STABILITY IN MULTICRITERIA LOCATION PROBLEMS

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Abstract. In this paper, the stability in multicriteria location problems is considered. Given a family of parameterized multicriteria location problems, the (weak) Pareto optimal value/solution mapping is defined as the set-valued mapping which associates to each parameter value the set of all (weak) Pareto optimal values/solutions of the problem. Sufficient conditions for the upper and lower semicontinuity of the (weak) Pareto optimal value/solution mapping are obtained.

1 Introduction and preliminaries In a general continuous location model, finitely many points called demand points in $\mathbb{R}^n$, modeling existing facilities or customers, are given. Then a problem to locate a new facility in $\mathbb{R}^n$ is called a single facility location problem. This problem is usually formulated as a minimization problem with an objective function involving distances between the facility and demand points. Let $\boldsymbol{d}_i \in \mathbb{R}^n$, $i = 1, 2, \cdots, \ell$ be demand points, and we consider $\boldsymbol{d} \equiv (\boldsymbol{d}_1, \boldsymbol{d}_2, \cdots, \boldsymbol{d}_\ell) \in \mathbb{R}^{n\ell}$ as the parameter. We put $I \equiv \{1, 2, \cdots, \ell\}$. In this paper, we consider a *multicriteria location problem* formulated as follows:

$$(\mathrm{P}_{\boldsymbol{d}}) \qquad \left| \begin{array}{ll} \min & \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) \equiv (\gamma_{\boldsymbol{d}_1}(\boldsymbol{x} - \boldsymbol{d}_1), \gamma_{\boldsymbol{d}_2}(\boldsymbol{x} - \boldsymbol{d}_2), \cdots, \gamma_{\boldsymbol{d}_\ell}(\boldsymbol{x} - \boldsymbol{d}_\ell)) \\ \text{s.t.} & \boldsymbol{x} \in X(\boldsymbol{d}) \end{array} \right.$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the variable location of the facility, and X is a set-valued mapping from $\mathbb{R}^{n\ell}$ to $\mathbb{R}^n$ (which associates to each parameter $\boldsymbol{d} \in \mathbb{R}^{n\ell}$ a set $X(\boldsymbol{d}) \subset \mathbb{R}^n$ and we denote it as $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$), and $\gamma_{\boldsymbol{d}_i} : \mathbb{R}^n \to \mathbb{R}, i \in I$ are gauges for each parameter $\boldsymbol{d} = (\boldsymbol{d}_1, \boldsymbol{d}_2, \cdots, \boldsymbol{d}_\ell) \in \mathbb{R}^{n\ell}$. For each $i \in I$ and $\boldsymbol{d}_i \in \mathbb{R}^n$, let $B_i(\boldsymbol{d}_i) \subset \mathbb{R}^n$ be a compact convex set containing the origin in its interior, and it is assumed that the gauge $\gamma_{\boldsymbol{d}_i} : \mathbb{R}^n \to \mathbb{R}$ is defined as follows:

$$\gamma_{\boldsymbol{d}_i}(\boldsymbol{x}) \equiv \inf\{r > 0 : \boldsymbol{x} \in rB_i(\boldsymbol{d}_i)\}, \quad \boldsymbol{x} \in \mathbb{R}^n.$$

Then $B_i : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n, i \in I$ are set-valued mappings. For each $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{d}_i \in \mathbb{R}^n, i \in I$, the value $\gamma_{\boldsymbol{d}_i}(\boldsymbol{x} - \boldsymbol{d}_i)$ represents the distance from $\boldsymbol{d}_i$ to $\boldsymbol{x}$. For details, see [4] and references therein in multicriteria location problems, see [2], [4], [5] and [7] in gauges and see [3] and [6] in set-valued mappings.

Let us consider the following motivating example of the multicriteria location problem $(\mathrm{P}_{\boldsymbol{d}})$. A company is going to develop a new product of some kind. Suppose that a product is determined by n characteristic values, and that each product is represented as a point $\boldsymbol{x} \in \mathbb{R}^n$, and that each $\boldsymbol{d}_i, i \in I$ represents the preference of customer i for the products. Suppose also that for each $i \in I$, the value $\gamma_{\boldsymbol{d}_i}(\boldsymbol{x} - \boldsymbol{d}_i)$ represents the distance from the preference of customer i, $\boldsymbol{d}_i$, to the product $\boldsymbol{x} \in \mathbb{R}^n$, and that any customer prefers a product near the preference of the customer. In this case, the multicriteria location problem $(\mathrm{P}_{\boldsymbol{d}})$ is a problem to find a new product which is near preferences of customers as much as possible, where the feasible region $X(\boldsymbol{d})$ represents the set of all products which the

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company can produce. Then a (weak) Pareto optimal solution, which will be defined in Definition 1 later, is the new product to be produced for the company. In fact, since a (weak) Pareto optimal solution is not unique in general, the company needs to choose one product among (weak) Pareto optimal solutions by using, for example, trade-off analysis. The multicriteria location problem $(\mathrm{P}_{\boldsymbol{d}})$ can also be used in order to evaluate existing products. If an existing product is not a (weak) Pareto optimal solution, then the company should stop producing the existing product because it means that there exists another product which customers prefer its product to the existing product. Gauges $\gamma_{\boldsymbol{d}_i}, i \in I$ are asymmetric, which means that their unit balls are not symmetric around the origin, and their gauges depend on demand points $\boldsymbol{d}_i, i \in I$. It is suitable to consider different distance measures (gauges), which are asymmetric and depend on demand points, in the following situation. For customers $i_1, i_2 \in I$ with $i_1 \neq i_2$, consider the j_1-th and j_2-th characters of the products for $j_1, j_2 \in \{1, 2, \cdots, n\}$ with $j_1 \neq j_2$. For customer i_1, let $d_{i_1}^{j_1} \in \mathbb{R}$ be the preference value for the j_1-th character of the products. Suppose that for $\varepsilon > 0$ and the j_1-th character of the products, customer i_1 prefers the value $d_{i_1}^{j_1} + \varepsilon$ to the value $d_{i_1}^{j_1}$ a little and prefers the value $d_{i_1}^{j_1}$ to the value $d_{i_1}^{j_1} - \varepsilon$ very much. In this case, it is suitable to use a gauge in order to measure the distance from the preference of customer i_1 to a product because of the asymmetricity. Suppose that the j_1-th character of the products is important for customer i_1 but not for customer i_2, and that the j_2-th character of the products is important for customer i_2 but not for customer i_1. In this case, it is suitable to use different gauges in order to measure distances from preferences of customers to a product because important characters of the products are different according to customers. On the other hand, demand points $\boldsymbol{d}_i, i \in I$, which represent preferences of customers for the products, may be estimators based on some data in general. Such estimated preferences of customers for the products may be different from true preferences of customers for the products. Let (P_1) and (P_2) be multicriteria location problems with estimated and true preferences of customers as demand points, respectively. In this case, the difference between (weak) Pareto optimal solutions of (P_1) and those of (P_2) is very important. Therefore, the stability in multicriteria location problems $(\mathrm{P}_{\boldsymbol{d}})$ with respect to demand points, that is, the stability of (weak) Pareto optimal solutions with respect to the parameter $\boldsymbol{d}$ is very important .

In this paper, a family of parameterized multicriteria location problems $(\mathrm{P}_{\boldsymbol{d}})$ is considered, and the stability of Pareto and weak Pareto optimal values/solutions is investigated, where we consider demand points as the parameter and distance measures depend on the parameter. First, some auxiliary results are given in order to investigate the stability of Pareto and weak Pareto optimal values/solutions. Next, the continuity of the weak Pareto optimal value mapping, the continuity of the Pareto optimal value mapping, the continuity of the weak Pareto optimal solution mapping, and the continuity of the Pareto optimal solution mapping are investigated as the stability of Pareto and weak Pareto optimal values/solutions. Finally, some conclusions are given.

Definition 1. Let $\boldsymbol{d} \equiv (\boldsymbol{d}_1, \boldsymbol{d}_2, \cdots, \boldsymbol{d}_\ell) \in \mathbb{R}^{n\ell}$.

(i) A point $\boldsymbol{x}_0 \in \mathbb{R}^n$ is called a *Pareto optimal solution* of $(\mathrm{P}_{\boldsymbol{d}})$ if there is no $\boldsymbol{x} \in X(\boldsymbol{d})$ such that $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) \leq \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{d})$ and $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) \neq \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{d})$, where $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) \leq \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{d})$ means that $\gamma_{\boldsymbol{d}_i}(\boldsymbol{x} - \boldsymbol{d}_i) \leq \gamma_{\boldsymbol{d}_i}(\boldsymbol{x}_0 - \boldsymbol{d}_i), i \in I$. If $\boldsymbol{x}_0 \in \mathbb{R}^n$ is a Pareto optimal solution of $(\mathrm{P}_{\boldsymbol{d}})$, then $\boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{d})$ is called a *Pareto optimal value* of $(\mathrm{P}_{\boldsymbol{d}})$.

(ii) A point $\boldsymbol{x}_0 \in \mathbb{R}^n$ is called a *weak Pareto optimal solution* of $(\mathrm{P}_{\boldsymbol{d}})$ if there is no $\boldsymbol{x} \in X(\boldsymbol{d})$ such that $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) < \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{d})$, where $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) < \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{d})$ means that $\gamma_{\boldsymbol{d}_i}(\boldsymbol{x} - \boldsymbol{d}_i) < \gamma_{\boldsymbol{d}_i}(\boldsymbol{x}_0 - \boldsymbol{d}_i)$, $i \in I$. If $\boldsymbol{x}_0 \in \mathbb{R}^n$ is a weak Pareto optimal solution of $(\mathrm{P}_{\boldsymbol{d}})$, then $\boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{d})$ is called a *weak Pareto optimal value* of $(\mathrm{P}_{\boldsymbol{d}})$.

(iii) A point $\boldsymbol{x}_0 \in \mathbb{R}^n$ is called a *strictly Pareto optimal solution* of $(\mathrm{P}_{\boldsymbol{d}})$ if there is no $\boldsymbol{x} \in X(\boldsymbol{d})$ such that $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) \leq \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{d})$ and $\boldsymbol{x} \neq \boldsymbol{x}_0$.

(iv) A point $\boldsymbol{x}_0 \in \mathbb{R}^n$ is called an *alternately Pareto optimal solution* of $(\mathrm{P}_{\boldsymbol{d}})$ if $\boldsymbol{x}_0$ is a Pareto optimal solution of $(\mathrm{P}_{\boldsymbol{d}})$ and not a strictly Pareto optimal solution of $(\mathrm{P}_{\boldsymbol{d}})$.

