ON DUPLICATIVE ALGEBRAS

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Abstract. In this paper we introduce a class of binary systems (groupoids, algebras) $(X; *)$ on a set X for which $x*y = (x*z)*(y*z)$ holds for all $x, y, z \in X$. These duplicative algebras include surprisingly large classes of examples, including the B-algebras which are closely related to groups as well as the left zero semigroup. In order to study the structure theory of such algebras, we introduce a graph $\Gamma_D(X)$ whose components have right ideal characteristics and which determine certain related subalgebras which are B-algebras.

1. Introduction

The BCK and BCI-algebras were originally introduced by Y. Imai and K. Iséki as algebraic structures for abstract algebras ([1, 2]). It has been founded that that the class of BCK-algebras is a proper subclass of the class of BCI-algebras ([5]). In [3, 4], Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [7], J. Neggers and H. S. Kim introduced the notion of d-algebras, i.e., (I) $x*x = 0$; (IV) $0*x = 0$; (V) $x*y = 0$ and $y*x = 0$ imply $x = y$, which is another useful generalization of BCK-algebras, and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations between d-algebras and oriented digraphs. Y. B. Jun, E. H. Roh and H. S. Kim introduced a new notion, called an BH-algebra, i.e., (I), (II) $x * 0 = x$ and (V), which is a generalization of $BCH/BCI/BCK$-algebras ([6]). They also defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. Recently, J. Neggers and H. S. Kim introduced the notion of B-algebras, i.e., (I), (II) $x * 0 = x$ and (III) $(x * y) * z = x * (z * (0 * y))$ for all x, y, z in X, and established some of its fundamental properties ([8]). In this paper we introduce a class of algebras which is related to the class of B-algebras but which is also a vast generation determined by a single law. As we shall see, it turns out that it is possible to define a related graph whose components have right ideal characteristics including that of being subalgebras with some other special properties noted below.

2. Duplicative Algebras

An algebra $(X; *)$ is said to be a *duplicative algebra* if for any $x, y, z \in X$,

$$x * y = (x * z) * (y * z) \tag{2.1}$$

Obviously, there is a dual duplicative law:

$$x * y = (z * x) * (z * y)$$

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for any $x, y, z \in X$, but the corresponding theory will mostly be analogous to the discussion we shall develop in the case of duplicative algebras.

Example 2.1. If $(X; *)$ is the left zero semigroup, i.e., $x * y = x$ for all $x, y \in X$, then also $x * y = (x * z) * (y * z)$ for any $x, y, z \in X$, i.e., the left zero semigroup is a duplicative algebra.

Example 2.2. If $(X; \cdot)$ is a group, and if $(X; *)$ is defined by $x * y := x \cdot y^{-1}$, then $(x*z)*(y*z) = (x \cdot z^{-1}) \cdot (y \cdot z^{-1})^{-1} = x \cdot z^{-1} \cdot z \cdot y^{-1} = x*y$, and $(X; *)$ is a duplicative algebra.

The following proposition immediately follows from Example 2.2.

Proposition 2.3. *Let $(G; \cdot)$ be a group and $\alpha \in G$ (fixed). If we define $x * y := xy^{-1}\alpha$, $x, y \in X$, then $(G; *)$ is a duplicative algebra.*

Proof. Straightforward. □

Example 2.4. Let $\mathbf{R}$ be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on $\mathbf{R}$ by

$$x * y := \frac{n(x - y)}{n + y}$$

Then $(\mathbf{R}; *)$ is a duplicative algebra.

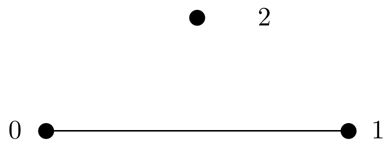
Example 2.5. Let $X := \{0, 1, 2\}$ be a set with the following table:

$*$	0	1	2
0	0	0	0
1	0	0	0
2	2	2	2

Then $(\mathbf{X}; *)$ is a duplicative algebra.

Example 2.6. Let $\mathbf{R}$ be a set of all real numbers and $x, y \in \mathbf{R}$. If we define $x * y := x - y + \sqrt{3}$, then $(\mathbf{R}; *)$ is a duplicative algebra.

Given a duplicative algebra $(X; *)$, define the graph $\Gamma_D(X)$ to consist of a vertex set $V = X$ and $(x, y) \in E$ an edge, provided $x * X \cap y * X \neq \emptyset$. Thus, in Example 2.1, $x * X = \{x\}$ for any $x \in X$, and $(x, y) \in E$ if and only if $x = y$. On the other hand, in Example 2.2, for any $x, y \in X$, $x * ((e * y) * (e * x)) = x((ey^{-1})(ex^{-1})^{-1})^{-1} = y$ and $x * X = X$ so that $x * X \cap y * X \neq \emptyset$ and $(x, y) \in E$. In the one case we have $\Gamma_D(X)$ with singleton components, while in the second case $\Gamma_D(X)$ is a complete graph (including loops at vertices). Moreover, the graph $\Gamma_D(X)$ of Example 2.5 is one of the form:



If X is finite, then the incidence matrix of $\Gamma_D(X)$ in Example 2.1 is the identity matrix, while the incidence matrix in Example 2.2 is the matrix $U = (u_{ij})$ where $u_{ij} = 1$ for all i and j. A graph Γ is *duplicative* if it is $\Gamma_D(X)$ for some duplicative algebra $(X; *)$.

