STONE-WEIERSTRASS TYPE THEOREMS FOR ALGEBRAS CONTAINING CONTINUOUS UNBOUNDED FUNCTIONS

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Abstract. Let X be a topological space, $\mathfrak{S}$ a cover of X and $C_b(X, \mathbb{K}; \mathfrak{S})$ the algebra of all $\mathbb{K}$-valued continuous functions on X, which are bounded on every $S \in \mathfrak{S}$. Necessary and sufficient conditions for a subalgebra $\mathcal{A}$ of $C_b(X, \mathbb{K}; \mathfrak{S})$ to be dense in $C_b(X, \mathbb{K}; \mathfrak{S})$ in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence and in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$ on $C_b(X, \mathbb{K}; \mathfrak{S})$ are given. Also, necessary and sufficient conditions for the completeness of the topological algebras $(C_b(X, \mathbb{K}; \mathfrak{S}), \tau_{\mathfrak{S}})$ and $(C_b(X, \mathbb{K}; \mathfrak{S}), \beta_{\mathfrak{S}})$ are given.

1 Introduction Let X be a topological space and $\mathbb{K}$ one of the fields $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers. The algebra $C_b(X, \mathbb{K})$ of all $\mathbb{K}$-valued continuous and bounded functions on X is one of the most studied objects in modern analysis. Usually it has been equipped either with the topology τ_X of uniform convergence or with the strict topology β_X. It is well-known that the structure of $(C_b(X), \tau_X)$ is quite complicated. For example, its Gelfand space (the set of all maximal regular ideals equipped with the relative weak*-topology) is homeomorphic to the Stone-Cech compactification $\beta(X)$ of X. Another difficulty is based on the fact that $(C_b(X), \tau_X)$ does not satisfy the Stone-Weierstrass property, i.e., a pointseparating subalgebra (if $\mathbb{K} = \mathbb{C}$, then a self-adjoint subalgebra), which is bounded away from zero, is not necessarily uniformly dense in $C_b(X)$. In fact, in order that a subalgebra is dense in $(C_b(X), \tau_X)$ one has to replace the property separation of points by separation of zero-sets. On the other hand, $(C_b(X), \beta_X)$ is in some sense easier to handle than $(C_b(X), \tau_X)$. For example, its Gelfand space is homeomorphic to X and it also satisfies the Stone-Weierstrass property. However, $(C_b(X), \beta_X)$ is not necessarily complete whereas $(C_b(X), \tau_X)$ always has this property.

The structure of $C_b(X)$, when equipped with the uniform or with the strict topology, has been generalized in the literature in several ways. In [6] and [7] proper subalgebras of $C_b(X)$ and their Stone-Weierstrass type properties and ideal structures were developed. In the present paper, Stone-Weierstrass properties as well as completeness properties of algebras of continuous functions containing also unbounded functions (in both topologies described above) are considered. These results are important by the remark made in [5]. Namely, Stone-Weierstrass property is closely related to the Gelfand representations of several kinds of topological algebras. In particular, the Gelfand representations of locally A-convex and uniformly locally A-convex algebras can be described by means of such function algebras (see [4]). Also, the results of this paper are needed in [3], where a description of the ideal structure of algebras containing continuous unbounded functions is given.

2 Preliminaries

1. Let X be a topological space, $\mathfrak{S}$ a cover[1] of X and $\mathbb{K}$ one of the fields $\mathbb{R}$ of real numbers

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[1] That is, $\mathfrak{S}$ is a collection of subsets of X, whose union is X. The cover $\mathfrak{S}$ of X is a *closed cover*, if every $S \in \mathfrak{S}$ is closed in X, and a *compact cover*, if every $S \in \mathfrak{S}$ is compact in X.

or $\mathbb{C}$ of complex numbers. We will denote by $C(X,\mathbb{K})$ the set of all $\mathbb{K}$-valued continuous functions on X, by $C_b(X,\mathbb{K})$ the subset of $C(X,\mathbb{K})$ consisting of bounded functions, and by $C_b(X,\mathbb{K};\mathfrak{S})$ the subset of $C(X,\mathbb{K})$ consisting of functions, which are bounded on every $S\in\mathfrak{S}$. Obviously

$$C_b(X,\mathbb{K})\subset C_b(X,\mathbb{K};\mathfrak{S})\subset C(X,\mathbb{K}).$$

Herewith, $C_b(X,\mathbb{K};\mathfrak{S})=C_b(X,\mathbb{K})$, if $\mathfrak{S}$ has only a finite number of elements (in particular, if $\mathfrak{S}=\{X\}$), and $C_b(X,\mathbb{K};\mathfrak{S})=C(X,\mathbb{K})$, if $\mathfrak{S}$ consists of *bounding subsets*[2] of X, that is, of such subsets $S\subset X$ for which $f(S)$ is bounded in $\mathbb{K}$ for every $f\in C(X,\mathbb{K})$. It is easy to see that $C_b(X,\mathbb{K})$, $C_b(X,\mathbb{K};\mathfrak{S})$ and $C(X,\mathbb{K})$ are algebras over $\mathbb{K}$ with respect to the pointwise algebraic operations on X.

