FIXED POINTS FOR MULTIVALUED CONTRACTIONS WITH RESPECT TO A w-DISTANCE.

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Abstract. The aim of this paper is to obtain a generalization of some well known fixed point theorems for multivalued mappings of contractive type in the framework of complete metric spaces, by using the concept of w-distance.

1. Introduction.

Due to the simplicity and usefulness of the well known Banach contraction principle, there is a vast amount of literature dealing with technical extensions and generalizations of it.

In particular, in 1976, J. Caristi [1] proved a celebrated generalization of the Banach contraction principle. Caristi's Theorem is in some sense equivalent to the famous Ekeland variational principle, a useful tool in Optimization Theory which has extensive applications in the fields of mathematics such as variational inequalities, optimization, control theory and differential equations. W. Takahashi proved in 1991 (see [11]) an important nonconvex optimization theorem which allows to obtain as corollaries both Caristi's Theorem and Ekeland's variational principle.

On the other hand, Banach contraction principle was extended for multivalued mappings. One of the first results in this setting is S.B. Nadler's theorem given in 1969 in [8]. It is worthwhile to note here that from Takahashi nonconvex optimization Theorem one can derive Nadler's Theorem as a corollary.

Many modifications and generalizations of Nadler's Theorem have been developed in successive years. These generalizations are weakening of the contractive nature of the map, but very often with some additional requirements, as for instance to take compact values. Among many others, there are fixed point results for set-valued mappings of generalized contractive type due to S. Reich (1972) [9], L. Ćirić (1972) [2], V. M. Sehgal and R. E. Smithson (1980) [10]. In these papers the fixed point results are proved without using minimization techniques.

Nevertheless, in [7] (1989) N. Mizoguchi and W. Takahashi presented a set-valued version of J. Caristi's fixed point theorem and then they used it to deduce I. Ekeland's variational principle, and consequently they generalized again Nadler's fixed point theorem. Moreover they partially solved a conjecture of S. Reich in [9].

In a recent paper [3], Y. Feng and S. Liu defined a different kind of contractivity for multivalued mappings, which focuses the requirements on some orbits of the mapping under consideration. The main fixed point theorem is also a proper generalization of Nadler's Theorem. They also gave fixed point theorems for multi-valued Caristi type mappings.

Very recently, inspired by Mizoguchi-Takahashi and Feng-Liu's ideas, D. Klim and D. Wardowski [5] obtained a further generalization of the previous fixed point results given in [9, 7, 3].

On the other hand in 1996, O. Kada, T. Suzuki and W. Takahashi [4] introduced the concept of w-distance on a metric space and by using this new notion they obtained an improvement of the

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Takahashi's nonconvex optimization Theorem, as well as generalizations of J. Caristi's fixed point theorem and Ekeland's variational principle. They also gave fixed point theorems for single valued mappings of (w-)contractive type.

In this paper, we will use w-distances in order to obtain a multivalued version of a fixed point theorem due to Kada, Suzuki and Takahashi [Theorem 4 of [4]] which allows us to give a generalization of Theorem 2.1 of [5]. Our main theorem, when particularized for the case of single valued mappings, recaptures a fixed point result given in [12].

2. Preliminaries

Throughout this paper we assume that (X, d) is a complete metric space and we will use the following notation:

- $P(X)$ is the collection of all nonempty subsets of X.
- $P_{cl}(X)$ is the collection of all nonempty closed subsets of X.
- $P_{b,cl}(X)$ is the collection of all nonempty bounded closed subsets of X.
- $P_c(X)$ is the collection of all nonempty compact subsets of X.

The map $H : P_{b,cl}(X) \times P_{b,cl}(X) \to \mathbb{R}^+$ defined by

$$H(A, B) := \max\{\sup_{x\in B} d(x, A), \sup_{y\in A} d(y, B)\},$$

where $d(x, A) = \inf_{y\in A} d(x, y)$, is known as the Hausdorff metric induced by d.

An point $x \in X$ is said to be a *fixed point of* a multivalued mapping $T : X \to P(X)$, if $x \in T(x)$.

