(\xi)\,d\xi = u(y)$$

where we have used the fact

$$P(\xi^2)\,\widehat{G}(\xi) = 1 \qquad (-\infty < \xi < \infty).$$

(2) is shown by putting $u(x) = G(x,y)$ in (4.2). This completes the proof of Theorem 4.1. ■

5 Sobolev inequality and the best constant

In this section, we prove Theorem 1.2.

Proof of Theorem 1.2 Applying Schwarz inequality to (4.2) and using (4.3), we have

$$|\,u(y)\,|^2 \le \|\,u\,\|_H^2\,\|\,G(x,y)\,\|_H^2 = G(0)\,\|\,u\,\|_H^2$$

Taking the supremum with respect to y $(-\infty < y < \infty)$, we have

$$\left(\sup_{-\infty<y<\infty}|\,u(y)\,|\right)^2 \le G(0)\,\|\,u\,\|_H^2.$$

Hence, we can take a positive constant C such that the following Sobolev inequality holds for any function $u(x) \in H$.

$$\left(\sup_{-\infty<y<\infty}|\,u(y)\,|\right)^2 \le C\,\|\,u\,\|_H^2. \tag{5.1}$$

The best constant $C(N)$ among such C obviously satisfies

$$C(N) \leq G(0). \tag{5.2}$$

In the second place, for any fixed y_0 $(-\infty < y_0 < \infty)$, we apply this inequality (5.1) to $u(x) = G(x, y_0) \in H$ and have

$$\left(\sup_{-\infty<y<\infty} |\, G(y, y_0)\, | \right)^2 \leq C(N) \,\|\, G(x, y_0) \,\|_H^2 \;=\; C(N)\, G(0)$$

Combining this and trivial inequality

$$G(0)^2 \;=\; |\, G(y_0, y_0)\, |^2 \;\leq\; \left(\sup_{-\infty<y<\infty} |\, G(y, y_0)\, | \right)^2$$

we have $G(0) \leq C(N)$. Together with (5.2), it is concluded that $C(N) = G(0)$ and that $G(x, y_0)$ is a best function for arbitrarily fixed y_0, that is,

$$\left(\sup_{-\infty<y<\infty} |\, G(y, y_0)\, | \right)^2 \;=\; C(N) ||\, G(x, y_0) \,||_H^2$$

This proved Theorem 1.2. ■

6 Infimum of Sobolev functional

In this section, we prove Theorem 1.1(2) concerning the infimum of Sobolev functional $S(u)$.

Proof of Theorem 1.1(2) Let $h(x,t)$ be a heat kernel given as follows.

$$h(x,t) \;=\; \frac{1}{\sqrt{4\pi t}} \exp\left(-x^2/(4t)\right) \qquad (-\infty < x < \infty, \quad 0 < t < \infty)$$

It is easy to see that

$$\sup_{-\infty<y<\infty} |\, h(y,t)\, | \;=\; h(0,t) \;=\; \frac{1}{\sqrt{4\pi t}}$$

holds. Since Fourier transform of $h(x,t)$ is given by

$$\widehat{h}(\xi, t) \;=\; e^{-\xi^2 t} \qquad (-\infty < \xi < \infty, \quad 0 < t < \infty)$$

its Sobolev energy is calculated as

$$\|\, h(x,t) \,\|_H^2 \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\xi^2)\, e^{-2\xi^2 t} d\xi \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{j=0}^{N} p_j\, \xi^{2(N-j)} e^{-2\xi^2 t} d\xi \;=$$
$$\frac{1}{2\pi} \sum_{j=0}^{N} p_j \left(-\frac{1}{2}\, \partial_t \right)^{N-j} \int_{-\infty}^{\infty} e^{-2\xi^2 t} d\xi \;=\; \frac{1}{2\sqrt{2\pi}} \sum_{j=0}^{N} p_j \left(-\frac{1}{2}\, \partial_t \right)^{N-j} t^{-1/2}.$$

Hence we have

$$S(h(x,t)) \;=\; \left(\frac{1}{4\pi t} \right) \Bigg/ \left(\frac{1}{2\sqrt{2\pi}} \sum_{j=0}^{N} p_j \left(-\frac{1}{2}\, \partial_t \right)^{N-j} t^{-1/2} \right) \;\longrightarrow\; 0 \qquad (t \to +0)$$

which completes the proof of Theorem 1.1(2). ■

Another proof of Theorem 1.1(2) For $A > 0$, we introduce a function

$$u(A;x) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sqrt{-1}x\xi}\,\widehat{u}(A;\xi)\,d\xi \qquad (-\infty < x < \infty)$$

where

$$\widehat{u}(A;\xi) \;=\; \begin{cases} \widehat{G}(\xi) & (|\xi| \geq A) \\ 0 & (|\xi| < A) \end{cases}$$

It is easy to see

$$\sup_{-\infty<y<\infty} |\,u(A;y)\,| \;=\; u(A;0) \;=\; \frac{1}{2\pi}\int_{|\xi|\geq A} \widehat{G}(\xi)\,d\xi$$