For details in Pareto optimality, see [1], [3] and [4].

We define a set-valued mapping $F : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^{\ell}$ as $F(\boldsymbol{d}) \equiv \boldsymbol{f}(X(\boldsymbol{d}), \boldsymbol{d}), \boldsymbol{d} \in \mathbb{R}^{n\ell}$, where $\boldsymbol{f}(X(\boldsymbol{d}), \boldsymbol{d}) \equiv \{\boldsymbol{y} \in \mathbb{R}^{\ell} : \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}), \boldsymbol{x} \in X(\boldsymbol{d})\}$.

For multicriteria location problems $(\mathrm{P}_{\boldsymbol{d}})$, we define set-valued mappings $M : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^{\ell}, WM : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^{\ell}, MS : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^{n}, WMS : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^{n}$ as

$$\begin{aligned} M(\boldsymbol{d}) &\equiv \{\boldsymbol{y} \in F(\boldsymbol{d}) : (\boldsymbol{y} - \mathbb{R}^{\ell}_{+}) \cap F(\boldsymbol{d}) = \{\boldsymbol{y}\}\}, \\ WM(\boldsymbol{d}) &\equiv \{\boldsymbol{y} \in F(\boldsymbol{d}) : (\boldsymbol{y} - \operatorname{int} \mathbb{R}^{\ell}_{+}) \cap F(\boldsymbol{d}) = \emptyset\}, \\ MS(\boldsymbol{d}) &\equiv \{\boldsymbol{x} \in X(\boldsymbol{d}) : \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) \in M(\boldsymbol{d})\}, \\ WMS(\boldsymbol{d}) &\equiv \{\boldsymbol{x} \in X(\boldsymbol{d}) : \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{d}) \in WM(\boldsymbol{d})\}, \end{aligned}$$

and they are called, respectively, the *Pareto optimal value mapping*, the *weak Pareto optimal value mapping*, the *Pareto optimal solution mapping*, the *weak Pareto optimal solution mapping*, where $\mathbb{R}^{\ell}_{+} \equiv \{\boldsymbol{y} \in \mathbb{R}^{\ell} : \boldsymbol{y} \geq \boldsymbol{0}\}$ is the non-negative orthant of $\mathbb{R}^{\ell}$, and $\operatorname{int} \mathbb{R}^{\ell}_{+}$ is the interior of $\mathbb{R}^{\ell}_{+}$, that is, $\operatorname{int} \mathbb{R}^{\ell}_{+} = \{\boldsymbol{y} \in \mathbb{R}^{\ell} : \boldsymbol{y} > \boldsymbol{0}\}$. For each parameter $\boldsymbol{d} \in \mathbb{R}^{n\ell}$, $M(\boldsymbol{d}), WM(\boldsymbol{d}), MS(\boldsymbol{d})$ and $WMS(\boldsymbol{d})$ are, respectively, sets of all Pareto optimal values, weak Pareto optimal values, Pareto optimal solutions and weak Pareto optimal solutions of $(\mathrm{P}_{\boldsymbol{d}})$.

Definition 2.(See [1].) A set $A \subset \mathbb{R}^{\ell}$ is said to be $\mathbb{R}^{\ell}_{+}$-*compact* if the section $(\boldsymbol{y} - \mathbb{R}^{\ell}_{+}) \cap A$ is compact for any $\boldsymbol{y} \in A$.

For a set $A \subset \mathbb{R}^{\ell}$, $A_N \equiv \{\boldsymbol{y} \in A : (\boldsymbol{y} - \mathbb{R}^{\ell}_{+}) \cap A = \{\boldsymbol{y}\}\}$ is called the *nondominated set* of A. For each $\boldsymbol{d} \in \mathbb{R}^{n\ell}$, $M(\boldsymbol{d})$ is the nondominated set of $F(\boldsymbol{d})$, that is, $M(\boldsymbol{d}) = (F(\boldsymbol{d}))_N$.

Definition 3.(See [1].) For a set $A \subset \mathbb{R}^{\ell}$, the nondominated set of A, A_N, is said to be *externally stable* if for each $\boldsymbol{y} \in A \setminus A_N$, there exists $\overline{\boldsymbol{y}} \in A_N$ such that $\boldsymbol{y} \in \overline{\boldsymbol{y}} + \mathbb{R}^{\ell}_{+}$.

Definition 4.(See [4].) For a compact convex set $B \subset \mathbb{R}^n$ containing the origin in its interior, assume that the gauge $\gamma : \mathbb{R}^n \to \mathbb{R}$ is defined as $\gamma(\boldsymbol{x}) \equiv \inf\{r > 0 : \boldsymbol{x} \in rB\}, \boldsymbol{x} \in \mathbb{R}^n$. Then the gauge γ is said to be *strictly convex* if

$$\gamma(\lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2) < \lambda\gamma(\boldsymbol{x}_1) + (1 - \lambda)\gamma(\boldsymbol{x}_2)$$

for any $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$ with $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$, which are not on the same half line emanating from the origin, and any λ with $0 < \lambda < 1$.

Definition 5.(See [6].) Let $S : \mathbb{R}^p \rightsquigarrow \mathbb{R}^q$ be a set-valued mapping, and let $\overline{\boldsymbol{u}} \in \mathbb{R}^p$.

(i) S is said to be *upper semicontinuous* at $\overline{\boldsymbol{u}}$ if $\{\boldsymbol{u}_k\} \subset \mathbb{R}^p$, $\boldsymbol{u}_k \to \overline{\boldsymbol{u}}$, $\{\boldsymbol{v}_k\} \subset \mathbb{R}^q$, $\boldsymbol{v}_k \to \overline{\boldsymbol{v}}$, $\boldsymbol{v}_k \in S(\boldsymbol{u}_k)$ $(k \in \mathbb{N})$ imply that $\overline{\boldsymbol{v}} \in S(\overline{\boldsymbol{u}})$, where $\mathbb{N}$ is the set of all natural numbers. S is said to be upper semicontinuous (on $\mathbb{R}^p$) if S is upper semicontinuous at any $\boldsymbol{u} \in \mathbb{R}^p$.

(ii) S is said to be *lower semicontinuous* at $\overline{\boldsymbol{u}}$ if $\{\boldsymbol{u}_k\} \subset \mathbb{R}^p$, $\boldsymbol{u}_k \to \overline{\boldsymbol{u}}$, $\overline{\boldsymbol{v}} \in S(\overline{\boldsymbol{u}})$ imply the existence of $k_0 \in \mathbb{N}$ and $\{\boldsymbol{v}_k\} \subset \mathbb{R}^q$ such that $\boldsymbol{v}_k \to \overline{\boldsymbol{v}}$ and $\boldsymbol{v}_k \in S(\boldsymbol{u}_k)$ $(k \geq k_0, k \in \mathbb{N})$. S is said to be lower semicontinuous (on $\mathbb{R}^p$) if S is lower semicontinuous at any $\boldsymbol{u} \in \mathbb{R}^p$.

(iii) S is said to be *continuous* at $\overline{\boldsymbol{u}}$ if S is both upper and lower semicontinuous at $\overline{\boldsymbol{u}}$. S is said to be continuous (on $\mathbb{R}^p$) if S is continuous at any $\boldsymbol{u} \in \mathbb{R}^p$.

Definition 6.(See [6].) Let $S : \mathbb{R}^p \rightsquigarrow \mathbb{R}^q$ be a set-valued mapping, and let $\overline{\boldsymbol{u}} \in \mathbb{R}^p$. S is said to be *locally bounded* at $\overline{\boldsymbol{u}}$ if there exists a neighborhood $U \subset \mathbb{R}^p$ of $\overline{\boldsymbol{u}}$ such that $S(U) \subset \mathbb{R}^q$ is bounded, where $S(U) \equiv \cup_{\boldsymbol{u} \in U} S(\boldsymbol{u})$. S is said to be locally bounded (on $\mathbb{R}^p$) if S is locally bounded at any $\boldsymbol{u} \in \mathbb{R}^p$.

2 Auxiliary results

In this section, some auxiliary results are given in order to investigate the stability of Pareto and weak Pareto optimal values/solutions.

First, we investigate the continuity of the objective function $\boldsymbol{f} : \mathbb{R}^n \times \mathbb{R}^{n\ell} \to \mathbb{R}^\ell$ for multicriteria location problems ($\mathrm{P}_{\boldsymbol{d}}$).

Theorem 1. For each $i \in I$, we define $f_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as $f_i(\boldsymbol{x}, \boldsymbol{d}_i) \equiv \gamma_{\boldsymbol{d}_i}(\boldsymbol{x} - \boldsymbol{d}_i)$, $(\boldsymbol{x}, \boldsymbol{d}_i) \in \mathbb{R}^n \times \mathbb{R}^n$. If $B_i : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$, $i \in I$ are continuous and locally bounded, then $f_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $i \in I$ are continuous.

Proof. Fix any $i \in I$ and $\overline{\boldsymbol{x}} \in \mathbb{R}^n$ and $\overline{\boldsymbol{d}}_i \in \mathbb{R}^n$. We shall show that f_i is continuous at $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{d}}_i) \in \mathbb{R}^n \times \mathbb{R}^n$. Let $\{\boldsymbol{x}_k\} \subset \mathbb{R}^n$ be any sequence which converges to $\overline{\boldsymbol{x}}$, and $\{\boldsymbol{d}_{ik}\} \subset \mathbb{R}^n$ be any sequence which converges to $\overline{\boldsymbol{d}}_i$.