2. Let X be a topological space. It is said that a subset $\mathcal{A}$ of $C(X,\mathbb{K})$

a) *separates the points of* X, if for any pair (x_1,x_2) of distinct points of X there exists $f\in\mathcal{A}$ such that $f(x_1)\neq f(x_2)$;

b) *separates the zero-sets*[3] *of* X, if for any pair (Z_1,Z_2) of disjoint zero-sets of X there exists an $\mathbb{R}$-valued $f\in\mathcal{A}$ such that $f(Z_1)$ and $f(Z_2)$ have disjoint closures in $\mathbb{R}$;

c) is *bounded away from zero*, if for every $x\in X$ there exists $f\in\mathcal{A}$ such that $f(x)\neq 0$;

and in the case when $\mathbb{K}=\mathbb{C}$

d) is *self-adjoint*, if $\bar{f}\in\mathcal{A}$ for every $f\in\mathcal{A}$ (here $\bar{f}$ denotes the complex conjugate of f).

3. Let X be a topological space and $\mathfrak{S}$ a cover of X. It is said (see [2], p. 5) that $(X,\mathfrak{S})$ has the *extension property*, if for every $S\in\mathfrak{S}$ and $f\in C_b(S,\mathbb{K})$ there exists $g\in C_b(X,\mathbb{K};\mathfrak{S})$ such that $g_{|S}=f$. It is known that $(X,\mathfrak{S})$ has the extension property, for example, if either X is a completely regular Hausdorff space and every $S\in\mathfrak{S}$ is compact in X (see [14], p. 43) or X is a normal space and every $S\in\mathfrak{S}$ is closed in X (Tietze's extension theorem).

4. Let X be a topological space and $\mathfrak{S}$ a cover of X. We will say that $(X,\mathfrak{S})$ has the *weak extension property*, if for arbitrary $\epsilon>0$, $S\in\mathfrak{S}$ and $f\in C_b(S,\mathbb{K})$ there exists $g\in C_b(X,\mathbb{K};\mathfrak{S})$ such that $\sup_{x\in S}|f(x)-g(x)|<\epsilon$. If $(X,\mathfrak{S})$ has the extension property, then $(X,\mathfrak{S})$ also has the weak extension property.

5. Let X be a topological space, $\mathfrak{S}$ a cover of X and $S_0^+(X)$ the set of all non-negative upper semicontinuous real-valued functions on X, which vanish at infinity. We will denote by τ_X the *topology of uniform convergence* on $C_b(X,\mathbb{K})$ (defined by the norm $\|\ \|_X$, where

$$\|f\|_X=\sup_{x\in X}|f(x)|$$

for every $f\in C_b(X,\mathbb{K})$), by β_X the *strict topology* on $C_b(X,\mathbb{K})$ (defined by the system $\{p_v : v\in S_0^+(X)\}$ of seminorms, where

$$p_v(f)=\sup_{x\in X}v(x)|f(x)|$$

for every $f\in C_b(X,\mathbb{K})$), by $\tau_{\mathfrak{S}}$ the *topology of* $\mathfrak{S}$*-convergence* on $C_b(X,\mathbb{K};\mathfrak{S})$ (defined by the system $\{p_S : S\in\mathfrak{S}\}$ of seminorms, where

$$p_S(f)=\sup_{x\in S}|f(x)|$$

[2]For example, pseudocompact (in particular, compact) sets.

[3]A subset Z of X is called a zero-set of X, if $Z=\{x\in X : f(x)=0\}$ for some $f\in C(X,\mathbb{R})$.

for every $f \in C_b(X,\mathbb{K};\mathfrak{S})$), and by $\beta_{\mathfrak{S}}$ the $\mathfrak{S}$-*strict topology* on $C_b(X,\mathbb{K};\mathfrak{S})$ (defined by the system $\{p_{S,v_S} : S \in \mathfrak{S}, v_S \in S_0^+(S)\}$ of seminorms, where

$$p_{S,v_S}(f) = \sup_{x\in S} v_S(x)|f(x)|$$

for every $f \in C_b(X,\mathbb{K};\mathfrak{S})$). We clearly have $\beta_X \subset \tau_X$ on $C_b(X,\mathbb{K})$ and $\beta_{\mathfrak{S}} \subset \tau_{\mathfrak{S}}$ on $C_b(X,\mathbb{K};\mathfrak{S})$.