A mapping $f : X \to \mathbb{R}$ is said to be lower semi-continuous (lsc for short), if for any sequence (x_n) in X and $x \in X$ with $x_n \to x$, the inequality $f(x) \le \liminf_{n\to\infty} f(x_n)$ holds.

Given a mapping $T : X \to P(X)$, $b \in (0, 1]$ and $x \in X$ we use the notation

$$I_b^x = \{y \in T(x) : \ b\, d(x, y) \le d(x, T(x)\}.$$

As we pointed out in the introduction, the first fixed point result for multivalued contraction mappings in metric spaces was given by S.B. Nadler. He proved the following:

Theorem 2.1 ([8]). *Let (X, d) be a complete metric space and let $T : X \to P_{b,cl}(X)$. Assume that there exists $c \in]0, 1[$ such that*

$$H(T(x), T(y)) \le c\, d(x, y),$$

for every $x, y \in X$. Then T has a fixed point.

Among many articles which deal with this type of fixed point theorems we refer the reader to [3, 5, 7, 9]). An extension of Nadler's theorem has recently been obtained by Y. Feng and S. Liu as follows.

Theorem 2.2 ([3]). *Let (X, d) be a complete metric space and let $T : X \to P_{cl}(X)$. Assume that the following conditions hold:*

1. *The map $f : X \to \mathbb{R}$ defined by $f(x) := d(x, T(x))$, $x \in X$ is lsc;*
2. *There exists $b, c \in (0, 1)$ with $c < b$, for every $x \in X$ there exists $y \in I_b^x$ such that $d(y, T(y)) \le c\, d(x, y)$.*

Then T has a fixed point.

D.Klim and D. Wardowski proved in [5] the following result.

Theorem 2.3 ([5]). *Let (X, d) be a complete metric space and let $T : X \to P_{cl}(X)$ be a multivalued mapping. Assume that the following conditions hold:*

1. *The map $f : X \to \mathbb{R}$ defined by $f(x) := d(x, T(x))$, $x \in X$ is lsc.*
2. *There exist $b \in (0, 1)$ and $\varphi : [0, +\infty[\to [0, b)$ such that*
    - *(2i) For each $t \in [0, +\infty[$, $\limsup_{r\to t^+} \varphi(r) < b$.*
    - *(2ii) For every $x \in X$ there exists $y \in I_b^x$ such that $d(y, T(y)) \le \varphi(d(x, y))d(x, y)$.*

Then T has a fixed point.

The concept of w-distance was introduced in [4] as follows:

A mapping $w : X \times X \to \mathbb{R}^+$ is said to be w-distance on the metric space (X, d) if the following axioms are satisfied:

1. For any $x, y, z \in X$ the inequality $w(x, z) \leq w(x, y) + w(y, z)$ holds.
2. For every $x \in X$, the map $w(x, .) : X \to \mathbb{R}^+$ is lsc.
3. For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $w(z, x) \leq \delta$ and $w(z, y) \leq \delta$, then $d(x, y) \leq \varepsilon$.

Examples of non trivial w-distances can be found in [4].

A crucial result in order to obtain fixed point theorems by using a $w-$ distance is the following:

Lemma 2.4 (Lemma 1,[4])**.** *Let (X, d) be a metric space, and let w be a w-distance on X. Let (x_n) and (y_n) be two sequences in X, let (α_n) and (β_n) be sequences in $[0, +\infty[$ converging to zero and let $x, y, z \in X$. Then the following hold:*

1. *If $w(x_n, y) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$.*
2. *If $w(x_n, y_n) \leq \alpha_n$ and $w(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z.*
3. *If $w(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.*
4. *If $w(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence.*

The above lemma is used to prove the following generalization of Caristi's fixed point theorem [1]:

Theorem 2.5 (Theorem 4,[4])**.** *Let (X, d) be a complete metric space, let w be a w-distance on X and let $T : X \to X$ be a mapping such that there exists $r \in [0, 1)$ satisfying*

$$w(Tx, T^2x) \leq r\, w(x, Tx),$$

for every $x \in X$, and that

$$\inf\{w(x, y) + w(x, T(x)) :\ x \in X\} > 0,$$

for every $y \in X$ with $y \neq T(y)$. Then T has a fixed point. Moreover, if $v = Tv$, then $w(v, v) = 0$.