(i) First, we consider the case $\overline{\boldsymbol{x}} = \overline{\boldsymbol{d}}_i$. Then

$$f_i(\overline{\boldsymbol{x}}, \overline{\boldsymbol{d}}_i) = \gamma_{\overline{\boldsymbol{d}}_i}(\overline{\boldsymbol{x}} - \overline{\boldsymbol{d}}_i) = 0,$$
$$f_i(\boldsymbol{x}_k, \boldsymbol{d}_{ik}) = \gamma_{\boldsymbol{d}_{ik}}(\boldsymbol{x}_k - \boldsymbol{d}_{ik}), \quad k \in \mathbb{N}.$$

Since B_i is convex-valued (which means that $B_i(\boldsymbol{d}_i)$ is convex for any $\boldsymbol{d}_i \in \mathbb{R}^n$) and $\mathrm{int}\, B_i(\overline{\boldsymbol{d}}_i) \neq \emptyset$ and B_i is continuous (especially lower semicontinuous), there exists a neighborhood $U(\overline{\boldsymbol{d}}_i, \boldsymbol{0}) \subset \mathbb{R}^n \times \mathbb{R}^n$ of $(\overline{\boldsymbol{d}}_i, \boldsymbol{0}) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $U(\overline{\boldsymbol{d}}_i, \boldsymbol{0}) \subset \mathrm{gph}\, B_i$ from Theorem 5.9 in [6], where $\mathrm{gph}\, B_i \equiv \{(\boldsymbol{d}_i, \boldsymbol{x}) \in \mathbb{R}^n \times \mathbb{R}^n : \boldsymbol{x} \in B_i(\boldsymbol{d}_i)\}$. Without loss of generality, assume that $U(\overline{\boldsymbol{d}}_i, \boldsymbol{0}) = V(\overline{\boldsymbol{d}}_i) \times W$ for some neighborhood $V(\overline{\boldsymbol{d}}_i) \subset \mathbb{R}^n$ of $\overline{\boldsymbol{d}}_i$ and some compact convex set $W \subset \mathbb{R}^n$ containing the origin in its interior. In this case, $W \subset B_i(\boldsymbol{d}_i)$ for any $\boldsymbol{d}_i \in V(\overline{\boldsymbol{d}}_i)$. Since $\boldsymbol{d}_{ik} \to \overline{\boldsymbol{d}}_i$, there exists $k_0 \in \mathbb{N}$ such that $\boldsymbol{d}_{ik} \in V(\overline{\boldsymbol{d}}_i)$ for any $k \geq k_0$. Now, we define a gauge $\gamma_0 : \mathbb{R}^n \to \mathbb{R}$ as $\gamma_0(\boldsymbol{x}) \equiv \inf\{r > 0 : \boldsymbol{x} \in rW\}, \boldsymbol{x} \in \mathbb{R}^n$. For each $k \geq k_0$, since $W \subset B_i(\boldsymbol{d}_{ik})$, we have

$$0 \leq f_i(\boldsymbol{x}_k, \boldsymbol{d}_{ik}) = \gamma_{\boldsymbol{d}_{ik}}(\boldsymbol{x}_k - \boldsymbol{d}_{ik}) \leq \gamma_0(\boldsymbol{x}_k - \boldsymbol{d}_{ik}).$$

When $k \to \infty$, since $\gamma_0(\boldsymbol{x}_k - \boldsymbol{d}_{ik}) \to \gamma_0(\overline{\boldsymbol{x}} - \overline{\boldsymbol{d}}_i) = 0$, we have $f_i(\boldsymbol{x}_k, \boldsymbol{d}_{ik}) \to 0 = f_i(\overline{\boldsymbol{x}}, \overline{\boldsymbol{d}}_i)$. Therefore, f_i is continuous at $(\overline{\boldsymbol{x}}, \overline{\boldsymbol{d}}_i)$.

(ii) Next, we consider the case $\overline{\boldsymbol{x}} \neq \overline{\boldsymbol{d}}_i$. For each $k \in \mathbb{N}$, since

$$\gamma_{\boldsymbol{d}_{ik}}(\boldsymbol{x}_k - \boldsymbol{d}_{ik}) \leq \gamma_{\boldsymbol{d}_{ik}}(\boldsymbol{x}_k - \boldsymbol{d}_{ik} - (\overline{\boldsymbol{x}} - \overline{\boldsymbol{d}}_i)) + \gamma_{\boldsymbol{d}_{ik}}(\overline{\boldsymbol{x}} - \overline{\boldsymbol{d}}_i),$$
$$\gamma_{\boldsymbol{d}_{ik}}(\overline{\boldsymbol{x}} - \overline{\boldsymbol{d}}_i) \leq \gamma_{\boldsymbol{d}_{ik}}(\overline{\boldsymbol{x}} - \overline{\boldsymbol{d}}_i - (\boldsymbol{x}_k - \boldsymbol{d}_{ik})) + \gamma_{\boldsymbol{d}_{ik}}(\boldsymbol{x}_k - \boldsymbol{d}_{ik})$$

by triangular inequality for gauges, we have

$$-\gamma_{\boldsymbol{d}_{ik}}((\overline{\boldsymbol{x}} - \boldsymbol{x}_k + 2\boldsymbol{d}_{ik} - \overline{\boldsymbol{d}}_i) - \boldsymbol{d}_{ik}) \leq \gamma_{\boldsymbol{d}_{ik}}(\boldsymbol{x}_k - \boldsymbol{d}_{ik}) - \gamma_{\boldsymbol{d}_{ik}}(\overline{\boldsymbol{x}} - \overline{\boldsymbol{d}}_i) \leq \gamma_{\boldsymbol{d}_{ik}}((\boldsymbol{x}_k - \overline{\boldsymbol{x}} + \overline{\boldsymbol{d}}_i) - \boldsymbol{d}_{ik}).$$

From the result of (i),

$$\lim_{k \to \infty} \{\gamma_{\boldsymbol{d}_{ik}}(\boldsymbol{x}_k - \boldsymbol{d}_{ik}) - \gamma_{\boldsymbol{d}_{ik}}(\overline{\boldsymbol{x}} - \overline{\boldsymbol{d}}_i)\} = 0.$$

Thus, if it can be shown that

$$\lim_{k\to\infty}\gamma_{\boldsymbol{d}_{ik}}(\overline{\boldsymbol{x}}-\overline{\boldsymbol{d}}_i)=\gamma_{\overline{\boldsymbol{d}}_i}(\overline{\boldsymbol{x}}-\overline{\boldsymbol{d}}_i), \tag{1}$$

then

$$\begin{aligned}\lim_{k\to\infty} f_i(\boldsymbol{x}_k,\boldsymbol{d}_{ik}) &= \lim_{k\to\infty}\gamma_{\boldsymbol{d}_{ik}}(\boldsymbol{x}_k-\boldsymbol{d}_{ik})\\ &= \lim_{k\to\infty}\left[\left\{\gamma_{\boldsymbol{d}_{ik}}(\boldsymbol{x}_k-\boldsymbol{d}_{ik})-\gamma_{\boldsymbol{d}_{ik}}(\overline{\boldsymbol{x}}-\overline{\boldsymbol{d}}_i)\right\}+\gamma_{\boldsymbol{d}_{ik}}(\overline{\boldsymbol{x}}-\overline{\boldsymbol{d}}_i)\right]\\ &= 0+\gamma_{\overline{\boldsymbol{d}}_i}(\overline{\boldsymbol{x}}-\overline{\boldsymbol{d}}_i)\\ &= f_i(\overline{\boldsymbol{x}},\overline{\boldsymbol{d}}_i)\end{aligned}$$

and f_i is continuous at $(\overline{\boldsymbol{x}},\overline{\boldsymbol{d}}_i)$. In order to show (1), it is sufficient to show that

$$\lim_{k\to\infty}\gamma_{\boldsymbol{d}_{ik}}\left(\frac{\overline{\boldsymbol{x}}-\overline{\boldsymbol{d}}_i}{\gamma_{\overline{\boldsymbol{d}}_i}(\overline{\boldsymbol{x}}-\overline{\boldsymbol{d}}_i)}\right)=1. \tag{2}$$

Note that

$$\overline{\boldsymbol{x}}_0\equiv\frac{\overline{\boldsymbol{x}}-\overline{\boldsymbol{d}}_i}{\gamma_{\overline{\boldsymbol{d}}_i}(\overline{\boldsymbol{x}}-\overline{\boldsymbol{d}}_i)}\in \mathrm{bd}\ B_i(\overline{\boldsymbol{d}}_i)\subset B_i(\overline{\boldsymbol{d}}_i)$$

where bd $B_i(\overline{\boldsymbol{d}}_i)$ is the boundary of $B_i(\overline{\boldsymbol{d}}_i)$. For each $k\in\mathbb{N}$, since bd $B_i(\boldsymbol{d}_{ik})\cap\{\lambda\overline{\boldsymbol{x}}_0:\lambda\geq 0\}=\{\overline{\boldsymbol{x}}_k\}$ for some $\overline{\boldsymbol{x}}_k\in\mathbb{R}^n$, we have

$$\gamma_{\boldsymbol{d}_{ik}}(\overline{\boldsymbol{x}}_0)=\frac{\|\overline{\boldsymbol{x}}_0\|}{\|\overline{\boldsymbol{x}}_k\|}$$

where $\|\cdot\|$ is Euclidean norm (defined on $\mathbb{R}^n$). Thus, in order to show (2), it is sufficient to show that

$$\lim_{k\to\infty}\overline{\boldsymbol{x}}_k=\overline{\boldsymbol{x}}_0.$$

Since B_i is locally bounded and $\boldsymbol{d}_{ik}\to\overline{\boldsymbol{d}}_i$, $\{\overline{\boldsymbol{x}}_k\}$ is bounded. Thus, $\{\overline{\boldsymbol{x}}_k\}$ has a convergent subsequence. Let $\{\overline{\boldsymbol{x}}_{k'}\}$ be any convergent subsequence of $\{\overline{\boldsymbol{x}}_k\}$, and $\overline{\boldsymbol{x}}_0'$ be its limit. Since B_i is continuous (especially upper semicontinuous), we have $\overline{\boldsymbol{x}}_0'\in B_i(\overline{\boldsymbol{d}}_i)$. Thus, $\overline{\boldsymbol{x}}_0'=\mu\overline{\boldsymbol{x}}_0$ for some μ with $0\leq\mu\leq 1$. Since B_i is continuous (especially lower semicontinuous), there exist $k_1\in\mathbb{N}$ and $\{\widehat{\boldsymbol{x}}_k\}\subset\mathbb{R}^n$ such that $\widehat{\boldsymbol{x}}_k\to\overline{\boldsymbol{x}}_0$ and $\widehat{\boldsymbol{x}}_k\in B_i(\boldsymbol{d}_{ik}), k\geq k_1$. Again, since B_i is continuous (especially lower semicontinuous), there exist $\varepsilon>0$ and $k_2\in\mathbb{N}$ such that $V\equiv\{\boldsymbol{x}\in\mathbb{R}^n:\|\boldsymbol{x}\|<\varepsilon\}\subset$ int $B_i(\boldsymbol{d}_{ik})$ for any $k\geq k_2$ by the same argument in (i). If $\mu=0$, then $\overline{\boldsymbol{x}}_{k'}\in V\subset$ int $B_i(\boldsymbol{d}_{ik'})$ for sufficiently large $k'\in\mathbb{N}$, which contradicts that $\overline{\boldsymbol{x}}_{k'}\in$ bd $B_i(\boldsymbol{d}_{ik'})$. Thus, $\mu>0$. Suppose that $\mu<1$. Since

$$\widehat{\boldsymbol{x}}_{k'}\to\overline{\boldsymbol{x}}_0,\quad \overline{\boldsymbol{x}}_{k'}\to\overline{\boldsymbol{x}}_0'=\mu\overline{\boldsymbol{x}}_0,\quad 0<\mu<1,\quad \{\overline{\boldsymbol{x}}_{k'}\}\subset\{\lambda\overline{\boldsymbol{x}}_0:\lambda\geq 0\},$$

for sufficiently large $k'\in\mathbb{N}$, we have $\widehat{\boldsymbol{x}}_{k'}\neq\overline{\boldsymbol{x}}_{k'}$ and

$$\{\widehat{\boldsymbol{x}}_{k'}+\lambda(\overline{\boldsymbol{x}}_{k'}-\widehat{\boldsymbol{x}}_{k'}):\lambda\geq 1\}\cap V\neq\emptyset$$

and $\widehat{\boldsymbol{x}}_{k'}\in B_i(\boldsymbol{d}_{ik'}), V\subset$ int $B_i(\boldsymbol{d}_{ik'})$ and $\overline{\boldsymbol{x}}_{k'}\in$ int $B_i(\boldsymbol{d}_{ik'})$ from the convexity of $B_i(\boldsymbol{d}_{ik'})$, which contradicts that $\overline{\boldsymbol{x}}_{k'}\in$ bd $B_i(\boldsymbol{d}_{ik'})$. Thus, $\mu=1$ and $\overline{\boldsymbol{x}}_0'=\overline{\boldsymbol{x}}_0$. Namely, $\overline{\boldsymbol{x}}_0$ is a unique accumulation point of $\{\overline{\boldsymbol{x}}_k\}$. Therefore, $\{\overline{\boldsymbol{x}}_k\}$ converges to $\overline{\boldsymbol{x}}_0$. □