It is easy to see that the multiplication on algebras $C_b(X,\mathbb{K})$ and $C_b(X,\mathbb{K};\mathfrak{S})$ is jointly continuous with respect to all topologies mentioned above. Herewith, the well-known topological algebras $(C_b(X,\mathbb{K}),\tau_X)$, $(C_b(X,\mathbb{K}),\beta_X)$ and $(C(X,\mathbb{K}),\tau_{\mathfrak{K}})$ (here $\mathfrak{K}$ denotes the collection of all compact subsets of X, and so $\tau_{\mathfrak{K}}$ is the usual compact-open topology on $C(X,\mathbb{K})$) are special cases of the topological algebras $(C_b(X,\mathbb{K};\mathfrak{S}),\tau_{\mathfrak{S}})$ and $(C_b(X,\mathbb{K};\mathfrak{S}),\beta_{\mathfrak{S}})$.

6. Necessary and sufficient conditions for a subalgebra $\mathcal{A}$ of $C_b(X,\mathbb{K};\mathfrak{S})$ to be dense in $C_b(X,\mathbb{K};\mathfrak{S})$ in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence and in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$ are given. Also, necessary and sufficient conditions for the completeness of $(C_b(X,\mathbb{K};\mathfrak{S}),\tau_{\mathfrak{S}})$ and $(C_b(X,\mathbb{K};\mathfrak{S}),\beta_{\mathfrak{S}})$ are given in the present paper.

3 Stone-Weierstrass type theorems for algebras of bounded continuous functions The classical Stone-Weierstrass theorem gives sufficient conditions for a subalgebra $\mathcal{A}$ of $C(X,\mathbb{K})$ to be dense in $C(X,\mathbb{K})$ in the topology τ_X of uniform convergence with X a compact Hausdorff space. This result and its several generalizations (see for example [6] and [23], p. 119) are applicable in many areas of mathematics. We shall now represent these generalizations, as they are needed later in this paper.

Theorem 3.1. (see [20], Theorem 2.5 for[4] $\mathbb{K}=\mathbb{R}$ and [1], Theorem 1 for $\mathbb{K}=\mathbb{C}$) *Let X be a topological space. A subalgebra (if $\mathbb{K}=\mathbb{C}$, then a self-adjoint subalgebra) $\mathcal{A}$ of $C_b(X,\mathbb{K})$ is dense in $C_b(X,\mathbb{K})$ in the topology τ_X of uniform convergence if and only if*

a) *$\mathcal{A}$ separates the zero-sets of X;*

b) *there exists $f\in\mathcal{A}$ such that* $\inf_{x\in X}|f(x)|>0$.

To describe dense subalgebras of $C_b(X,\mathbb{K})$ in the strict topology β_X, we need the following results:

Lemma 3.1. *Let X be a topological space, $\mathcal{A}$ a subalgebra of $C_b(X,\mathbb{R})$ and $f\in C_b(X,\mathbb{R})$ arbitrary. Then f belongs to the closure of $\mathcal{A}$ in the strict topology β_X if and only if the following conditions are satisfied:*

a) *for every pair (x,y) of distinct points of X such that $f(x)\neq f(y)$ there exists $g\in\mathcal{A}$ such that $g(x)\neq g(y)$;*

b) *for every $x\in X$ such that $f(x)\neq 0$ there exists $g\in\mathcal{A}$ such that $g(x)\neq 0$.*

Proof. See [22], p. 69. ∎

Lemma 3.2. *Let X be a topological space and $\mathcal{A}$ a self-adjoint subalgebra of $C_b(X,\mathbb{C})$. Then the set $\Re\mathcal{A}$ of real parts $\Re f$ of functions $f\in\mathcal{A}$ is a subalgebra of $C_b(X,\mathbb{R})$ and $\mathcal{A}=\Re\mathcal{A}+i\Re\mathcal{A}$.*

Proof. See [19], pp. 47–48. ∎

[4]See also [11], Theorem 6; [21], Corollary 1; [9] and [12].

Theorem 3.2. *Let X be a topological space. A subalgebra (if $\mathbb{K} = \mathbb{C}$, then a self-adjoint subalgebra) $\mathcal{A}$ of $C_b(X, \mathbb{K})$ is dense in $C_b(X, \mathbb{K})$ in the strict topology β_X, if (when X is a completely regular Hausdorff space, then if and only if)*

a) *$\mathcal{A}$ separates the points of X;*

b) *$\mathcal{A}$ is bounded away from zero.*

Proof. If $\mathbb{K} = \mathbb{R}$, then by [13], Theorem 3.1[5], the conditions a) and b) are sufficient to imply that $\mathcal{A}$ is dense in $C_b(X, \mathbb{R})$ in the strict topology β_X. Let now $\mathbb{K} = \mathbb{C}$ and suppose that $\mathcal{A}$ satisfies the conditions a) and b). Then for any two distinct points x and y of X there exists $f, f' \in \mathcal{A}$ such that $f(x) \neq f(y)$ and $f'(x) \neq 0$. By Lemma 3.2, $f = g + ih$ and $f' = g' + ih'$, where g, g', h and h' belong to $\Re\mathcal{A}$. Obviously $g(x) \neq g(y)$ or $h(x) \neq h(y)$ and $g'(x) \neq 0$ or $h'(x) \neq 0$. Since $\Re\mathcal{A}$ is a subalgebra of $C_b(X, \mathbb{R})$, the first part of Theorem 3.2 implies that $\Re\mathcal{A}$ is dense in $C_b(X, \mathbb{R})$ in the strict topology β_X. Now, for a given $f \in C_b(X, \mathbb{C})$, the real part $\Re f$ and the imaginary part $\Im f$ of f belong to $C_b(X, \mathbb{R})$, and so