The purpose of this paper is to obtain a multivalued version of Theorem 2.5 and to extend Theorem 2.2 in terms of a w-distance.

3. Fixed Point Results

In order to proceed, we shall first give the following definition.

Definition 3.1. *Let $T : X \to P(X)$ be a multivalued mapping, let w be a w-distance on X. Define the function $f : X \to \mathbb{R}$ by $f(x) := D_w(x, T(x))$, where $D_w(x, T(x)) = \inf\{w(x, y) :\ y \in T(x)\}$.*

For each $b \in [0, 1]$ we define the set $I^x_{b,w} := \{y \in T(x) :\ b\, w(x, y) \leq D_w(x, T(x))\}$.

Remark 3.2. *If $T : X \to P_{cl}(X)$ is a multivalued mapping and $0 < b < 1$, it is clear that, for every $x \in X$, the set $I^x_{b,w}$ is nonempty.*

Next, we shall present a fixed point theorem for multivalued mappings on a complete metric space endowed with a w-distance.

Theorem 3.3. *Let (X, d) be a complete metric space, let w be a w-distance on X, and let $T : X \to P_{cl}(X)$ be a multivalued mapping. Assume that the following conditions hold:*

1. *There exist $b \in (0, 1)$ and $\varphi : [0, +\infty[\to [0, b)$ such that*

   (1i) for each $t \in [0, +\infty[$,

   $$\limsup_{r \to t^+} \varphi(r) < b\,;$$

   (1ii) for every $x \in X$, there exists $y \in I^x_{b,w}$ such that

   $$D_w(y, T(y)) \leq \varphi(w(x, y))w(x, y)\,;$$

2. *for every $y \in X$ with $y \notin T(y)$,*

   $$\inf\{w(x, y) + D_w(x, T(x)) :\ x \in X\} > 0.$$

Then T has a fixed point.

Proof. Given an element $x_0 \in X$, by hypotheses (1ii) there exists $x_1 \in T(x_0)$ which satisfies the following two conditions:

$$bw(x_0, x_1) \leq D_w(x_0, T(x_0)) \tag{3.1}$$

and

$$D_w(x_1, T(x_1)) \leq \varphi(w(x_0, x_1))w(x_0, x_1). \tag{3.2}$$

Inequalities (3.1) and (3.2) yield

$$\begin{aligned} D_w(x_0, T(x_0)) - D_w(x_1, T(x_1)) \quad &\geq bw(x_0, x_1) - \varphi(w(x_0, x_1))w(x_0, x_1) \\ &= (b - \varphi(w(x_0, x_1)))w(x_0, x_1) \geq 0. \end{aligned}$$

Now, let us argue in the same sense as above. Given x_1, there exists $x_2 \in T(x_1)$ such that

$$bw(x_1, x_2) \leq D_w(x_1, T(x_1)) \tag{3.3}$$

and

$$D_w(x_2, T(x_2)) \leq \varphi(w(x_1, x_2))w(x_1, x_2). \tag{3.4}$$

By (3.3) and (3.4) we have:

$$\begin{aligned} D_w(x_1, T(x_1)) - D_w(x_2, T(x_2)) \quad &\geq bw(x_1, x_2) - \varphi(w(x_1, x_2))w(x_1, x_2) \\ &= (b - \varphi(w(x_1, x_2))w(x_1, x_2) \geq 0. \end{aligned}$$

Moreover, from (3.2) and (3.3) we derive the following inequality:

$$w(x_1, x_2) \leq \frac{1}{b} D_w(x_1, T(x_1)) \leq \frac{1}{b}\varphi(w(x_0, x_1))w(x_0, x_1) \leq w(x_0, x_1).$$

By an inductive process, it is not difficult to obtain a sequence (x_n) of elements of X satisfying the following conditions:

- (i) For every $n \in \mathbb{N}$, $x_{n+1} \in T(x_n)$;
- (ii) $bw(x_n, x_{n+1}) \leq D_w(x_n, T(x_n))$;
- (iii) $D_w(x_{n+1}, T(x_{n+1})) \leq \varphi(w(x_n, x_{n+1}))w(x_n, x_{n+1})$.