Throughout this paper, it is assumed that the objective function $\boldsymbol{f} : \mathbb{R}^n \times \mathbb{R}^{n\ell} \to \mathbb{R}^\ell$ for multicriteria location problems ($\mathrm{P}_{\boldsymbol{d}}$) is continuous. For example, $\boldsymbol{f} : \mathbb{R}^n \times \mathbb{R}^{n\ell} \to \mathbb{R}^\ell$ is continuous if $B_i : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n, i \in I$ are continuous and locally bounded from Theorem 1.

Next, we investigate the continuity of the set-valued mapping $F : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^\ell$.

Lemma 1. Let $\overline{\boldsymbol{d}}_0 \in \mathbb{R}^{n\ell}$, and assume that $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is lower semicontinuous at $\overline{\boldsymbol{d}}_0$. Then $F : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^\ell$ is lower semicontinuous at $\overline{\boldsymbol{d}}_0$.

Proof. Let $\{\overline{\boldsymbol{d}}_k\} \subset \mathbb{R}^{n\ell}$ be any sequence which converges to $\overline{\boldsymbol{d}}_0$, and fix any $\overline{\boldsymbol{y}}_0 \in F(\overline{\boldsymbol{d}}_0)$. Then there exists $\overline{\boldsymbol{x}}_0 \in X(\overline{\boldsymbol{d}}_0)$ such that $\overline{\boldsymbol{y}}_0 = \boldsymbol{f}(\overline{\boldsymbol{x}}_0, \overline{\boldsymbol{d}}_0)$. Since X is lower semicontinuous at $\overline{\boldsymbol{d}}_0$, there exist $\{\overline{\boldsymbol{x}}_k\} \subset \mathbb{R}^n$, which converges to $\overline{\boldsymbol{x}}_0$, and $k_0 \in \mathbb{N}$ such that $\overline{\boldsymbol{x}}_k \in X(\overline{\boldsymbol{d}}_k)$ for any $k \geq k_0$. For each $k \in \mathbb{N}$, we put $\overline{\boldsymbol{y}}_k \equiv \boldsymbol{f}(\overline{\boldsymbol{x}}_k, \overline{\boldsymbol{d}}_k)$. Then $\overline{\boldsymbol{y}}_k \in F(\overline{\boldsymbol{d}}_k)$ for any $k \geq k_0$. Since $\boldsymbol{f}$ is continuous, we have

$$\lim_{k\to\infty} \overline{\boldsymbol{y}}_k = \lim_{k\to\infty} \boldsymbol{f}(\overline{\boldsymbol{x}}_k, \overline{\boldsymbol{d}}_k) = \boldsymbol{f}(\overline{\boldsymbol{x}}_0, \overline{\boldsymbol{d}}_0) = \overline{\boldsymbol{y}}_0.$$

Therefore, F is lower semicontinuous at $\overline{\boldsymbol{d}}_0$. □

Lemma 2. Let $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$, and assume that $B_{i_0} : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is locally bounded at $\overline{\boldsymbol{d}}_{0i_0}$ for some $i_0 \in I$, and that $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is upper semicontinuous at $\overline{\boldsymbol{d}}_0$. Then $F : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^\ell$ is upper semicontinuous at $\overline{\boldsymbol{d}}_0$.

Proof. Let $\{\overline{\boldsymbol{d}}_k\} \subset \mathbb{R}^{n\ell}$ be any sequence which converges to $\overline{\boldsymbol{d}}_0$, and $\{\overline{\boldsymbol{y}}_k\} \subset \mathbb{R}^\ell$ be any sequence which converges to $\overline{\boldsymbol{y}}_0 \in \mathbb{R}^\ell$ such that $\overline{\boldsymbol{y}}_k \in F(\overline{\boldsymbol{d}}_k), k \in \mathbb{N}$. Then for each $k \in \mathbb{N}$, there exists $\overline{\boldsymbol{x}}_k \in X(\overline{\boldsymbol{d}}_k)$ such that $\overline{\boldsymbol{y}}_k = \boldsymbol{f}(\overline{\boldsymbol{x}}_k, \overline{\boldsymbol{d}}_k)$. For each $k \in \mathbb{N}$, if we put $\overline{\boldsymbol{y}}_k \equiv (\overline{y}_{k1}, \overline{y}_{k2}, \cdots, \overline{y}_{k\ell}) \in \mathbb{R}^\ell$ and $\overline{\boldsymbol{d}}_k \equiv (\overline{\boldsymbol{d}}_{k1}, \overline{\boldsymbol{d}}_{k2}, \cdots, \overline{\boldsymbol{d}}_{k\ell}) \in \mathbb{R}^{n\ell}$, then $\overline{y}_{ki} = \gamma_{\overline{\boldsymbol{d}}_{ki}}(\overline{\boldsymbol{x}}_k - \overline{\boldsymbol{d}}_{ki}), i \in I$. Since $\overline{\boldsymbol{y}}_k \to \overline{\boldsymbol{y}}_0$, $\{\overline{\boldsymbol{y}}_k\}$ is bounded. Thus, there exists $P > 0$ such that

$$0 \leq \overline{y}_{ki} = \gamma_{\overline{\boldsymbol{d}}_{ki}}(\overline{\boldsymbol{x}}_k - \overline{\boldsymbol{d}}_{ki}) \leq P, \quad k \in \mathbb{N},\ i \in I. \tag{3}$$

Now, we shall show that $\{\overline{\boldsymbol{x}}_k - \overline{\boldsymbol{d}}_{ki_0}\}$ is bounded. Since B_{i_0} is locally bounded at $\overline{\boldsymbol{d}}_{0i_0}$, there exists a neighborhood U of $\overline{\boldsymbol{d}}_{0i_0}$ such that $B_{i_0}(U) = \cup_{\boldsymbol{d}_{i_0} \in U} B_{i_0}(\boldsymbol{d}_{i_0})$ is bounded. Since $\overline{\boldsymbol{d}}_{ki_0} \to \overline{\boldsymbol{d}}_{0i_0}$, there exists $k_0 \in \mathbb{N}$ such that $\overline{\boldsymbol{d}}_{ki_0} \in U$ for any $k \geq k_0$. Thus, for sufficiently large $u_0 \in \mathbb{R}$,

$$\bigcup_{k \geq k_0} B_{i_0}(\overline{\boldsymbol{d}}_{ki_0}) \subset \bigcup_{\boldsymbol{d}_{i_0} \in U} B_{i_0}(\boldsymbol{d}_{i_0}) \subset U_0 \equiv \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq u_0\}.$$

For U_0, which is a compact convex set containing the origin in its interior, we define the gauge $\gamma_{U_0} : \mathbb{R}^n \to \mathbb{R}$ as $\gamma_{U_0}(\boldsymbol{x}) \equiv \inf\{r > 0 : \boldsymbol{x} \in rU_0\}, \boldsymbol{x} \in \mathbb{R}^n$. Since $B_{i_0}(\overline{\boldsymbol{d}}_{ki_0}) \subset U_0, k \geq k_0$, we have

$$0 \leq \gamma_{U_0}(\overline{\boldsymbol{x}}_k - \overline{\boldsymbol{d}}_{ki_0}) \leq \gamma_{\overline{\boldsymbol{d}}_{ki_0}}(\overline{\boldsymbol{x}}_k - \overline{\boldsymbol{d}}_{ki_0}) \leq P, \quad k \geq k_0$$

from (3). Thus, $\{\overline{\boldsymbol{x}}_k - \overline{\boldsymbol{d}}_{ki_0}\}$ is bounded. Since $\overline{\boldsymbol{d}}_{ki_0} \to \overline{\boldsymbol{d}}_{0i_0}$, $\{\overline{\boldsymbol{d}}_{ki_0}\}$ is bounded. Thus, $\{\overline{\boldsymbol{x}}_k\}$ is also bounded. Thus, there exists a subsequence $\{\overline{\boldsymbol{x}}_{k'}\} \subset \{\overline{\boldsymbol{x}}_k\}$ which converges to $\overline{\boldsymbol{x}}_0 \in \mathbb{R}^n$. Since X is upper semicontinuous at $\overline{\boldsymbol{d}}_0$, we have $\overline{\boldsymbol{x}}_0 \in X(\overline{\boldsymbol{d}}_0)$. Since $\boldsymbol{f}$ is continuous, we have

$$\overline{\boldsymbol{y}}_0 = \lim_{k'\to\infty} \overline{\boldsymbol{y}}_{k'} = \lim_{k'\to\infty} \boldsymbol{f}(\overline{\boldsymbol{x}}_{k'}, \overline{\boldsymbol{d}}_{k'}) = \boldsymbol{f}(\overline{\boldsymbol{x}}_0, \overline{\boldsymbol{d}}_0) \in \boldsymbol{f}(X(\overline{\boldsymbol{d}}_0), \overline{\boldsymbol{d}}_0) = F(\overline{\boldsymbol{d}}_0).$$

□

From Lemma 1 and 2, the following lemma is obtained.