Proposition 2.7. *Let $\Gamma_D(X)$ be a duplicative graph of a duplicative algebra $(X; *)$. If (x, y) is an edge in $\Gamma_D(X)$, i.e., $(x, y) \in E$, then $x * x = y * y$.*

Proof. If $(x, y) \in E$, then there exists an $u \in X$ such that $x * a = y * b = u$ for some $a, b \in X$. Since X is duplicative, $x*x = (x*a)*(x*a) = u*u$ and $y*y = (y*b)*(y*b) = u*u$, proving that $x * x = y * y$. □

Hence, if x and y are in the same connected component Γ_i of $\Gamma_D(X)$, then there exists a path $(x, t_1), (t_1, t_2), \cdots, (t_n, y)$ connecting x and y, so that $x*x = t_1*t_1 = \cdots = t_n*t_n = y*y$. Let θ_i be this common element for the ith component Γ_i of $\Gamma_D(X)$.

Proposition 2.8. *If Γ_i and Γ_j are components of $\Gamma_D(X)$ and $x \in \Gamma_i, y \in \Gamma_j$, then $x * y = (x * y) * \theta_j = \theta_i * (y * x)$.*

Proof. Since $x * x = \theta_i$ and $y * y = \theta_j$, we have $x * y = (x * x) * (y * x) = \theta_i * (y * x)$ and $x * y = (x * y) * (y * y) = (x * y) * \theta_j$. □

Corollary 2.9. *Let Γ_i be a component of $\Gamma_D(X)$. If $x, y \in \Gamma_i$, then $x*y = \theta_i*(y*x) = (x * y) * \theta_i$.*

Proposition 2.10. *Let Γ_i be a component of $\Gamma_D(X)$. If $x \in \Gamma_i$, then $(x, \theta_i) \in E$.*

Proof. If $x \in \Gamma_i$, then $\theta_i = x * x = (x * x) * (x * x) = \theta_i * \theta_i \in \theta_i * X$. Since $x * x \in x * X$, $(x * X) \cap (\theta_i * X) \neq \emptyset$, proving that $(x, \theta_i) \in E$. □

Proposition 2.11. *If Γ_i is a component of a duplicative graph $\Gamma_D(X)$ then $\Gamma_i * X \subseteq \Gamma_i$.*

Proof. If $x \in \Gamma_i$ then $\theta_i = x * x = (x * z) * (x * z)$ for any $z \in X$. This means that $(x * X) \cap ((x * z) * X) \neq \emptyset$, i.e., $(x, x * z) \in E$, proving that $x * z \in \Gamma_i$. □

It follows from Proposition 2.11 that $\Gamma_i * \Gamma_i \subseteq \Gamma_i$, i.e., $(\Gamma_i; *)$ is a duplicative algebra which contains a special element θ_i such that $y * y = \theta_i$ for all $y \in \Gamma_i$. Furthermore, $u * v = \theta_i * (v * u) = (u * v) * \theta_i$ for all $u, v \in \Gamma_i$. We denote $C_i := \{u * v \mid u, v \in \Gamma_i\}$. Since $C_i * C_i \subseteq C_i \subseteq \Gamma_i$ and $\theta_i * \theta_i = \theta_i$, $(C_i; *)$ is a subalgebra of a duplicative algebra $(\Gamma_i; *)$ containing θ_i.

Proposition 2.12. *If $x \in C_i$, then $x * \theta_i = x$.*

Proof. If $x \in C_i$, then $x = u * v$ for some $u, v \in \Gamma_i$. It follows from Proposition 2.8 that $x * \theta_i = (u * v) * \theta_i = u * v = x$. □

Consider the following identity:

$$(x * y) * z = ((x * y) * w) * (z * w), \tag{2.2}$$

where $x, y, z \in C_i$ and $w \in X$. If we let $w := x * y$ in (2.2), then

$$\begin{aligned}(x * y) * z &= ((x * y) * (x * y)) * (z * (x * y)) \\ &= \theta_i * (z * (x * y))\end{aligned} \tag{2.3}$$

