RELATIVE REARRANGEMENT AND ITS APPLICATIONS

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Abstract. We would like to show a precise review of a part of the work [2]. As materials, we take up Rearrangement, Neumann Problem, Equilibrium Equation, and Chemotaxis System. We also explain the outline of the Sobolev embedding described in the work [1]. Finally we investigate the relation between the Rearrangement and Conditionally Convergent Integrals.

Introduction Equilibrium equation $-\Delta \mathrm{u}(x) = \mathrm{G}(u, u_*, u'_*, b_{*u})$ describes "plasma" confined in the machine Stellarator [2] p.173. We find rearrangements(RA) u_*, b_{*u} of measurable functions $\mathrm{u}(x), \mathrm{b}(x)$ in the right side of the equation. We would like to show the meaning of the rearrangements (in the form with Examples), and the relation between the RA and conditionally convergent integrals such as A-integral (Theorems A∼C in Section 1). Various inequalities relating to the RA are found in the work [2]. We would like to prove the estimations appearing in the Neumann problem [2] p.125. The outline of the results on the Chemotaxis system with the term $\partial_t u$ in [2] p.234 is also given. The problem relating to the Sobolev embeddings, found in the work [1] p.115, p.129, are explained. We express here Theorems in [2] in the form with the same numbers as [2].

1 Rearrangement and A-Integral

(I) Rearrangement

(i) Let u: $\Omega\left(\subseteq \boldsymbol{R}^N\right) \to \boldsymbol{R}$, measurable, and $\mathrm{m}(t) = \mathrm{m}_u(t) = |u > t| \equiv \text{measure}\{x; \mathrm{u}(x) > t\}$. $\mathrm{u}_*(s) = \mathrm{Inf}\{t \in \boldsymbol{R}, \mathrm{m}(t) \leqq s\}, s \in \Omega_*$, is called "decreasing rearrangement(RA) of u". $\Omega_* = [0, |\Omega|)$.

(ii) $\mathrm{F}(t, u) = ts + \int_\Omega (u-t)_+ dx, \mathrm{F}(\mathrm{u}_*(s), u) = \int_{[0,s]} \mathrm{u}_*(\sigma)\, d\sigma, (u-t)_+ = \mathrm{Max}\{0, (u-t)\}$. $\mathrm{F}'(\cdot, u; v) = \lim_{\lambda \to 0, \lambda > 0}\{(\mathrm{F}(\cdot, u + \lambda v) - \mathrm{F}(\cdot, u))/\lambda\}$. $\mathrm{v}_{*u}(s) = (d/ds)\, \mathrm{F}'(\cdot, u; v)$ is "relative RA of v with respect to u".

R-formula. $\mathrm{v}_{*u}(s) = (d/ds)_+ \mathrm{w}(s)$ for $\mathrm{F}(\mathrm{u}_*(s), u)$, [2] p.33, p.40; $\mathrm{w}(s) = \int_{\{u > \mathrm{u}_*(s)\}} \mathrm{v}(x)\, dx + \int_{[0, s - |u > \mathrm{u}_*(s)|]} \left(\mathrm{v}|_{\{u = \mathrm{u}_*(s)\}}\right)_* (\sigma)\, d\sigma$.

(iii) u has a step at t. if $|u = t| > 0$. D_u is the set of steps, and $\mathrm{P}(u) = \cup_{t \in \mathrm{D}_u}\{u = t\}$ is the plateau of u. If $|\mathrm{P}(u)| = |\Omega|, \mathrm{u}(x)$ is called "step function".

Remark 1. $\mathrm{u}(x), \mathrm{v}(x)$ in $\boldsymbol{L}^1(\Omega)$ are treated in [2] p.40. When $\mathrm{u}(x)$ and $\mathrm{v}(x)$ are step functions, δ-like sigularities do not appear in $\mathrm{v}_{*u}(s)$ given from Definition. Then $(d/ds)_+$ must be used.

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(iv) $u_{\sim}(x) = \mathrm{u}_*\left(\alpha_N |x|^N\right)$, $\mathrm{N}\alpha_N$: surface area of the unit ball in $\boldsymbol{R}^N$, is called spherical RA of u.

Example 1.