Lemma 3. Let $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$, and assume that $B_{i_0} : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is locally bounded at $\overline{\boldsymbol{d}}_{0i_0}$ for some $i_0 \in I$, and that $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous at $\overline{\boldsymbol{d}}_0$. Then $F : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^\ell$ is continuous at $\overline{\boldsymbol{d}}_0$.

For example, if we define $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ as $X(\boldsymbol{d}) \equiv A, \boldsymbol{d} \in \mathbb{R}^{n\ell}$ for a closed set $A \subset \mathbb{R}^n$, then $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous.

Example 1. We define $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ as

$$X(\boldsymbol{d}) \equiv \text{co}\ \{\boldsymbol{d}_1, \boldsymbol{d}_2, \cdots, \boldsymbol{d}_\ell\}, \quad \boldsymbol{d} = (\boldsymbol{d}_1, \boldsymbol{d}_2, \cdots, \boldsymbol{d}_\ell) \in \mathbb{R}^{n\ell}$$

where co $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \cdots, \boldsymbol{d}_\ell\}$ is the convex hull of $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \cdots, \boldsymbol{d}_\ell\}$. Then it can be seen that $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous.

Example 2.(See Example 5.10 in [6].) Assume that $g_j : \mathbb{R}^n \times \mathbb{R}^{n\ell} \to \mathbb{R}, j = 1, 2, \cdots, m$ are continuous, and that $g_j(\boldsymbol{x}, \boldsymbol{d})$ is convex in $\boldsymbol{x} \in \mathbb{R}^n$ for each $j \in \{1, 2, \cdots, m\}$ and $\boldsymbol{d} \in \mathbb{R}^{n\ell}$. We define $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ as

$$X(\boldsymbol{d}) \equiv \{\boldsymbol{x} \in \mathbb{R}^n : g_j(\boldsymbol{x}, \boldsymbol{d}) \leq 0, j = 1, 2, \cdots, m\}, \quad \boldsymbol{d} \in \mathbb{R}^{n\ell}.$$

Then for $\overline{\boldsymbol{d}} \in \mathbb{R}^{n\ell}$, if there exists $\overline{\boldsymbol{x}} \in \mathbb{R}^n$ such that $g_j(\overline{\boldsymbol{x}}, \overline{\boldsymbol{d}}) < 0, j = 1, 2, \cdots, m$, then $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous not only at $\overline{\boldsymbol{d}}$ but at every $\boldsymbol{d}$ in some neighborhood of $\overline{\boldsymbol{d}}$.

Now, we refer to the following theorem.

Theorem 2.(Theorem 2.21 in [1].) Let $A \subset \mathbb{R}^\ell$ be a nonempty $\mathbb{R}^\ell_+$-compact set. Then the nondominated set of A, A_N, is externally stable.

Lemma 4. Let $\overline{\boldsymbol{d}}_0 \in \mathbb{R}^{n\ell}$, and assume that $X(\overline{\boldsymbol{d}}_0)$ is closed. Then $F(\overline{\boldsymbol{d}}_0)$ is $\mathbb{R}^\ell_+$-compact.

Proof. First, we shall show that $F(\overline{\boldsymbol{d}}_0)$ is closed. Let $\{\overline{\boldsymbol{y}}_k\} \subset F(\overline{\boldsymbol{d}}_0)$ be any convergent sequence, and $\overline{\boldsymbol{y}}_0 \in \mathbb{R}^\ell$ be its limit. For each $k \in \mathbb{N}$, there exists $\overline{\boldsymbol{x}}_k \in X(\overline{\boldsymbol{d}}_0)$ such that $\overline{\boldsymbol{y}}_k = \boldsymbol{f}(\overline{\boldsymbol{x}}_k, \overline{\boldsymbol{d}}_0)$. For each $k \in \mathbb{N}$, if we put $\overline{\boldsymbol{y}}_k \equiv (\overline{y}_{k1}, \overline{y}_{k2}, \cdots, \overline{y}_{k\ell}) \in \mathbb{R}^\ell$ and $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$, then $\overline{y}_{ki} = \gamma_{\overline{\boldsymbol{d}}_{0i}}(\overline{\boldsymbol{x}}_k - \overline{\boldsymbol{d}}_{0i}), i \in I$. Since $\overline{\boldsymbol{y}}_k \to \overline{\boldsymbol{y}}_0$, $\{\overline{\boldsymbol{y}}_k\}$ is bounded. Thus, there exists $P > 0$ such that

$$0 \leq \overline{y}_{ki} = \gamma_{\overline{\boldsymbol{d}}_{0i}}(\overline{\boldsymbol{x}}_k - \overline{\boldsymbol{d}}_{0i}) \leq P, \quad k \in \mathbb{N},\ i \in I.$$

Thus, $\{\overline{\boldsymbol{x}}_k\}$ is bounded, and there exists a subsequence $\{\overline{\boldsymbol{x}}_{k'}\} \subset \{\overline{\boldsymbol{x}}_k\}$ which converges to $\overline{\boldsymbol{x}}_0 \in \mathbb{R}^n$. Since $X(\overline{\boldsymbol{d}}_0)$ is closed, $\overline{\boldsymbol{x}}_0 \in X(\overline{\boldsymbol{d}}_0)$. Since $\boldsymbol{f}$ is continuous, we have

$$\overline{\boldsymbol{y}}_0 = \lim_{k' \to \infty} \overline{\boldsymbol{y}}_{k'} = \lim_{k' \to \infty} \boldsymbol{f}(\overline{\boldsymbol{x}}_{k'}, \overline{\boldsymbol{d}}_0) = \boldsymbol{f}(\overline{\boldsymbol{x}}_0, \overline{\boldsymbol{d}}_0) \in \boldsymbol{f}(X(\overline{\boldsymbol{d}}_0), \overline{\boldsymbol{d}}_0) = F(\overline{\boldsymbol{d}}_0).$$

Therefore, $F(\overline{\boldsymbol{d}}_0)$ is closed.

Next, we shall show that any section of $F(\overline{\boldsymbol{d}}_0)$ is compact. Fix any $\boldsymbol{y} \in F(\overline{\boldsymbol{d}}_0)$. Since $F(\overline{\boldsymbol{d}}_0)$ is closed, $(\boldsymbol{y} - \mathbb{R}^\ell_+) \cap F(\overline{\boldsymbol{d}}_0)$ is also closed. On the other hand, since, $F(\overline{\boldsymbol{d}}_0) \subset \mathbb{R}^\ell_+$, we have $(\boldsymbol{y} - \mathbb{R}^\ell_+) \cap F(\overline{\boldsymbol{d}}_0) \subset (\boldsymbol{y} - \mathbb{R}^\ell_+) \cap \mathbb{R}^\ell_+$. Since $(\boldsymbol{y} - \mathbb{R}^\ell_+) \cap \mathbb{R}^\ell_+$ is bounded, $(\boldsymbol{y} - \mathbb{R}^\ell_+) \cap F(\overline{\boldsymbol{d}}_0)$ is also bounded. Therefore, the section $(\boldsymbol{y} - \mathbb{R}^\ell_+) \cap F(\overline{\boldsymbol{d}}_0)$ is compact. □

From Theorem 2 and Lemma 4, the following theorem is obtained.

Theorem 3. Let $\overline{\boldsymbol{d}}_0 \in \mathbb{R}^{n\ell}$, and assume that $X(\overline{\boldsymbol{d}}_0)$ is closed. Then $M(\overline{\boldsymbol{d}}_0)$ is externally stable.

The following theorem gives sufficient conditions for that the set of all weak Pareto optimal solutions coincides with the set of all Pareto optimal solutions and that there does not exist any alternately Pareto optimal solution.

Theorem 4. Let $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$, and assume that $X(\overline{\boldsymbol{d}}_0)$ is convex, and that all gauges $\gamma_{\overline{\boldsymbol{d}}_{0i}}, i \in I$ are strictly convex. Then $WMS(\overline{\boldsymbol{d}}_0) = MS(\overline{\boldsymbol{d}}_0)$, and there does not exist any alternately Pareto optimal solution of $(\mathrm{P}_{\overline{\boldsymbol{d}}_0})$.

Proof. Let $SMS(\overline{\boldsymbol{d}}_0)$ be the set of all strictly Pareto optimal solutions of $(\mathrm{P}_{\overline{\boldsymbol{d}}_0})$. From Definition 1, $SMS(\overline{\boldsymbol{d}}_0) \subset MS(\overline{\boldsymbol{d}}_0) \subset WMS(\overline{\boldsymbol{d}}_0)$. Thus, in order to show that $WMS(\overline{\boldsymbol{d}}_0) = MS(\overline{\boldsymbol{d}}_0)$ and there does not exist any alternately Pareto optimal solution of $(\mathrm{P}_{\overline{\boldsymbol{d}}_0})$, it is sufficient to show that $WMS(\overline{\boldsymbol{d}}_0) \subset SMS(\overline{\boldsymbol{d}}_0)$. Fix any $\boldsymbol{x}_0 \in WMS(\overline{\boldsymbol{d}}_0)$ and $\boldsymbol{x}_1 \in X(\overline{\boldsymbol{d}}_0)$ with $\boldsymbol{x}_1 \neq \boldsymbol{x}_0$. We put $\boldsymbol{x}_2 \equiv \frac{\boldsymbol{x}_0+\boldsymbol{x}_1}{2}$. Then there exists $i \in I$ such that $\gamma_{\overline{\boldsymbol{d}}_{0i}}(\boldsymbol{x}_0 - \overline{\boldsymbol{d}}_{0i}) \leq \gamma_{\overline{\boldsymbol{d}}_{0i}}(\boldsymbol{x}_2 - \overline{\boldsymbol{d}}_{0i})$. If $\boldsymbol{x}_0 - \overline{\boldsymbol{d}}_{0i}$ and $\boldsymbol{x}_2 - \overline{\boldsymbol{d}}_{0i}$ are on the same line passing through the origin, then $\gamma_{\overline{\boldsymbol{d}}_{0i}}(\boldsymbol{x}_0 - \overline{\boldsymbol{d}}_{0i}) \leq \gamma_{\overline{\boldsymbol{d}}_{0i}}(\boldsymbol{x}_2 - \overline{\boldsymbol{d}}_{0i}) < \gamma_{\overline{\boldsymbol{d}}_{0i}}(\boldsymbol{x}_1 - \overline{\boldsymbol{d}}_{0i})$. If they are not on the same line passing through the origin, then $\gamma_{\overline{\boldsymbol{d}}_{0i}}(\boldsymbol{x}_0 - \overline{\boldsymbol{d}}_{0i}) < \gamma_{\overline{\boldsymbol{d}}_{0i}}(\boldsymbol{x}_1 - \overline{\boldsymbol{d}}_{0i})$ from the strict convexity of a function $t \mapsto \gamma_{\overline{\boldsymbol{d}}_{0i}}((1-t)\boldsymbol{x}_0 + t\boldsymbol{x}_1 - \overline{\boldsymbol{d}}_{0i})$ defined on $\{t \in \mathbb{R} : 0 \leq t \leq 1\}$. Therefore, $\boldsymbol{x}_0 \in SMS(\overline{\boldsymbol{d}}_0)$. □

3 Main results

In this section, by using the results given in the previous section, the continuity of the weak Pareto optimal value mapping, the continuity of the Pareto optimal value mapping, the continuity of the weak Pareto optimal solution mapping, and the continuity of the Pareto optimal solution mapping are investigated as the stability of Pareto and weak Pareto optimal values/solutions.