If we let $z := \theta_i$ in (2.3) then by Corollary 2.9 we have

$$(x * y) * \theta_i = \theta_i * (\theta_i * (x * y)) = x * y \tag{2.4}$$

If we let $y := \theta_i$ in (2.4) then

$$\begin{aligned} x &= x * \theta_i && [\text{ by Proposition 2.12}] \\ &= (x * \theta_i) * \theta_i && [\text{ by Corollary 2.9}] \\ &= \theta_i * (\theta_i * (x * \theta_i)) && [\text{ by (2.4)}] \\ &= \theta_i * (\theta_i * x) && [\text{ by Proposition 2.12}] \end{aligned}$$

If we let $w := \theta_i * y$ in (2.2), then

$$\begin{aligned}(x * y) * z &= ((x * y) * (\theta_i * y)) * (z * (\theta_i * y)) \\ &= (x * \theta_i) * (z * (\theta_i * y)) \\ &= x * (z * (\theta_i * y)).\end{aligned}$$

We summarize:

Theorem 2.13. *Let $\Gamma_D(X)$ be a duplicative graph of a duplicative algebra $(X; *)$ and Γ_i be a component of $\Gamma_D(X)$. Then $C_i = \{u * v \mid u, v \in \Gamma_i\}$ is a B-algebra.*

Proposition 2.14. *Every B-algebra $(X; *, 0)$ is duplicative.*

Proof. For any $x, y, z \in X$, we have

$$\begin{aligned}(y * z) * (0 * z) &= y * ((0 * z) * (0 * z)) \\ &= y * 0 \\ &= y.\end{aligned}$$

It follows that

$$\begin{aligned}(x * z) * (y * z) &= x * ((y * z) * (0 * z)) \\ &= x * y,\end{aligned}$$

proving that X is duplicative. □

3. Polynomial Duplicative Algebras

Let $(X;+,\cdot)$ be a field. An algebra $(X;*)$ is said to be *linear* if $x*y = A + Bx + Cy$, where $A, B, C \in X$, for any $x, y \in X$. A duplicative algebra $(X;*)$ is said to be *linear* if it is linear.

Theorem 3.1. *Let $(X;+,\cdot)$ be a field. Then every linear duplicative algebra $(X;*)$ has one of the following forms: $x*y = A; x*y = A + x - y; x*y = x$, where $A \in X$.*

Proof. Define $x*y = A + Bx + Cy$, $A, B, C \in X$. Then $(x*z)*(y*z) = A + B(x*z) + C(y*z) = A(1+B+C) + B^2x + CBy + (CB + C^2)z$. Since $(X;*)$ is duplicative we obtain: $A(1+B+C) = A$, $(B+C)C = 0$, $B^2 = B, CB = C$. If $A \neq 0$, then $1+B+C = 1$, i.e., $B + C = 0$. If $B = C = 0$, then we obtain $x*y = A$. If $B = -C \neq 0$, since $B^2 = B$, we have $B = 1, C = -1$, i.e., $x*y = A + x - y$, $A \neq 0$. Assume $A = 0$. If $C = 0$, then we have either $A = B = C = 0$ or $A = C = 0, B = 1$, i.e., $x*y = 0$ or $x*y = x$. If $C \neq 0$, then $B + C = 0, CB = C$ imply $B = 1, C = -1$, i.e., $x*y = x - y$. This completes the proof. □

An algebra $(X;*)$ is said to be *quadratic* if for all x, y in X, $x*y$ is defined by $x*y = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$, where $a_1, \ldots, a_6 \in X$. Similarly, we can define cubic polynomial algebras or polynomial algebra of higher degrees. Next, we give an example of a quadratic duplicative algebra to provide a very small partial answer to the question whether there are any other possibilities in this class of algebras. With the answer positive, it becomes a question to identify those polynomial algebras $(X;*)$ which are duplicative.

Example 3.2. Let $X := Z/(3)$ be a field, i.e., $X = \{0, 1, -1\}$. Then $x^2(x^2-1) = 0$ for all $x \in X$ and thus $(x*z)*(y*z) = x^2 * y^2 = x^4 = x^2 = x*y$. This proves that $(X;*)$ is a quadratic duplicative algebra, where $x*y = x^2$ for any $x, y \in X = Z/(3)$.

Higher degree examples may also be constructed for particular fields. For example, if $X := Z/(11)$, then $(x^5)^5 = (x^{11})^2 \cdot x^3 = x^5$ and thus $x*y = x^5$ yields $(x*z)*(y*z) = x^5*y^5 = (x^5)^5 = x^5 = x*y$. Also, in $X := Z/(11)$, $(x^6)^6 = (x^{11})^3 \cdot x^3 = x^6$ so that $x*y = x^6$ works as well. If we find an integer $\alpha \in Z_p$, where p is a prime, such that $x^{2\alpha} = x^\alpha$ for any $x \in Z_p$, then this leads to construct a duplicative algebra $(Z_p;*)$ where $x*y = x^\alpha$ on the field Z_p in a similar fashion.

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