(a) $\mathrm{u}(x) = \left(1 - x^2\right)_+$, $\Omega = (-1, 1)$, $\mathrm{m}(t) = 2(1-t)^{1/2}$,
$\mathrm{u}_*(s) = 1 - s^2/4$, $ts + \int_\Omega (u - t)_+ \, dx = ts + (4/3)(1-t)^{3/2}$.

(b) $\mathrm{v}(x) = \left(1 - x^2\right)^2$, $\mathrm{w}(x) = \int_{\{u > \mathrm{u}_*(s)\}} \mathrm{v}(x)\, dx = s - s^3/6 + s^5/80$,
$\mathrm{v}_{*u}(s) = \left(1 - s^2/4\right)^2 = \mathrm{v}_*(s)$.

(c) $\mathrm{u}(x) = 2$ for $1 \leqq |x| < 2$, $= 1$ for $|x| < 1$; $\Omega = (-2, 2)$;
$\mathrm{m}_u(t) = 0$ for $t \geqq 2$, $= 2$ for $2 > t \geqq 1$, $= 4$ for $1 > t$;
$\mathrm{u}_*(s) = 2$ for $0 \leqq s < 2$, $= 1$ for $2 \leqq s < 4$; $\Omega_* = [0, 4)$.

Remark 2. $\int_\Omega \mathrm{u}(x)\, dx = -\int_{(-\infty,\infty)} t d\mathrm{m}_u(t)/dt = \int_{[0,|\Omega|)} \mathrm{u}_*(s)\, ds$ **(I)** holds for the above (a), (c) in Example 1.

Example 2. $\mathrm{u}(x, y) = \left(1 - (x-5)^2/4 - (y-7)^2/9\right)_+$, $\mathrm{m}(t) = 6\pi(1-t)$, $\mathrm{u}_*(s) = 1 - s/6\pi$, $\mathrm{u}_{\sim}(x, y) = 1 - \left(x^2 + y^2\right)/6$.

Example 3. Denote by $\mathrm{u}(x) = 0$ for $|x| \leqq 1/2$, $= 1/2$ for $1/2 < |x| \leqq 1$. $\mathrm{u}_*(s) = 1/2$ for $0 \leqq s < 1$, $= 0$ for $1 \leqq s < 2$. For $\mathrm{v}(x) = \left(1 - x^2\right)^2$, $\mathrm{v}_{*u}(s) = \left(1 - (s+1)^2/4\right)^2$ for $0 \leqq s < 1$, $= \left(1 - (s-1)^2/4\right)^2$ for $1 \leqq s < 2$.

Remark 3. Example 3 is given by using the R-formula in (ii). $u(x)$ plays the effective role to give a measure preserving transform in Ω_*.

(II) Conditionally Convergent Integrals such as A-integral.
Measurable fuction $\mathrm{f}(x)$ defined on an open bounded set Ω is A-integrable, if the following conditions are satisfied:

(a) $\mathrm{meas}\{x; |\mathrm{f}(x)| > n, x \in \Omega\} = O\left((n \cdot \log_e n)^{-1}\right)$.

(b) There exists a finite limit $\lim_{n\to\infty} \int_\Omega [\mathrm{f}]_n(x)\, dx = \mathrm{A}\!-\!\int_\Omega \mathrm{f}(x)\, dx$; $[\mathrm{f}]_n(x) = \mathrm{f}(x)$, if $|\mathrm{f}(x)| \leqq n$, and $[\mathrm{f}]_n(x) = 0$, if $|\mathrm{f}(x)| > n$.
Since $\mathrm{f}(x)$ is measurable, the function $\mathrm{m}_f(t), -\infty < t < \infty$, can be defined. We obtain the following Theorem A from the Definition of $\mathrm{f}_*(s)$, $s \in [0, |\Omega|)$.

Theorem A. A-integral $\mathrm{A}\text{-}\int_\Omega \mathrm{f}(x)\, dx = \lim_{n\to\infty} \int_\Omega [\mathrm{f}]_n(x)\, dx$ is given by $\mathrm{A}\text{-}\int_\Omega \mathrm{f}(x)\, dx = \lim_{n\to\infty} \int_{\mathrm{I}(n)} \mathrm{f}_*(s)\, ds$, where the interval $\mathrm{I}(n) = \left[\mathrm{f}_*^{-1}(n), \mathrm{f}_*^{-1}(-n)\right]$.