First, we investigate the continuity of the weak Pareto optimal value mapping $WM : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^{\ell}$.

Theorem 5. Let $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$, and assume that $B_{i_0} : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is locally bounded at $\overline{\boldsymbol{d}}_{0i_0}$ for some $i_0 \in I$, and that $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous at $\overline{\boldsymbol{d}}_0$. Then the weak Pareto optimal value mapping $WM : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^{\ell}$ is upper semicontinuous at $\overline{\boldsymbol{d}}_0$.

Proof. Let $\{\overline{\boldsymbol{d}}_k\} \subset \mathbb{R}^{n\ell}$ be any sequence which converges to $\overline{\boldsymbol{d}}_0$, and $\{\overline{\boldsymbol{y}}_k\}$ be any sequence which converges to $\overline{\boldsymbol{y}}_0 \in \mathbb{R}^{\ell}$ such that $\overline{\boldsymbol{y}}_k \in WM(\overline{\boldsymbol{d}}_k), k \in \mathbb{N}$. From Lemma 3, F is continuous (especially upper semicontinuous) at $\overline{\boldsymbol{d}}_0$. Thus, $\overline{\boldsymbol{y}}_0 \in F(\overline{\boldsymbol{d}}_0)$. Suppose that $\overline{\boldsymbol{y}}_0 \notin WM(\overline{\boldsymbol{d}}_0)$. Then there exists $\widehat{\boldsymbol{y}} \in F(\overline{\boldsymbol{d}}_0)$ such that $\widehat{\boldsymbol{y}} < \overline{\boldsymbol{y}}_0$. Since F is continuous (especially lower semicontinuous) at $\overline{\boldsymbol{d}}_0$, there exist $\{\widehat{\boldsymbol{y}}_k\} \subset \mathbb{R}^{\ell}$, which converges to $\widehat{\boldsymbol{y}}$, and $k_0 \in \mathbb{N}$ such that $\widehat{\boldsymbol{y}}_k \in F(\overline{\boldsymbol{d}}_k)$ for any $k \geq k_0$. Then for any sufficiently large $k \in \mathbb{N}$, $\widehat{\boldsymbol{y}}_k < \overline{\boldsymbol{y}}_k$, which contradicts that $\overline{\boldsymbol{y}}_k \in WM(\overline{\boldsymbol{d}}_k)$. □

The following theorem gives sufficient conditions for the weak Pareto optimal value mapping $WM : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^{\ell}$ to be continuous.

Theorem 6. Let $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$, and assume that $B_{i_0} : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is locally bounded at $\overline{\boldsymbol{d}}_{0i_0}$ for some $i_0 \in I$, and that $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous at $\overline{\boldsymbol{d}}_0$. Furthermore, assume that there exists a neighborhood $U(\overline{\boldsymbol{d}}_0)$ of $\overline{\boldsymbol{d}}_0$ such that $X(\boldsymbol{d})$ is closed for any $\boldsymbol{d} \in U(\overline{\boldsymbol{d}}_0)$. Then if $WM(\overline{\boldsymbol{d}}_0) = M(\overline{\boldsymbol{d}}_0)$, then the weak Pareto optimal value mapping $WM : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^{\ell}$ is continuous at $\overline{\boldsymbol{d}}_0$.

Proof. From Theorem 5, it is sufficient to show that WM is lower semicontinuous at $\overline{\boldsymbol{d}}_0$. Let $\{\overline{\boldsymbol{d}}_k\} \subset \mathbb{R}^{n\ell}$ be any sequence which converges to $\overline{\boldsymbol{d}}_0$, and fix any $\overline{\boldsymbol{y}}_0 \in WM(\overline{\boldsymbol{d}}_0) \subset F(\overline{\boldsymbol{d}}_0)$. Since F is continuous (especially lower semicontinuous) at $\overline{\boldsymbol{d}}_0$ from Lemma 3, there exist $\{\overline{\boldsymbol{y}}_k\} \subset \mathbb{R}^{\ell}$, which converges to $\overline{\boldsymbol{y}}_0$, and $k_0 \in \mathbb{N}$ such that $\overline{\boldsymbol{y}}_k \in F(\overline{\boldsymbol{d}}_k)$ for any $k \geq k_0$. If $\overline{\boldsymbol{y}}_k \in WM(\overline{\boldsymbol{d}}_k)$ for any sufficiently large $k \in \mathbb{N}$, then WM is lower semicontinuous at

$\overline{\boldsymbol{d}}_0$. Thus, without loss of generality, assume that $\overline{\boldsymbol{y}}_k \notin WM(\overline{\boldsymbol{d}}_k)$ for any $k \geq k_0$. From Theorem 3, $M(\overline{\boldsymbol{d}}_k)$ is externally stable for any sufficiently large $k \in \mathbb{N}$. Thus, there exists $\{\widehat{\boldsymbol{y}}_k\} \subset \mathbb{R}^\ell$ such that $\widehat{\boldsymbol{y}}_k \in WM(\overline{\boldsymbol{d}}_k)$ and $\widehat{\boldsymbol{y}}_k \leq \overline{\boldsymbol{y}}_k$ for any sufficiently large $k \in \mathbb{N}$. In order to show that WM is lower semicontinuous at $\overline{\boldsymbol{d}}_0$, it is sufficient to show that $\{\widehat{\boldsymbol{y}}_k\}$ converges to $\overline{\boldsymbol{y}}_0$. Since $\overline{\boldsymbol{y}}_k \to \overline{\boldsymbol{y}}_0$, $\{\overline{\boldsymbol{y}}_k\}$ is bounded. Thus, there exists $\boldsymbol{y}_0 \in \mathbb{R}^\ell_+$ such that $\overline{\boldsymbol{y}}_k \in (\boldsymbol{y}_0 - \mathbb{R}^\ell_+) \cap \mathbb{R}^\ell_+$ for any $k \geq k_0$. For any sufficiently large $k \in \mathbb{N}$, since $\widehat{\boldsymbol{y}}_k \leq \overline{\boldsymbol{y}}_k$ and $\widehat{\boldsymbol{y}}_k \in WM(\overline{\boldsymbol{d}}_k) \subset F(\overline{\boldsymbol{d}}_k)$, we have $\widehat{\boldsymbol{y}}_k \in (\boldsymbol{y}_0 - \mathbb{R}^\ell_+) \cap \mathbb{R}^\ell_+$. Thus, $\{\widehat{\boldsymbol{y}}_k\}$ is bounded, and $\{\widehat{\boldsymbol{y}}_k\}$ has a convergent subsequence. Let $\{\widehat{\boldsymbol{y}}_{k'}\}$ be any convergent subsequence of $\{\widehat{\boldsymbol{y}}_k\}$, and $\widehat{\boldsymbol{y}}_0 \in \mathbb{R}^\ell$ be its limit. Then $\widehat{\boldsymbol{y}}_0 \leq \overline{\boldsymbol{y}}_0$. Since F is continuous (especially upper semicontinuous) at $\overline{\boldsymbol{d}}_0$ from Lemma 3, we have $\widehat{\boldsymbol{y}}_0 \in F(\overline{\boldsymbol{d}}_0)$. Thus, $\widehat{\boldsymbol{y}}_0 = \overline{\boldsymbol{y}}_0$. Namely, $\overline{\boldsymbol{y}}_0$ is a unique accumulation point of $\{\widehat{\boldsymbol{y}}_k\}$. Therefore, $\{\widehat{\boldsymbol{y}}_k\}$ converges to $\overline{\boldsymbol{y}}_0$. □

For example, the condition "$WM(\overline{\boldsymbol{d}}_0) = M(\overline{\boldsymbol{d}}_0)$" in Theorem 6 holds if $X(\overline{\boldsymbol{d}}_0)$ is convex and all $\gamma_{\overline{\boldsymbol{d}}_{0i}}, i \in I$ are strictly convex from Theorem 4, where $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$.

Next, we investigate the continuity of the Pareto optimal value mapping $M : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^\ell$.

Theorem 7. Let $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$, and assume that $B_{i_0} : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is locally bounded at $\overline{\boldsymbol{d}}_{0i_0}$ for some $i_0 \in I$, and that $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous at $\overline{\boldsymbol{d}}_0$. Furthermore, assume that there exists a neighborhood $U(\overline{\boldsymbol{d}}_0)$ of $\overline{\boldsymbol{d}}_0$ such that $X(\boldsymbol{d})$ is closed for any $\boldsymbol{d} \in U(\overline{\boldsymbol{d}}_0)$. Then the Pareto optimal value mapping $M : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^\ell$ is lower semicontinuous at $\overline{\boldsymbol{d}}_0$.