On the other hand, we have

$$\|\,u(A;x)\,\|_H^2 \;=\; \frac{1}{2\pi}\int_{|\xi|\geq A} P(\xi^2)\;|\widehat{u}(A;\xi)|^2\,d\xi \;=\; \frac{1}{2\pi}\int_{|\xi|\geq A} \widehat{G}(\xi)\,d\xi \;=\; u(A;0).$$

Considering that $\widehat{G}(\xi) \in L^1(-\infty,\infty)$, we conclude that

$$S(u(A;x)) \;=\; u(A;0) \quad\longrightarrow\quad 0 \qquad (A \to \infty).$$

■

7 Explicit forms of best constants and functions

In this section, we find explicit forms the best constants $C(N)$ and best functions in simple cases $N = 1, 2, 3$. Although some of the results, where all the characteristic roots a_j are real, are obtained in our previous paper [1], we also list them for the sake of self-containedness.

In the simplest case $N = 1$, or equivalently $(L, M) = (1, 0)$, Sobolev space is given as follows.

$$H \;=\; H(1) \;=\; \left\{ u(x) \;\middle|\; u(x),\; u'(x) \,\in\, L^2(-\infty,\infty) \right\} \tag{7.1}$$

Corresponding Sobolev inner product is

$$(u,v)_H \;=\; \int_{-\infty}^{\infty} \Big[u'(x)\,\overline{v}'(x) \;+\; p_1\,u(x)\,\overline{v}(x) \Big]\,dx \tag{7.2}$$

where $p_1 = a_0^2$. Moreover we note that $q_1 = a_0$. We here rewrite $a_0 = a$ $(0 < a < \infty)$ for the sake of simplicity.

As a special case of Theorem 1.2 we have the next theorem.

Theorem 7.1 *For any function $u(x) \in H(1)$, there exists a positive constant C which is independent of $u(x)$ such that the following Sobolev inequality holds.*

$$\Big(\sup_{-\infty<y<\infty} |\,u(y)\,| \Big)^2 \leq C \int_{-\infty}^{\infty} \Big[\, |\,u'(x)\,|^2 \;+\; p_1\;|\,u(x)\,|^2 \,\Big]\,dx \tag{7.3}$$

Among such C the best constant is

$$C(1) \;=\; \frac{1}{2a} \;=\; \frac{1}{2q_1} \tag{7.4}$$

If we replace C by $C(1)$ in (7.3), the equality holds for $u(x) = c\,G(x-y)$ $(-\infty < x < \infty)$ where y is an arbitrary real number and c is an arbitrary complex number.

Green function $G(x)$ is given by the following formula.

$$G(x) \;=\; \frac{1}{2a}\,e^{-a|x|} \qquad (-\infty < x < \infty) \tag{7.5}$$

In the second place, we treat the case $N = 2$, or equivalently $(L, M) = (2, 0)$ or $(0, 1)$. In these cases, we consider Sobolev space

$$H \;=\; H(2) \;=\; \left\{ u(x) \;\middle|\; u(x),\; u'(x),\; u''(x) \;\in\; L^2(-\infty, \infty) \right\} \tag{7.6}$$

Sobolev inner product

$$(u, v)_H \;=\; \int_{-\infty}^{\infty} \Big[u''(x)\,\overline{v}''(x) \;+\; p_1\,u'(x)\,\overline{v}'(x) \;+\; p_2\,u(x)\,\overline{v}(x) \Big]\,dx \tag{7.7}$$

where $p_1 = a_0^2 + a_1^2,\ p_2 = a_0^2 a_1^2$. We note that $q_1 = a_0 + a_1,\ q_2 = a_0 a_1$.

In the case $(L, M) = (2, 0)$, we put $a_0 = a,\ a_1 = b$ $(0 < a < b)$, then we have

$$p_1 = a^2 + b^2, \quad p_2 = a^2 b^2, \quad q_1 = a + b, \quad q_2 = ab$$

In the case $(L, M) = (0, 1)$, we put $a_0 = a + \sqrt{-1}\,b,\ a_1 = a - \sqrt{-1}\,b$ $(0 < a, b)$, then we have

$$p_1 = 2(a^2 - b^2), \quad p_2 = (a^2 + b^2)^2, \quad q_1 = 2a, \quad q_2 = a^2 + b^2$$

As a special case of Theorem 1.2 we have the next theorem.

Theorem 7.2 *For any function $u(x) \in H(2)$, there exists a positive constant C which is independent of $u(x)$ such that the following Sobolev inequality holds.*

$$\Big(\sup_{-\infty<y<\infty} |\,u(y)\,| \Big)^2 \le C \int_{-\infty}^{\infty} \Big[\, |\,u''(x)\,|^2 \;+\; p_1\,|\,u'(x)\,|^2 \;+\; p_2\,|\,u(x)\,|^2 \,\Big]\,dx \tag{7.8}$$

Among such C the best constant is

$$C(2) \;=\; \frac{1}{2a_0 a_1 (a_0 + a_1)} \;=\; \frac{1}{2q_1 q_2} \;=\; \begin{cases} \dfrac{1}{2ab(a+b)} & (L, M) = (2, 0) \\[2ex] \dfrac{1}{4a(a^2+b^2)} & (L, M) = (0, 1) \end{cases} \tag{7.9}$$

If we replace C by $C(2)$ in (7.8), the equality holds for $u(x) = c\,G(x-y)$ $(-\infty < x < \infty)$ where y is an arbitrary real number and c is an arbitrary complex number.