Applying the above properties (i),(ii) and (iii), we have that for each $n \in \mathbb{N}$ the following inequalities

$$\begin{cases} D_w(x_n, T(x_n)) \geq D_w(x_{n+1}, T(x_{n+1})), \\ w(x_n, x_{n+1}) \leq w(x_{n-1}, x_n) \end{cases} \tag{3.5}$$

hold. Expression (3.5) implies that both $(D_w(x_n, T(x_n))$ and $(w(x_n, x_{n+1}))$ are decreasing sequences of nonnegative real numbers. Therefore they are convergent.

Since $(w(x_n, x_{n+1}))$ is a convergent sequence, there exists $t \in [0, \infty[$ such that $\lim_{n\to\infty} w(x_n, x_{n+1}) = t$. Hence, by hypothesis $(1i)$ we may find $q \in [0, b)$ such that

$$\limsup_{n\to\infty} \varphi(w(x_n, x_{n+1})) \leq \limsup_{r\to t^+} \varphi(r) = q.$$

Thus, given $b_0 \in (q, b)$ there exists $n_0 \in \mathbb{N}$ such that

$$\varphi(w(x_n, x_{n+1})) < b_0, \text{ for any } n \geq n_0. \tag{3.6}$$

Consequently, for any $n > n_0$, we have:

$$D_w(x_n, T(x_n)) - D_w(x_{n+1}, T(x_{n+1})) \geq [b - \varphi(w(x_n, x_{n+1}))]w(x_n, x_{n+1}) \geq \alpha w(x_n.x_{n+1}), \tag{3.7}$$

where $\alpha = b - b_0$.

For each $n > n_0$, inequalities (ii), (iii) and (3.6) yield

$$
\begin{aligned}
D_w(x_{n+1},T(x_{n+1})) \leq\ & \varphi(w(x_n,x_{n+1}))w(x_n,x_{n+1}) \leq b_0\tfrac{1}{b}D_w(x_n,T(x_n))\\
& \leq ...\\
& \leq (\tfrac{b_0}{b})^{n-n_0}D_w(x_0,T(x_0)).
\end{aligned} \tag{3.8}
$$

Obviously $\lim_{n\to\infty}(\frac{b_0}{b})^{n-n_0}=0$, since $b_0<b$. This means that $\lim_{n\to\infty}D_w(x_n,T(x_n))=0$. On the other hand, if $m>n>n_0$, by (3.7), we obtain

$$
\begin{aligned}
w(x_n,x_m) &= \textstyle\sum_{s=n}^{m-1} w(x_s,x_{s+1})\\
&\leq \tfrac{1}{\alpha}\textstyle\sum_{s=n}^{m-1}(D_w(x_s,T(x_s))-D_w(x_{s+1},T(x_{s+1}))\\
&= \tfrac{1}{\alpha}(D_w(x_n,T(x_n))-D_w(x_m,T(x_m))\\
&\leq \tfrac{1}{\alpha}D_w(x_n,T(x_n)).
\end{aligned}
$$

The above argument allows us to apply condition (3) of Lemma 2.4 and thus we have that (x_n) is a Cauchy sequence in (X,d). Since X is complete, (x_n) is a convergent sequence. Let $z\in X$ be the limit of the sequence (x_n).

Assume that $z\notin T(z)$. Since for each $x\in X$ the mapping $w(x,.):X\to[0,+\infty[$ is lsc, for every $n>n_0$ we derive

$$w(x_n,z)\leq \liminf_{m\to\infty} w(x_n,x_m)\leq \frac{1}{\alpha}D_w(x_n,T(x_n)).$$

Therefore by hypothesis (2) and by using the above inequality, we obtain

$$
\begin{aligned}
0 &< \inf\{w(x,z)+D_w(x,T(x)) : x\in X\}\\
&\leq \inf\{w(x_n,z)+D_w(x_n,T(x_n)) :\ n>n_0\}\\
&\leq \inf\{\tfrac{2}{\alpha}D_w(x_n,T(x_n)) : n>n_0\}\\
&= \lim_{n\to\infty}\tfrac{2}{\alpha}D_w(x_n,T(x_n))=0.
\end{aligned}
$$

This is a contradiction. Thus we conclude that $z\in T(z)$.