Proof Let $\{\overline{\boldsymbol{d}}_k\} \subset \mathbb{R}^{n\ell}$ be any sequence which converges to $\overline{\boldsymbol{d}}_0$, and fix any $\overline{\boldsymbol{y}}_0 \in M(\overline{\boldsymbol{d}}_0) \subset F(\overline{\boldsymbol{d}}_0)$. Since F is continuous (especially lower semicontinuous) at $\overline{\boldsymbol{d}}_0$ from Lemma 3, there exist $\{\overline{\boldsymbol{y}}_k\} \subset \mathbb{R}^\ell$, which converges to $\overline{\boldsymbol{y}}_0$, and $k_0 \in \mathbb{N}$ such that $\overline{\boldsymbol{y}}_k \in F(\overline{\boldsymbol{d}}_k)$ for any $k \geq k_0$. If $\overline{\boldsymbol{y}}_k \in M(\overline{\boldsymbol{d}}_k)$ for any sufficiently large $k \in \mathbb{N}$, M is lower semicontinuous at $\overline{\boldsymbol{d}}_0$. Thus, without loss of generality, assume that $\overline{\boldsymbol{y}}_k \notin M(\overline{\boldsymbol{d}}_k)$ for any $k \geq k_0$. From Theorem 3, $M(\overline{\boldsymbol{d}}_k)$ is externally stable for any sufficiently large $k \in \mathbb{N}$. Thus, there exists $\{\widehat{\boldsymbol{y}}_k\} \subset \mathbb{R}^\ell$ such that $\widehat{\boldsymbol{y}}_k \in M(\overline{\boldsymbol{d}}_k)$ and $\widehat{\boldsymbol{y}}_k \leq \overline{\boldsymbol{y}}_k$ for any sufficiently large $k \in \mathbb{N}$. In order to show that M is lower semicontinuous at $\overline{\boldsymbol{d}}_0$, it is sufficient to show that $\{\widehat{\boldsymbol{y}}_k\}$ converges to $\overline{\boldsymbol{y}}_0$. Since $\overline{\boldsymbol{y}}_k \to \overline{\boldsymbol{y}}_0$, $\{\overline{\boldsymbol{y}}_k\}$ is bounded. Thus, there exists $\boldsymbol{y}_0 \in \mathbb{R}^\ell_+$ such that $\overline{\boldsymbol{y}}_k \in (\boldsymbol{y}_0 - \mathbb{R}^\ell_+) \cap \mathbb{R}^\ell_+$ for any $k \geq k_0$. Thus, for any sufficiently large $k \in \mathbb{N}$, since $\widehat{\boldsymbol{y}}_k \leq \overline{\boldsymbol{y}}_k$ and $\widehat{\boldsymbol{y}}_k \in M(\overline{\boldsymbol{d}}_k) \subset F(\overline{\boldsymbol{d}}_k)$, we have $\widehat{\boldsymbol{y}}_k \in (\boldsymbol{y}_0 - \mathbb{R}^\ell_+) \cap \mathbb{R}^\ell_+$. Thus, $\{\widehat{\boldsymbol{y}}_k\}$ is bounded, and $\{\widehat{\boldsymbol{y}}_k\}$ has a convergent subsequence. Let $\{\widehat{\boldsymbol{y}}_{k'}\}$ be any convergent subsequence of $\{\widehat{\boldsymbol{y}}_k\}$, and $\widehat{\boldsymbol{y}}_0 \in \mathbb{R}^\ell$ be its limit. Then $\widehat{\boldsymbol{y}}_0 \leq \overline{\boldsymbol{y}}_0$. Since F is continuous (especially upper semicontinuous) at $\overline{\boldsymbol{d}}_0$ from Lemma 3, we have $\widehat{\boldsymbol{y}}_0 \in F(\overline{\boldsymbol{d}}_0)$. Thus, $\widehat{\boldsymbol{y}}_0 = \overline{\boldsymbol{y}}_0$. Namely, $\overline{\boldsymbol{y}}_0$ is a unique accumulation point of $\{\widehat{\boldsymbol{y}}_k\}$. Therefore, $\{\widehat{\boldsymbol{y}}_k\}$ converges to $\overline{\boldsymbol{y}}_0$. □

Theorem 8. Let $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$, and assume that $B_{i_0} : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is locally bounded at $\overline{\boldsymbol{d}}_{0i_0}$ for some $i_0 \in I$, and that $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous at $\overline{\boldsymbol{d}}_0$. Furthermore, assume that there exists a neighborhood $U(\overline{\boldsymbol{d}}_0)$ of $\overline{\boldsymbol{d}}_0$ such that $X(\boldsymbol{d})$ is closed for any $\boldsymbol{d} \in U(\overline{\boldsymbol{d}}_0)$. Then if $WM(\overline{\boldsymbol{d}}_0) = M(\overline{\boldsymbol{d}}_0)$, then the Pareto optimal value mapping $M : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^\ell$ is continuous at $\overline{\boldsymbol{d}}_0$.

Proof. From Theorem 7, it is sufficient to show that M is upper semicontinuous at $\overline{\boldsymbol{d}}_0$. Let $\{\overline{\boldsymbol{d}}_k\} \subset \mathbb{R}^{n\ell}$ be any sequence which converges to $\overline{\boldsymbol{d}}_0$, and $\{\overline{\boldsymbol{y}}_k\}$ be any sequence which converges to $\overline{\boldsymbol{y}}_0 \in \mathbb{R}^\ell$ such that $\overline{\boldsymbol{y}}_k \in M(\overline{\boldsymbol{d}}_k), k \in \mathbb{N}$. Since F is continuous (especially upper semicontinuous) at $\overline{\boldsymbol{d}}_0$ from Lemma 3, we have $\overline{\boldsymbol{y}}_0 \in F(\overline{\boldsymbol{d}}_0)$. Suppose that $\overline{\boldsymbol{y}}_0 \notin M(\overline{\boldsymbol{d}}_0) = WM(\overline{\boldsymbol{d}}_0)$. Then there exists $\widehat{\boldsymbol{y}} \in F(\overline{\boldsymbol{d}}_0)$ such that $\widehat{\boldsymbol{y}} < \overline{\boldsymbol{y}}_0$. Since F is continuous (especially lower semicontinuous) at $\overline{\boldsymbol{d}}_0$, there exist $\{\widehat{\boldsymbol{y}}_k\} \subset \mathbb{R}^\ell$, which converges to $\widehat{\boldsymbol{y}}$, and $k_0 \in \mathbb{N}$ such that $\widehat{\boldsymbol{y}}_k \in F(\overline{\boldsymbol{d}}_k)$ for any $k \geq k_0$. Then for any sufficiently large $k \in \mathbb{N}$, $\widehat{\boldsymbol{y}}_k < \overline{\boldsymbol{y}}_k$, which contradicts that $\overline{\boldsymbol{y}}_k \in M(\overline{\boldsymbol{d}}_k)$. □

Next, we investigate the continuity of the weak Pareto optimal solution mapping $WMS : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ and the Pareto optimal solution mapping $MS : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$.

Theorem 9. Let $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$, and assume that $B_{i_0} : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is locally bounded at $\overline{\boldsymbol{d}}_{0i_0}$ for some $i_0 \in I$, and that $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous at $\overline{\boldsymbol{d}}_0$. Then the weak Pareto optimal solution mapping $WMS : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is upper semicontinuous at $\overline{\boldsymbol{d}}_0$.

Proof. Let $\{\overline{\boldsymbol{d}}_k\} \subset \mathbb{R}^{n\ell}$ be any sequence which converges to $\overline{\boldsymbol{d}}_0$, and $\{\overline{\boldsymbol{x}}_k\}$ be any sequence which converges to $\overline{\boldsymbol{x}}_0 \in \mathbb{R}^n$ such that $\overline{\boldsymbol{x}}_k \in WMS(\overline{\boldsymbol{d}}_k), k \in \mathbb{N}$. Then for each $k \in \mathbb{N}$, we put $\overline{\boldsymbol{y}}_k \equiv \boldsymbol{f}(\overline{\boldsymbol{x}}_k, \overline{\boldsymbol{d}}_k) \in WM(\overline{\boldsymbol{d}}_k)$. Since X is continuous (especially upper semicontinuous) at $\overline{\boldsymbol{d}}_0$, we have $\overline{\boldsymbol{x}}_0 \in X(\overline{\boldsymbol{d}}_0)$. Since $\boldsymbol{f}$ is continuous and WM is upper semicontinuous at $\overline{\boldsymbol{d}}_0$ from Theorem 5, we have $\overline{\boldsymbol{y}}_0 \equiv \boldsymbol{f}(\overline{\boldsymbol{x}}_0, \overline{\boldsymbol{d}}_0) \in WM(\overline{\boldsymbol{d}}_0)$. Therefore, $\overline{\boldsymbol{x}}_0 \in WMS(\overline{\boldsymbol{d}}_0)$. □

Theorem 10. Let $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$, and assume that $B_{i_0} : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is locally bounded at $\overline{\boldsymbol{d}}_{0i_0}$ for some $i_0 \in I$, and that $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous at $\overline{\boldsymbol{d}}_0$. Furthermore, assume that there exists a neighborhood $U(\overline{\boldsymbol{d}}_0)$ of $\overline{\boldsymbol{d}}_0$ such that $X(\boldsymbol{d})$ is closed for any $\boldsymbol{d} \in U(\overline{\boldsymbol{d}}_0)$. Then if $WM(\overline{\boldsymbol{d}}_0) = M(\overline{\boldsymbol{d}}_0)$ and there does not exist any alternately Pareto optimal solution of $(\mathrm{P}_{\overline{\boldsymbol{d}}_0})$, then the weak Pareto optimal solution mapping $WMS : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous at $\overline{\boldsymbol{d}}_0$.

Proof. From Theorem 9, it is sufficient to show that WMS is lower semicontinuous at $\overline{\boldsymbol{d}}_0$. Let $\{\overline{\boldsymbol{d}}_k\} \subset \mathbb{R}^{n\ell}$ be any sequence which converges to $\overline{\boldsymbol{d}}_0$, and fix any $\overline{\boldsymbol{x}}_0 \in WMS(\overline{\boldsymbol{d}}_0)$. We put $\overline{\boldsymbol{y}}_0 \equiv \boldsymbol{f}(\overline{\boldsymbol{x}}_0, \overline{\boldsymbol{d}}_0)$. Then $\overline{\boldsymbol{y}}_0 \in WM(\overline{\boldsymbol{d}}_0)$. Since WM is continuous (especially lower semicontinuous) at $\overline{\boldsymbol{d}}_0$ from Theorem 6, there exist $\{\overline{\boldsymbol{y}}_k\} \subset \mathbb{R}^{\ell}$, which converges to $\overline{\boldsymbol{y}}_0$, and $k_0 \in \mathbb{N}$ such that $\overline{\boldsymbol{y}}_k \in WM(\overline{\boldsymbol{d}}_k)$ for any $k \geq k_0$. Let $\{\overline{\boldsymbol{x}}_k\} \subset \mathbb{R}^n$ be any sequence such that $\overline{\boldsymbol{y}}_k = \boldsymbol{f}(\overline{\boldsymbol{x}}_k, \overline{\boldsymbol{d}}_k)$ and $\overline{\boldsymbol{x}}_k \in WMS(\overline{\boldsymbol{d}}_k)$ for each $k \geq k_0$. In order to show that WMS is lower semicontinuous at $\overline{\boldsymbol{d}}_0$, it is sufficient to show that $\{\overline{\boldsymbol{x}}_k\}$ converges to $\overline{\boldsymbol{x}}_0$. By the same argument in the proof of Lemma 2, $\{\overline{\boldsymbol{x}}_k\}$ is bounded. Thus, $\{\overline{\boldsymbol{x}}_k\}$ has a convergent subsequence. Let $\{\overline{\boldsymbol{x}}_{k'}\}$ be any convergent subsequence of $\{\overline{\boldsymbol{x}}_k\}$, and $\overline{\boldsymbol{x}}'_0 \in \mathbb{R}^n$ be its limit. Since X is continuous (especially upper semicontinuous) at $\overline{\boldsymbol{d}}_0$, we have $\overline{\boldsymbol{x}}'_0 \in X(\overline{\boldsymbol{d}}_0)$. Since $\boldsymbol{f}$ is continuous, we have

$$\boldsymbol{f}(\overline{\boldsymbol{x}}_0, \overline{\boldsymbol{d}}_0) = \overline{\boldsymbol{y}}_0 = \lim_{k' \to \infty} \overline{\boldsymbol{y}}_{k'} = \lim_{k' \to \infty} \boldsymbol{f}(\overline{\boldsymbol{x}}_{k'}, \overline{\boldsymbol{d}}_{k'}) = \boldsymbol{f}(\overline{\boldsymbol{x}}'_0, \overline{\boldsymbol{d}}_0).$$