Green function $G(x)$ is given by the following formula.

$$G(x) = \frac{1}{a_1^2 - a_0^2} \left[\frac{1}{2a_0} e^{-a_0|x|} - \frac{1}{2a_1} e^{-a_1|x|} \right] =$$

$$\begin{cases} \dfrac{1}{b^2-a^2} \left[\dfrac{1}{2a} e^{-a|x|} - \dfrac{1}{2b} e^{-b|x|} \right] & (L,M)=(2,0) \\ \dfrac{1}{4ab(a^2+b^2)} e^{-a|x|} \left[a \sin(b|x|) + b \cos(b|x|) \right] & (L,M)=(0,1) \end{cases}$$

$$(-\infty < x < \infty) \tag{7.10}$$

Finally, we treat the case $N = 3$, or equivalently $(L, M) = (3, 0)$ or $(1, 1)$. In these cases, we consider Sobolev space

$$H = H(3) = \left\{ u(x) \;\middle|\; u(x),\ u'(x),\ u''(x),\ u'''(x) \in L^2(-\infty,\infty) \right\} \tag{7.11}$$

Sobolev inner product

$$(u,v)_H = \int_{-\infty}^{\infty} \left[u'''(x)\overline{v}'''(x) + p_1\, u''(x)\overline{v}''(x) + p_2\, u'(x)\overline{v}'(x) + p_3\, u(x)\overline{v}(x) \right] dx \tag{7.12}$$

where $p_1 = a_0^2 + a_1^2 + a_2^2$, $p_2 = a_0^2a_1^2 + a_1^2a_2^2 + a_2^2a_0^2$, $p_3 = a_0^2a_1^2a_2^2$.

In the case $(L, M) = (3, 0)$, we put $a_0 = a,\ a_1 = b,\ a_2 = c$ $(0 < a < b < c)$, then we have

$$p_1 = a^2 + b^2 + c^2, \quad p_2 = a^2b^2 + b^2c^2 + c^2a^2, \quad p_3 = a^2b^2c^2$$

In the case $(L, M) = (1, 1)$, we put $a_0 = a,\ a_{\{ {1 \atop 2}} = b \pm \sqrt{-1}\,c$ $(0 < a, b, c)$, then we have

$$p_1 = a^2 + 2b^2 - 2c^2, \quad p_2 = b^4 + c^4 + 2a^2b^2 + 2b^2c^2 - 2c^2a^2, \quad p_3 = a^2(b^2 + c^2)^2$$

As a special case of Theorem 1.2 we have the next theorem.

Theorem 7.3 *For any function $u(x) \in H(3)$, there exists a positive constant C which is independent of $u(x)$ such that the following Sobolev inequality holds.*

$$\left(\sup_{-\infty<y<\infty} |\,u(y)\,| \right)^2 \le$$

$$C \int_{-\infty}^{\infty} \left[|\,u'''(x)\,|^2 + p_1\, |\,u''(x)\,|^2 + p_2\, |\,u'(x)\,|^2 + p_3\, |\,u(x)\,|^2 \right] dx \tag{7.13}$$

Among such C the best constant is

$$C(3) = \frac{a_0 + a_1 + a_2}{2a_0a_1a_2(a_0 + a_1)(a_1 + a_2)(a_2 + a_0)} =$$

$$\begin{cases} \dfrac{a+b+c}{2abc(a+b)(b+c)(c+a)} & (L,M)=(3,0) \\ \dfrac{a+2b}{4ab(b^2+c^2)(a^2+b^2+c^2+2ab)} & (L,M)=(1,1) \end{cases} \tag{7.14}$$

If we replace C by $C(3)$ in (7.13), the equality holds for $u(x) = c\,G(x-y)$ $(-\infty < x < \infty)$ where y is an arbitrary real number and c is an arbitrary complex number.

If $(L, M) = (3, 0)$, Green function $G(x)$ is given by

$$G(x) = \frac{1}{2a(a^2-b^2)(a^2-c^2)}e^{-a|x|} + \frac{1}{2b(b^2-a^2)(b^2-c^2)}e^{-b|x|} + \frac{1}{2c(c^2-a^2)(c^2-b^2)}e^{-c|x|} \quad (-\infty < x < \infty) \tag{7.15}$$

and if $(L, M) = (1, 1)$

$$G(x) = \frac{1}{(a^2+b^2+c^2+2ab)(a^2+b^2+c^2-2ab)}\Bigg[\frac{1}{2a}\,e^{-a|x|} + \frac{1}{4bc(b^2+c^2)}\,e^{-b|x|}\big\{c(a^2-3b^2+c^2)\cos(c|x|) + b(a^2-b^2+3c^2)\sin(c|x|)\big\}\Bigg] \quad (-\infty < x < \infty) \tag{7.16}$$

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