□

Lemma 3.4. *Let (X,d) be a complete metric space, let w be a w-distance on X and let $T: X\to P_c(X)$ be a compact valued mapping. For every $x\in X$ there exists $y\in T(x)$ such that $w(x,y)=D_w(x,T(x))$.*

Proof. Given $x\in X$, define $d=D_w(x,T(x))=\inf\{w(x,y):\ y\in T(x)\}$. By definition of d, we know that for each $n\in\mathbb{N}$ there exists $y_n\in T(x)$ such that $d\leq w(x,y_n)\leq d+\frac{1}{n}$. Since $T(x)$ is a compact subset of X, without loss of generality we may assume that (y_n) converges to $y_0\in T(x)$. Using the lower semi-continuity of the mapping $w(x,.)$ we obtain that the inequality

$$d\leq w(x,y_0)\leq \liminf_{n\to\infty} w(x,y_n)\leq d$$

holds. This means that the proof is complete.

□

Remark 3.5. *The above lemma shows us that for every $x\in X$ the set $I^x_{1,w}$ is nonempty whenever T takes compact values.*

Theorem 3.6. *Let (X,d) be a complete metric space, let w be a w-distance on X, and let $T: X\to P_c(X)$ be a compact valued mapping. Assume that the following conditions hold:*

1. *There exists $\varphi:[0,+\infty[\to[0,1)$ such that*
   (1i) for each $t\in[0,+\infty[$, $\limsup_{r\to t^+}\varphi(r)<1$;
   (1ii) for every $x\in X$ there exists $y\in I^x_{1,w}$ such that $D_w(y,T(y))\leq\varphi(w(x,y))w(x,y)$.
2. *For every $y\in X$ with $y\notin T(y)$, $\inf\{w(x,y)+D_w(x,T(x)):\ x\in X\}>0$.*

Then T has a fixed point.

Proof. Since T takes compact values, by the previous lemma we may guarantee that, for every $x \in X$, the set $I_{1,w}^x$ is nonempty.

Finally, in order to obtain the result it is enough to argue as in the proof of Theorem 3.3.

□

An easy consequence of Theorem 3.6 is the following.

Corollary 3.7. *Let (X, d) be a complete metric space, let w be a w-distance on X, and let $T : X \to X$ be a mapping satisfying the following conditions:*

1. *There exists $\varphi : [0, +\infty[\to [0, 1)$ such that*
   (1i) for each $t \in [0, +\infty[$, $\limsup_{r\to t^+} \varphi(r) < 1$;
   (1ii) for every $x \in X$ $w(T(x), T^2(x)) \leq \varphi(w(x, T(x)))w(x, T(x))$.
2. *For every $y \in X$ with $y \neq T(y)$, $\inf\{w(x, y) + w(x, T(x)) : x \in X\} > 0$.*

Then T has a fixed point. Moreover, if $v = T(v)$, then $w(v, v) = 0$.

Proof. The existence of a fixed point is a direct consequence of Theorem 3.6.

On the other hand, if $T(v) = v$, then we have:

$$w(v, v) = w(T(v), T^2(v)) \leq \varphi(w(v, T(v)))w(v, T(v)) = \varphi(w(v, v))w(v, v),$$

and consequently $w(v, v) = 0$.

□

From this result we may recapture, among others, Corollary 3.8 of [12].

Corollary 3.8. *Let (X, d) be a complete metric space, let p be a w-distance on X, and let $T : X \to X$ be a mapping with the following conditions:*

1. *There exists $q \in [0, 1)$ such that for every $x, y \in X$,*
   $$p(T(x), T(y)) \leq q \max\{p(x, y), p(x, T(x), p(y, T(y)), p(x, T(y)), p(y, T(x))\}.$$
2. *For every $y \in X$ with $y \neq T(y)$, $\inf\{p(x, y) + p(x, T(x)) : x \in X\} > 0$.*

Then T has a fixed point.