Since there does not exist any alternately Pareto optimal solution of $(\mathrm{P}_{\overline{\boldsymbol{d}}_0})$ from the assumption, $\overline{\boldsymbol{x}}_0$ is a strictly Pareto optimal solution of $(\mathrm{P}_{\overline{\boldsymbol{d}}_0})$. Thus, since $\boldsymbol{f}(\overline{\boldsymbol{x}}'_0, \overline{\boldsymbol{d}}_0) = \boldsymbol{f}(\overline{\boldsymbol{x}}_0, \overline{\boldsymbol{d}}_0)$, we have $\overline{\boldsymbol{x}}'_0 = \overline{\boldsymbol{x}}_0$. Namely, $\overline{\boldsymbol{x}}_0$ is a unique accumulation point of $\{\overline{\boldsymbol{x}}_k\}$. Therefore, $\{\overline{\boldsymbol{x}}_k\}$ converges to $\overline{\boldsymbol{x}}_0$. □

For example, the condition "there does not exist any alternately Pareto optimal solution of $(\mathrm{P}_{\overline{\boldsymbol{d}}_0})$" in Theorem 10 holds if $X(\overline{\boldsymbol{d}}_0)$ is convex and all $\gamma_{\overline{\boldsymbol{d}}_{0i}}, i \in I$ are strictly convex from Theorem 4, where $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$.

Theorem 11. Let $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$, and assume that $B_{i_0} : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is locally bounded at $\overline{\boldsymbol{d}}_{0i_0}$ for some $i_0 \in I$, and that $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous at $\overline{\boldsymbol{d}}_0$. Furthermore, assume that there exists a neighborhood $U(\overline{\boldsymbol{d}}_0)$ of $\overline{\boldsymbol{d}}_0$ such that $X(\boldsymbol{d})$ is closed for any $\boldsymbol{d} \in U(\overline{\boldsymbol{d}}_0)$. Then if $WM(\overline{\boldsymbol{d}}_0) = M(\overline{\boldsymbol{d}}_0)$, then the Pareto optimal solution mapping $MS : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is upper semicontinuous at $\overline{\boldsymbol{d}}_0$.

Proof. Let $\{\overline{\boldsymbol{d}}_k\} \subset \mathbb{R}^{n\ell}$ be any sequence which converges to $\overline{\boldsymbol{d}}_0$, and $\{\overline{\boldsymbol{x}}_k\}$ be any sequence which converges to $\overline{\boldsymbol{x}}_0 \in \mathbb{R}^n$ such that $\overline{\boldsymbol{x}}_k \in MS(\overline{\boldsymbol{d}}_k), k \in \mathbb{N}$. Then for each $k \in \mathbb{N}$, we put $\overline{\boldsymbol{y}}_k \equiv \boldsymbol{f}(\overline{\boldsymbol{x}}_k, \overline{\boldsymbol{d}}_k) \in M(\overline{\boldsymbol{d}}_k)$. Since X is continuous (especially upper semicontinuous) at $\overline{\boldsymbol{d}}_0$, we have $\overline{\boldsymbol{x}}_0 \in X(\overline{\boldsymbol{d}}_0)$. Since $\boldsymbol{f}$ is continuous and M is continuous (especially upper semicontinuous) at $\overline{\boldsymbol{d}}_0$ from Theorem 8, we have $\overline{\boldsymbol{y}}_0 \equiv \boldsymbol{f}(\overline{\boldsymbol{x}}_0, \overline{\boldsymbol{d}}_0) \in M(\overline{\boldsymbol{d}}_0)$. Therefore, $\overline{\boldsymbol{x}}_0 \in MS(\overline{\boldsymbol{d}}_0)$. □

Theorem 12. Let $\overline{\boldsymbol{d}}_0 \equiv (\overline{\boldsymbol{d}}_{01}, \overline{\boldsymbol{d}}_{02}, \cdots, \overline{\boldsymbol{d}}_{0\ell}) \in \mathbb{R}^{n\ell}$, and assume that $B_{i_0} : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is locally bounded at $\overline{\boldsymbol{d}}_{0i_0}$ for some $i_0 \in I$, and that $X : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous at $\overline{\boldsymbol{d}}_0$. Furthermore, assume that there exists a neighborhood $U(\overline{\boldsymbol{d}}_0)$ of $\overline{\boldsymbol{d}}_0$ such that $X(\boldsymbol{d})$ is closed for any $\boldsymbol{d} \in U(\overline{\boldsymbol{d}}_0)$. Then if $WM(\overline{\boldsymbol{d}}_0) = M(\overline{\boldsymbol{d}}_0)$ and there does not exist any alternately Pareto optimal solution of $(\mathrm{P}_{\overline{\boldsymbol{d}}_0})$, then the Pareto optimal solution mapping $MS : \mathbb{R}^{n\ell} \rightsquigarrow \mathbb{R}^n$ is continuous at $\overline{\boldsymbol{d}}_0$.

Proof. From Theorem 11, it is sufficient to show that MS is lower semicontinuous at $\overline{\boldsymbol{d}}_0$. Let $\{\overline{\boldsymbol{d}}_k\} \subset \mathbb{R}^{n\ell}$ be any sequence which converges to $\overline{\boldsymbol{d}}_0$, and fix any $\overline{\boldsymbol{x}}_0 \in MS(\overline{\boldsymbol{d}}_0)$. We put $\overline{\boldsymbol{y}}_0 \equiv \boldsymbol{f}(\overline{\boldsymbol{x}}_0, \overline{\boldsymbol{d}}_0)$. Then $\overline{\boldsymbol{y}}_0 \in M(\overline{\boldsymbol{d}}_0)$. Since M is lower semicontinuous at $\overline{\boldsymbol{d}}_0$ from Theorem 7, there exist $\{\overline{\boldsymbol{y}}_k\} \subset \mathbb{R}^\ell$, which converges to $\overline{\boldsymbol{y}}_0$, and $k_0 \in \mathbb{N}$ such that $\overline{\boldsymbol{y}}_k \in M(\overline{\boldsymbol{d}}_k)$ for any $k \geq k_0$. Let $\{\overline{\boldsymbol{x}}_k\} \subset \mathbb{R}^n$ be any sequence such that $\overline{\boldsymbol{y}}_k = \boldsymbol{f}(\overline{\boldsymbol{x}}_k, \overline{\boldsymbol{d}}_k)$ and $\overline{\boldsymbol{x}}_k \in MS(\overline{\boldsymbol{d}}_k)$ for any $k \geq k_0$. In order to show that MS is lower semicontinuous at $\overline{\boldsymbol{d}}_0$, it is sufficient to show that $\{\overline{\boldsymbol{x}}_k\}$ converges to $\overline{\boldsymbol{x}}_0$. By the same argument in the proof of Lemma 2, $\{\overline{\boldsymbol{x}}_k\}$ is bounded. Thus, $\{\overline{\boldsymbol{x}}_k\}$ has a convergent subsequence. Let $\{\overline{\boldsymbol{x}}_{k'}\}$ be any convergent subsequence of $\{\overline{\boldsymbol{x}}_k\}$, and $\overline{\boldsymbol{x}}'_0 \in \mathbb{R}^n$ be its limit. Since X is continuous (especially upper semicontinuous) at $\overline{\boldsymbol{d}}_0$, we have $\overline{\boldsymbol{x}}'_0 \in X(\overline{\boldsymbol{d}}_0)$. Since $\boldsymbol{f}$ is continuous, we have

$$\boldsymbol{f}(\overline{\boldsymbol{x}}_0, \overline{\boldsymbol{d}}_0) = \overline{\boldsymbol{y}}_0 = \lim_{k' \to \infty} \overline{\boldsymbol{y}}_{k'} = \lim_{k' \to \infty} \boldsymbol{f}(\overline{\boldsymbol{x}}_{k'}, \overline{\boldsymbol{d}}_{k'}) = \boldsymbol{f}(\overline{\boldsymbol{x}}'_0, \overline{\boldsymbol{d}}_0).$$

Since there does not exist any alternately Pareto optimal solution of $(\mathrm{P}_{\overline{\boldsymbol{d}}_0})$ from the assumption, $\overline{\boldsymbol{x}}_0$ is a strictly Pareto optimal solution of $(\mathrm{P}_{\overline{\boldsymbol{d}}_0})$. Thus, since $\boldsymbol{f}(\overline{\boldsymbol{x}}'_0, \overline{\boldsymbol{d}}_0) = \boldsymbol{f}(\overline{\boldsymbol{x}}_0, \overline{\boldsymbol{d}}_0)$, we have $\overline{\boldsymbol{x}}'_0 = \overline{\boldsymbol{x}}_0$. Namely, $\overline{\boldsymbol{x}}_0$ is a unique accumulation point of $\{\overline{\boldsymbol{x}}_k\}$. Therefore, $\{\overline{\boldsymbol{x}}_k\}$ converges to $\overline{\boldsymbol{x}}_0$. □

4 Conclusions In this paper, we dealt with a family of parameterized multicriteria location problems, and investigated the stability of Pareto and weak Pareto optimal values/solutions, where we considered demand points as the parameter and distance measures depended on the parameter. First, we gave sufficient conditions for the objective function of the multicriteria location problems to be continuous as Theorem 1, and gave sufficient conditions for that the set of all weak Pareto optimal solutions coincides with the set of all Pareto optimal solutions and that there does not exist any alternately Pareto optimal solution as Theorem 4. Next, we investigated the continuity of the weak Pareto optimal value mapping as Theorem 5 and 6, the continuity of the Pareto optimal value mapping as Theorem 7 and 8, the continuity of the weak Pareto optimal solution mapping as Theorem 9 and 10, and the continuity of the Pareto optimal solution mapping as Theorem 11 and 12.

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