$$f = \Re f + i\Im f \in \mathrm{cl}(\Re\mathcal{A}) + i\mathrm{cl}(\Re\mathcal{A}) \subset \mathrm{cl}(\Re\mathcal{A} + i\Re\mathcal{A}) = \mathrm{cl}(\mathcal{A}).$$

Hence, $\mathcal{A}$ is dense in $C_b(X, \mathbb{C})$ in the strict topology β_X.

Suppose next that X is a completely regular Hausdorff space and $\mathcal{A}$ is a dense subalgebra of $C_b(X, \mathbb{K})$ in the strict topology β_X. Then any two distinct points x and y of X define a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(y) = 0$. As $f \in C_b(X, \mathbb{R})$ and $\mathcal{A}$ (in the complex case $\Re\mathcal{A}$)[6] is dense in $C_b(X, \mathbb{R})$, Lemma 3.1 implies the existence of $g, h \in \mathcal{A}$ such that $g(x) \neq g(y)$ and $h(x) \neq 0$. Hence, $\mathcal{A}$ satisfies the conditions a) and b). $\square$

Theorem 3.3. (see [19], p. 48) *Let X be a completely regular Hausdorff space. A subalgebra (if $\mathbb{K} = \mathbb{C}$, then a self-adjoint subalgebra) $\mathcal{A}$ of $C(X, \mathbb{K})$ is dense in $C(X, \mathbb{K})$ in the compact-open topology $\tau_{\mathfrak{K}}$ if and only if $\mathcal{A}$ satisfies the conditions* a) *and* b) *of* Theorem 3.2.

4 Stone-Weierstrass type theorems for algebras containing unbounded continuous functions To describe dense subalgebras of $C_b(X, \mathbb{K}; \mathfrak{S})$ in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence and in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$, we need the following two results.

Proposition 4.1. (In the case of topology $\tau_{\mathfrak{S}}$, see also [2], p. 5) *Let X be a topological space and $\mathfrak{S}$ a cover of X. A subalgebra $\mathcal{A}$ of $C_b(X, \mathbb{K}; \mathfrak{S})$ is dense in $C_b(X, \mathbb{K}; \mathfrak{S})$ in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence (respectively, in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$), if*

$$\mathcal{A}_S = \{f_{|S} : f \in \mathcal{A}\}$$

is dense in $C_b(S, \mathbb{K})$ in the topology τ_S of uniform convergence (respectively, in the strict topology β_S) for every $S \in \mathfrak{S}$.

Proof. Let $f \in C_b(X, \mathbb{K}; \mathfrak{S})$ be given and denote by $O(f)$ an arbitrary neighbourhood of f in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence (respectively, in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$) on $C_b(X, \mathbb{K}; \mathfrak{S})$. Then there exist $\epsilon > 0$ and $S \in \mathfrak{S}$ such that

$$\{g \in C_b(X, \mathbb{K}; \mathfrak{S}) : p_S(f - g) < \epsilon\} \subset O(f)$$

[5] See also [22], p. 70 and [8], p. 101.

[6] Suppose $\Re\mathcal{A}$ is not β_X-dense in $C_b(X, \mathbb{R})$. Then there exist $f_0 \in C_b(X, \mathbb{R})$ and a neighbourhood $O(f_0)$ of f_0 in the strict topology β_X on $C_b(X, \mathbb{R})$ such that $\Re\mathcal{A} \cap O(f_0)$ is empty. Further, if $g_0 = f_0 + if_0$ and $O(g_0) = O(f_0) + iO(f_0)$, then $g_0 \in C_b(X, \mathbb{C})$ and $O(g_0)$ is a neighbourhood of g_0 in the strict topology β_X on $C_b(X, \mathbb{C})$ such that $O(g_0) \cap \mathcal{A}$ is empty, a contradiction.