Proof. In order to obtain the proof it is enough to notice that the mapping $w : X \times X \to \mathbb{R}^+$ defined by

$$w(x, y) := \max\{\sup\{p(T^i x, T^j x) : i, j \in \mathbb{N} \cap \{0\}\}, p(x, y)\}$$

is a w-distance on X. Since in this case,

$$w(T(x), T^2(x)) \leq qw(x, T(x)).$$

Furthermore,

$$\inf\{w(x, u) + w(x, T(x)) : x \in X\} > 0$$

whenever $u \neq T(u)$. □

4. Examples

Example 4.1. (See [5], Example 3.1.)

Let $X = [0, 1]$ and $w : X \times X \to \mathbb{R}_+$, $w(x, y) = |x| + |y|$ for every $x, y \in [0, 1]$. Let $T : X \to P_c(X)$ be defined by

$$T(x) = \begin{cases} \{\frac{1}{2}x^2\}, & \text{for } x \in [0, \frac{15}{32}] \cup (\frac{15}{32}, 1], \\ \{\frac{17}{96}, \frac{1}{4}\}, & \text{for } x = \frac{15}{32}. \end{cases}$$

For $\varepsilon > 0$ small enough let $b = \frac{17}{30} + \varepsilon$, and let $\varphi : [0, \infty) \to [0, b)$ be the constant function $\varphi(r) \equiv \frac{17}{30}$.

In that case we trivially have that for each $t \in [0, \infty[$,

$$\limsup_{r\to t^+} \varphi(r) < b < 1$$

and condition (1i) of Theorem 3.3 is fulfilled.

If $x, y \neq \frac{15}{32}$, then:

$$w(T(x), T(y)) = w(\frac{1}{2}x^2, \frac{1}{2}y^2) = \frac{1}{2}(|x^2| + |y^2|) \leq \frac{1}{2}(|x| + |y|) \leq \frac{1}{2}w(x, y),$$

and moreover

$$D_w(T(\frac{15}{32}), T(y)) = min\{w(\frac{1}{2}y^2, \frac{17}{96}), w(\frac{1}{2}y^2, \frac{1}{4})\} \leq \frac{1}{2}y^2 + \frac{1}{4} \leq \frac{1}{2}|y| + \frac{17}{30}\frac{15}{32} \leq \frac{17}{30}w(\frac{15}{32}, y).$$

Then for every $x \in X$, $y \in I^x_{b,w}$, $D_w(T(x), T(y)) \leq \frac{17}{30}w(x, y)$, that is $D_w(T(x), T(y)) \leq \varphi(w(x, y))w(x, y)$. Thus, the hypothesis $(1ii)$ on Theorem 3.3 is also accomplished.

Finally, for $y \notin T(y)$, that is, for $y \neq 0$

$$inf\{w(x, y) + D_w(x, T(x)) : x \in [0, 1]\} \geq inf\{w(x, y) : x \in X\} = inf\{|x| + |y| : x \in X\} > |y| > 0.$$

Thus hypothesis (2) of Theorem 3.3 is also satisfied.

We remark that, as it is stated in [5], the mapping T of this example does not satisfy the assumptions of Theorem 2.2 .

Example 4.2. *(Hwei-Mei Ko, 1972, [6]).* Let $(\mathbb{R}^2, d)$ be the ordinary Euclidean two dimensional space. Let $C = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and let $T : C \to P_c(C)$ be the mapping given by

$$T((x_1, x_2)) := \mathrm{conv}(\{(0, 0), (x_1, 0), (0, x_2)\}).$$

Note that $T((x_1, x_2))$ is the triangle with vertices $(0, 0), (x_1, 0), (0, x_2)$, ant that, if $x_1x_2 = 0$, then $T((x_1, x_2))$ is a degenerate triangle.

It is clear that T has compact convex values and, according with [6], T is nonexpansive. The (nonconvex) set of fixed points of T is $W = \{(x_1, x_2) \in C : x_1x_2 = 0\}$. Moreover $T(0, 0) = \{(0, 0)\}$.