(respectively, there exist $\epsilon > 0$, $S \in \mathfrak{S}$ and $v_S \in S_0^+(S)$ such that

$$\{g \in C_b(X, \mathbb{K}; \mathfrak{S}) : p_{S,v_S}(f-g) < \epsilon\} \subset O(f)).$$

Since $\mathcal{A}_S$ is dense in $C_b(S, \mathbb{K})$ in the topology τ_S of uniform convergence (respectively, in the strict topology β_S), there exists $g \in \mathcal{A}$ such that

$$\|f_{|S} - g_{|S}\|_S < \epsilon \quad (\text{respectively, } p_{v_S}(f_{|S} - g_{|S}) < \epsilon).$$

As

$$p_S(f-g) = \|f_{|S} - g_{|S}\|_S \quad (\text{respectively, } p_{S,v_S}(f-g) = p_{v_S}(f_{|S} - g_{|S})),$$

$\mathcal{A} \cap O(f)$ is not empty. Hence, $\mathcal{A}$ is dense in $C_b(X, \mathbb{K}; \mathfrak{S})$ in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence and in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$. □

For the converse, we have the following:

Proposition 4.2. *Let X be a topological space, $\mathfrak{S}$ a cover of X and $\mathcal{A}$ a dense subalgebra of $C_b(X, \mathbb{K}; \mathfrak{S})$ in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence (respectively, in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$). Then for every $S \in \mathfrak{S}$*

a) *$\mathcal{A}_S$ is dense in $C_b(S, \mathbb{K})$ in the topology τ_S of uniform convergence if and only if $(X, \mathfrak{S})$ has the weak extension property;*

b) *$\mathcal{A}_S$ is dense in $C_b(S, \mathbb{K})$ in the strict topology β_S.*

Proof. a) It clearly suffices to show that if $(X, \mathfrak{S})$ has the weak extension property, then $\mathcal{A}_S$ is dense in $C_b(S, \mathbb{K})$ in the topology τ_S of uniform convergence. So, let $f \in C_b(S, \mathbb{K})$ and $\epsilon > 0$ be given. Then there exists $g \in C_b(X, \mathbb{K}; \mathfrak{S})$ such that

$$\sup_{x \in S} |f(x) - g(x)| = \|f - g_{|S}\|_S < \frac{\epsilon}{2}.$$

On the other hand, the set

$$O(g) = \{h \in C_b(X, \mathbb{K}; \mathfrak{S}) : p_S(g-h) < \frac{\epsilon}{2}\}$$

is a neighbourhood of g in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence on $C_b(X, \mathbb{K}; \mathfrak{S})$. Since $\mathcal{A}$ is $\tau_{\mathfrak{S}}$-dense in $C_b(X, \mathbb{K}; \mathfrak{S})$, the intersection $\mathcal{A} \cap O(g)$ is not empty. This implies the existence of $h \in \mathcal{A}$ such that

$$\|g_{|S} - h_{|S}\|_S = p_S(g-h) < \frac{\epsilon}{2}.$$

Thus,

$$\|f - h_{|S}\|_S \leq \|f - g_{|S}\|_S + \|g_{|S} - h_{|S}\|_S < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and so $\mathcal{A}_S$ is dense in $C_b(S, \mathbb{K})$ in the topology τ_S of uniform convergence.

b) Let $f \in C_b(S, \mathbb{K})$ be given and denote by $O(f)$ an arbitrary neighbourhood of f in the strict topology β_S on $C_b(S, \mathbb{K})$. Then there exist $v_S \in S_0^+(S)$ and $\epsilon > 0$ such that

$$\{g \in C_b(S, \mathbb{K}) : p_{v_S}(f-g) < \epsilon\} \subset O(f).$$

Moreover, the set $C_b(X, \mathbb{K}; \mathfrak{S})_S = \{g_{|S} : g \in C_b(X, \mathbb{K}; \mathfrak{S})\}$ is clearly a subalgebra (if $\mathbb{K} = \mathbb{C}$, then a self-adjoint subalgebra) of $C_b(S, \mathbb{K})$, which separates the points of S and is bounded away from zero. So, by Theorem 3.2, $C_b(X, \mathbb{K}; \mathfrak{S})_S$ is dense in $C_b(S, \mathbb{K})$ in the strict topology β_S. Thus, there exists $g \in C_b(X, \mathbb{K}; \mathfrak{S})$ such that

$$p_{v_S}(f - g_{|S}) < \frac{\epsilon}{2}.$$

Let now

$$O(g) = \{h \in C_b(X, \mathbb{K}; \mathfrak{S}) : p_{S,v_S}(g - h) < \frac{\epsilon}{2}\}.$$

Then $O(g)$ is a neighbourhood of g in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$ on $C_b(X, \mathbb{K}; \mathfrak{S})$. Since $\mathcal{A}$ is $\beta_{\mathfrak{S}}$-dense in $C_b(X, \mathbb{K}; \mathfrak{S})$, the intersection $\mathcal{A} \cap O(g)$ is not empty. Thus, there exists $h \in \mathcal{A}$ such that

$$p_{v_S}(g_{|S} - h_{|S}) = p_{S,v_S}(g - h) < \frac{\epsilon}{2},$$

and so

$$p_{v_S}(f - h_{|S}) \leq p_{v_S}(f - g_{|S}) + p_{v_S}(g_{|S} - h_{|S}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, $\mathcal{A}_S \cap O(f)$ is not empty, and therefore $\mathcal{A}_S$ is dense in $C_b(S, \mathbb{K})$ in the strict topology β_S. □

Theorem 4.1. *Let X be a topological space, $\mathfrak{S}$ a cover*[7] *of X and $\mathcal{A}$ a subalgebra (if $\mathbb{K} = \mathbb{C}$, then a self-adjoint subalgebra) of $C_b(X, \mathbb{K}; \mathfrak{S})$. Then $\mathcal{A}$ is dense in $C_b(X, \mathbb{K}; \mathfrak{S})$ in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence, if (when $(X, \mathfrak{S})$ has the weak extension property, then if and only if) for every $S \in \mathfrak{S}$*

a) *$\mathcal{A}_S$ separates the zero-sets of S;*

b) *there exists $f \in \mathcal{A}$ such that $\inf_{x \in S} |f(x)| > 0$.*

Proof. Theorem 4.1 holds by Propositions 4.1 and 4.2 and Theorem 3.1. □

Theorem 4.2. (see [2], Corollary 2) *Let X be a completely regular Hausdorff space and $\mathfrak{S}$ a compact cover of X, which is closed with respect to finite unions. A subalgebra (if $\mathbb{K} = \mathbb{C}$, then a self-adjoint subalgebra) $\mathcal{A}$ of $C(X, \mathbb{K})$ is dense in $C(X, \mathbb{K})$ in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence if and only if $\mathcal{A}$ separates the points of X and is bounded away from zero.*

Proof. Since X is a completely regular Hausdorff space and every $S \in \mathfrak{S}$ is compact, $(X, \mathfrak{S})$ has the extension property and $C_b(X, \mathbb{K}; \mathfrak{S}) = C(X, \mathbb{K})$. If $\mathcal{A}$ is dense in $C(X, \mathbb{K})$ in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence, then $\mathcal{A}_S$ is, by Proposition 4.2, dense in $C(S, \mathbb{K})$ in the topology τ_S of uniform convergence for every $S \in \mathfrak{S}$. Let now x and y be any two distinct points of X. Then there exists $S \in \mathfrak{S}$ such that $x, y \in S$. By the classical Stone-Weierstrass theorem, there exists $f, g \in \mathcal{A}$ such that $f_{|S}(x) \neq f_{|S}(y)$ and $g_{|S}(x) \neq 0$. Thus, $\mathcal{A}$ separates the points of X and is bounded away from zero.

Suppose next that $\mathcal{A}$ separates the points of X and is bounded away from zero. Then for every $S \in \mathfrak{S}$, $\mathcal{A}_S$ is a subalgebra (if $\mathbb{K} = \mathbb{C}$, then a self-adjoint subalgebra) of $C(S, \mathbb{K})$, which satisfies corresponding conditions. So, again by the classical Stone-Weierstrass theorem, $\mathcal{A}_S$ is τ_S-dense in $C(S, \mathbb{K})$ for every $S \in \mathfrak{S}$. Thus, by Proposition 4.1, $\mathcal{A}$ is dense in $C(X, \mathbb{K})$ in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence. □

[7] The case when $\mathfrak{S}$ consists of all pseudocompact C^*-embedded subsets of X (that is, of such subsets $S \subset X$ for which every $f \in C_b(S, \mathbb{R})$ can be extended to a function from $C_b(X, \mathbb{R})$) is considered in [20], Theorem 3.1, and the case in which X is a real Banach space and $\mathfrak{S}$ consists of all bounded subsets of X is considered in [17], Proposition 5.

Similarly to Theorem 4.2, we can prove (using Theorem 3.2 instead of the classical Stone-Weierstrass theorem) the following result:

Theorem 4.3. *Let X be a topological space and $\mathfrak{S}$ a cover of completely regular Hausdorff subsets of X, which is closed with respect to finite unions. A subalgebra (if $\mathbb{K} = \mathbb{C}$, then a self-adjoint subalgebra) $\mathcal{A}$ of $C_b(X, \mathbb{K}; \mathfrak{S})$ is dense in $C_b(X, \mathbb{K}; \mathfrak{S})$ in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$ if and only if $\mathcal{A}$ separates the points of X and is bounded away from zero.*

5 Completeness Let X be a topological space, $\mathfrak{S}$ a cover of X and $F_b(X, \mathbb{K}; \mathfrak{S})$ the set of all $\mathbb{K}$-valued functions on X, which are bounded on every $S \in \mathfrak{S}$. It is easy to see that $F_b(X, \mathbb{K}; \mathfrak{S})$ is an algebra over $\mathbb{K}$ with respect to the pointwise algebraic operations on X. If we endowe $F_b(X, \mathbb{K}; \mathfrak{S})$ with the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence or with the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$, then the multiplication on $F_b(X, \mathbb{K}; \mathfrak{S})$ is jointly continuous.