Let us observe that, for $x = (1, 1)$ and $y = (1, 0)$ one has

$$T(x) = \mathrm{conv}(\{(0, 0), (1, 0), (0, 1)\})$$

and

$$T(y) = \mathrm{conv}(\{(0, 0), (1, 0)\}) = [0, 1] \times \{0\}.$$

Therefore, it is easy to check that

$$H(T(x), T(y)) = 1 = d(x, y).$$

Thus, the mapping T is non contractive. In particular, T does not fall in the scope of the classical Nadler Theorem for contractive set-valued mappings.

Take $w\big((x_1, x_2), (y_1, y_2)\big) := \sqrt{y_1^2 + y_2^2} = \|(y_1, y_2)\|$. Following [4] it is easy to see that w is a w-distance in (C, d).

Notice that, for every $x \in C$, since $(0, 0) \in T(x)$,

$$D_w(x, T(x)) := \inf\{w(x, y) : y \in T(x)\} = \inf\{\|y\| : y \in T(x)\} = 0.$$

Therefore, for any fixed $b < 1$,

$$I^x_{b,w} := \{y \in T(x) : bw(x, y) \leq D_w(x, T(x))\} = \{y \in T(x) : bw(x, y) = 0\} = \{(0, 0)\}.$$

Let $\varphi : [0, +\infty) \to [0, b)$ the constant function given by $\varphi(r) = \frac{b}{2}$.

For every $x \in C$, $(0, 0)$ is the unique element of $I^x_{b,w}$. So

$$D_w((0, 0), T(0, 0)) = 0 \leq \frac{b}{2}w(x, (0, 0)) = 0.$$

Thus, condition (1ii) of Theorem 3.3 is satisfied.

For every $y \in C$ with $y \notin T(y)$ (which in turn implies that $y \neq (0, 0)$,

$$\inf_{x\in C}\{w(x, y) + D_w(x, T(x))\} = \inf_{x\in C}\{w(x, y)\} = \|y\| > 0.$$

Therefore, condition (2) of Theorem 3.3 is also satisfied.

Example 4.3. In this example, we are going to show that the hypothesis of Theorem 3.3 are almost the best even in the single valued case.

Consider the complete metric space (X, d), where X is the unit sphere of the Banach space $(c_0, \|.\|_\infty)$ and for each $x, y \in X$ the metric is given by $d(x, y) = \|x - y\|_\infty$.

Given a strictly decreasing sequence (a_n) such that $0 < a := \inf_{n\in\mathbb{N}} a_n\} < a_n \leq 1$ for every $n \in \mathbb{N}$, define for each $x = (x(n)),\ y = (y(n))$ in X,

$$w(x, y) = \max\{a_n|x(n) - y(n)| : \ n \in \mathbb{N}\}.$$

It is easy to see that w is a w-distance with respect to the distance d.

Let $T : X \to X$ be the mapping defined by $T(x(1), x(2), ..., x(n), ...) = (1, x(1), x(2), ..., x(n), ...)$. Then, we have:

1. T is well defined;
2. T is a fixed point free mapping;
3. T is a $\|.\|_\infty$-isometry.

However, with respect to the w-distance, the following inequality is satisfied:

$$w(T(x), T(y)) = \max\{a_{n+1}|x(n) - y(n)| : \ n \in \mathbb{N}\} < \max\{a_n|x(n) - y(n)| : \ n \in \mathbb{N}\} = w(x, y).$$

Finally, we prove that for every $x, y \in X$, $\inf\{w(x, y) + w(x, T(x)) : \ x \in X\} > 0$.

Otherwise, we may find a sequence (x_n) of elements of X such that $w(x_n, y) + w(x_n, T(x_n)) \to 0$ as $n \to \infty$.

By definition of w, we derive that $w(x_n, y) \to 0$ as $n \to \infty$ and thus we obtain that $x_n \to y$. Since T is a continuous mapping, we have that $T(x_n) \to T(y)$.

The properties of this w-distance allow us to have that in this case $w(y, T(y)) = 0$, and then $y = T(y)$. This is a contradiction

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