In this section we shall study completeness properties of $C_b(X, \mathbb{K}; \mathfrak{S})$, first in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence, and then in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$. The next useful lemma and the definition following it, are needed in several results of the section.

Lemma 5.1. *Let X be a completely regular Hausdorff space and $\mathfrak{S}$ a cover of X. Then $F_b(X, \mathbb{K}; \mathfrak{S})$ is complete in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence and in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$.*

Proof. For the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence, see [18], p. 71. For the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$ the proof is similar to the proof for the usual strict topology. □

Definition 5.1. Let X be a topological space and $\mathfrak{S}$ a cover of X. We will say that X is a $\mathfrak{S}_{\mathbb{R}}$-*space*, if from $f \in F_b(X, \mathbb{R}; \mathfrak{S})$ and $f_{|S} \in C(S, \mathbb{R})$ for every $S \in \mathfrak{S}$, it follows that $f \in C(X, \mathbb{R})$.

So in particular, when $\mathfrak{S} = \mathfrak{K}$, then a $\mathfrak{S}_{\mathbb{R}}$-space is just a usual $k_{\mathbb{R}}$-space, that is, such a topological space X for which every $f : X \to \mathbb{R}$ is continuous, if the restriction $f_{|K}$ is continuous for every compact subset K of X.

Proposition 5.1. *Let X be a completely regular Hausdorff space, $\mathfrak{S}$ a cover of X and*

$$\mathcal{A} = \{f \in F_b(X, \mathbb{K}; \mathfrak{S}) : f_{|S} \in C(S, \mathbb{K}) \text{ for every } S \in \mathfrak{S}\}.$$

Then $\mathcal{A}$ is complete in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence.

Proof. Let (f_α) be a Cauchy net in $(\mathcal{A}, \tau_{\mathfrak{S}})$. Then, by Lemma 5.1, (f_α) converges to a function $f \in F_b(X, \mathbb{K}; \mathfrak{S})$ in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence. In particular, for every $x \in X$ the net $(f_\alpha(x))$ converges to $f(x)$. Let now $S \in \mathfrak{S}$ be given. Then $(f_{\alpha|S})$ is clearly a Cauchy net in $(C_b(S, \mathbb{K}), \tau_S)$. So, by the completeness of $(C_b(S, \mathbb{K}), \tau_S)$, the net $(f_{\alpha|S})$ converges to a function $g_S \in C_b(S, \mathbb{K})$ in the topology τ_S of uniform convergence. In particular, for every $x \in S$ the net $(f_\alpha(x))$ converges to $g_S(x)$. Thus, $f_{|S} = g_S \in C(S, \mathbb{K})$, and so $\mathcal{A}$ is complete in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence. □

Corollary 5.1. *Let X be a completely regular Hausdorff space and $\mathfrak{S}$ a cover of X. If X is a $\mathfrak{S}_{\mathbb{R}}$-space, then $(C_b(X, \mathbb{K}; \mathfrak{S}), \tau_{\mathfrak{S}})$ is complete.*

Theorem 5.1. *Let X be a completely regular Hausdorff space, $\mathfrak{S}$ a cover of X, and suppose that $(X, \mathfrak{S})$ has the weak extension property. Then $(C_b(X, \mathbb{K}; \mathfrak{S}), \tau_{\mathfrak{S}})$ is complete if and only if X is a $\mathfrak{S}_{\mathbb{R}}$-space.*

Proof. By Corollary 5.1, it suffices to show that if $(C_b(X,\mathbb{K};\mathfrak{S}),\tau_{\mathfrak{S}})$ is complete, then X is a $\mathfrak{S}_{\mathbb{R}}$-space. For this, let[8] $f\in\mathcal{A}$ be given and denote by $O(f)$ an arbitrary neighbourhood of f in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence on $\mathcal{A}$. Then there exist $\epsilon>0$ and $S\in\mathfrak{S}$ such that

$$\{g\in\mathcal{A}:p_S(f-g)<\epsilon\}\subset O(f).$$

Since $(X,\mathfrak{S})$ has the weak extension property and $f_{|S}\in C_b(S,\mathbb{K})$, there exists $g\in C_b(X,\mathbb{K};\mathfrak{S})$ such that

$$p_S(f-g)=\sup_{x\in S}|f(x)-g(x)|<\epsilon.$$

Thus, the intersection $C_b(X,\mathbb{K};\mathfrak{S})\cap O(f)$ is not empty, and so $C_b(X,\mathbb{K};\mathfrak{S})$ is dense in $\mathcal{A}$ in the topology $\tau_{\mathfrak{S}}$ of $\mathfrak{S}$-convergence. However, since $(C_b(X,\mathbb{K};\mathfrak{S}),\tau_{\mathfrak{S}})$ is a complete subset of the Hausdorff space $\mathcal{A}$, we must have $C_b(X,\mathbb{K};\mathfrak{S})=\mathcal{A}$. Hence, X is a $\mathfrak{S}_{\mathbb{R}}$-space. ☐

Next we will study the completeness of $(C_b(X,\mathbb{K};\mathfrak{S}),\beta_{\mathfrak{S}})$. Note first that even for a completely regular Hausdorff space X, $(C_b(X,\mathbb{K}),\beta_X)$ is not necessarily complete. Namely,

$$(C_b(X,\mathbb{K}),\beta_X)_c=\{f\in F_b(X,\mathbb{K}):f_{|K}\in C(K,\mathbb{K})\text{ for every }K\in\mathfrak{K}\},$$

where $(C_b(X,\mathbb{K}),\beta_X)_c$ denotes the completion of $(C_b(X,\mathbb{K}),\beta_X)$ and $F_b(X,\mathbb{K})$ denotes the set of all $\mathbb{K}$-valued bounded functions on X (see [16], p. 27 and [15], p. 278). So in particular, $(C_b(X,\mathbb{K}),\beta_X)$ is complete if and only if X is a $k_{\mathbb{R}}$-space.

Proposition 5.2. *Let X be a completely regular Hausdorff space, $\mathfrak{S}$ a cover of X and*

$$\mathcal{B}=\{f\in F_b(X,\mathbb{K};\mathfrak{S}):f_{|S}\in(C_b(S,\mathbb{K}),\beta_S)_c\textit{ for every }S\in\mathfrak{S}\}.$$

Then $\mathcal{B}$ is complete in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$.

Proof. Let (f_α) be a Cauchy net in $(\mathcal{B},\beta_{\mathfrak{S}})$. Then, by Lemma 5.1, (f_α) converges to a function $f\in F_b(X,\mathbb{K};\mathfrak{S})$ in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$. In particular, for every $x\in X$ the net $(f_\alpha(x))$ converges to $f(x)$. Let now $S\in\mathfrak{S}$ be given. Then $(f_{\alpha|S})$ is clearly a Cauchy net in $(C_b(S,\mathbb{K}),\beta_S)_c$. So, it converges to a function $g_S\in(C_b(S,\mathbb{K}),\beta_S)_c$. In particular, for every $x\in S$ the net $(f_\alpha(x))$ converges to $g_S(x)$. Thus, $f_{|S}=g_S$, and so $\mathcal{B}$ is complete in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$. ☐

Theorem 5.2. *The completion of $(C_b(X,\mathbb{K};\mathfrak{S}),\beta_{\mathfrak{S}})$ is $(\mathcal{B},\beta_{\mathfrak{S}})$.*

Proof. By Proposition 5.2, we only have to prove that $C_b(X,\mathbb{K};\mathfrak{S})$ is dense in $\mathcal{B}$ in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$. For this, let $f\in\mathcal{B}$ be given and denote by $O(f)$ an arbitrary neighbourhood of f in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$ on $\mathcal{B}$. Then there exist $S\in\mathfrak{S}$, $v_S\in S_0^+(S)$ and $\epsilon>0$ such that

$$\{g\in\mathcal{B}:p_{S,v_S}(f-g)<\frac{\epsilon}{2}\}\subset O(f).$$

Further, since $f_{|S}\in(C_b(S,\mathbb{K}),\beta_S)_c$, there exists $g\in C_b(S,\mathbb{K})$ such that

$$p_{v_S}(f_{|S}-g)<\frac{\epsilon}{2}.$$

On the other hand, by Proposition 4.2 b), $C_b(X,\mathbb{K};\mathfrak{S})_S=\{f_{|S}:f\in C_b(X,\mathbb{K};\mathfrak{S})\}$ is dense in $C_b(S,\mathbb{K})$ in the strict topology β_S. Thus, there exists $h\in C_b(X,\mathbb{K};\mathfrak{S})$ such that

$$p_{v_S}(g-h_{|S})<\frac{\epsilon}{2}.$$

[8] The set $\mathcal{A}$ has been defined in Proposition 5.1.

Hence,

$$p_{S,v_S}(f-h) = p_{v_S}(f_{|S} - h_{|S}) \le p_{v_S}(f_{|S} - g) + p_{v_S}(g - h_{|S}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and so $C_b(X, \mathbb{K}; \mathfrak{S})$ is dense in $\mathcal{B}$ in the $\mathfrak{S}$-strict topology $\beta_{\mathfrak{S}}$. □

Corollary 5.2. *Let X be a completely regular Hausdorff space and $\mathfrak{S}$ a cover of $k_{\mathbb{R}}$-subsets of X (in the relative topology of X). Then $(C_b(X, \mathbb{K}; \mathfrak{S}), \beta_{\mathfrak{S}})$ is complete if and only if X is a $\mathfrak{S}_{\mathbb{R}}$-space.*

Proof. The result follows from Theorem 5.2 and the fact that for a given $S \in \mathfrak{S}$, $(C_b(S, \mathbb{K}), \beta_S)$ is complete if and only if S is a $k_{\mathbb{R}}$-space. □

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