Let $V_+ = |\mathrm{f}(x) > 0| < \infty, V_0 = |\mathrm{f}(x) = 0| > 0$, and $V_- = |\mathrm{f}(x) < 0| < \infty$. Since $\mathrm{m}_f(t) = |\mathrm{f}(x) > t|$, $\mathrm{m}_f(0) = \mathrm{m}_f(0+0) = V_+$ and $\mathrm{m}_f(0-0) = V_+ + V_0$. Since $\mathrm{f}_*(s) = \inf\{t \in \mathrm{R}; s \geqq m_f(t)\}$, then, $\mathrm{f}_*(V_+ + V_0 + 0) < 0$ and $\mathrm{f}_*(s) = 0$ for $V_+ \leqq s < V_+ + V_0$, hold. The rest part of the Proof is derived from the formula **(I)** in Remark 2.

A-integral, an extension of the Lebesgue integral, is the same as E.R.integral given by K.Kunugui [4] etc.. E.R. ν integral, is the Stieltjes type E.R.integral. E.R.ν integral by using $\nu(B) = \int_B \phi(x)\, dx$ with $\phi(x) \in \boldsymbol{L}^1(\Omega), \phi(x) > 0$, can be defined by $\mathrm{m}_f(t) = \int_{\{x|\mathrm{f}(x) > t\phi(x)\}} \phi(x)\, dx$. Improper integral can be also defined by using the infinite sequence

$\{\Omega_n; n = 1, 2, \cdots\}, \Omega_i \subset \Omega_{i+1}$. When the set $\{x; \mathrm{f}(x) \neq 0\}$ is infinite join of the nowhere dense parfect sets, these conditionally convergent integrals have various applications. As Example, Expression of the distributions by I.L.Bondi [5] p.131, and an interpretation of Causality and of Lorentz invariance in [6] p.548, are given.

Theorem B. If $\{(\mathrm{u}(x), \mathrm{v}(x)); x \in \Omega\left(\subseteq \boldsymbol{R}^N\right)\}$ satisfies $\mathrm{u}(x_1) \leqq \mathrm{u}(x_2) \leftrightarrows \mathrm{v}(x_1) \leqq \mathrm{v}(x_2)$ for $\forall (x_1, x_2) \in \Omega \times \Omega$, then $\mathrm{v}_{*\mathrm{u}}(s) = \mathrm{v}_*(s)$ holds for $\forall s \in \Omega_*$.

If $\mathrm{u}(x)$ is a step function, heorem B can be proved by using the R-formula $\mathrm{v}_{*\mathrm{u}}(s) = (d/ds)_+ \mathrm{w}(s)$ in (ii).

Suppose that $\mathrm{v}(x) \in \left(\boldsymbol{L}^1(\Omega)\right)^c$ (complement) is an A-integrable function defined on $\Omega\left(\subseteq \boldsymbol{R}^N\right)$. Let $\mathrm{n}(0) = 0$, and $\{\mathrm{n}(i); i = 1, 2, \cdot\} (\subseteq N)$ be an infinite increasing sequence such that $\left|\Omega_{n(i)}\right| > 0$ for $\Omega_{n(i)} = \{x \in \Omega; n(i-1) < |\mathrm{v}(x)| \leqq \mathrm{n}(i)\}$. Let $\mathrm{I}(1) = \left[0, \left|\Omega_{\mathrm{n}(1)}\right|\right)$, and $\mathrm{I}(i) := \left[\sum_{j=1\sim i-1} \left|\Omega_{n(j)}\right|, \sum_{j=1\sim i} \left|\Omega_{n(j)}\right|\right)$ for $i \geqq 2$.

Theorem C. Suppose that $\mathrm{v}(x) \in \left(\boldsymbol{L}^1(\Omega)\right)^c$ is an A-integrable function. Then there exists a step function $\mathrm{u}(x); x \in \Omega\left(\subseteq \boldsymbol{R}^N\right)$ such that A-$\int_\Omega \mathrm{v}(x)\,dx = \sum_{i \in \boldsymbol{N}} \int_{I(i)} \mathrm{v}_{*\mathrm{u}}(s)\,ds$ holds.

Proof.

Let $\mathrm{u}(x) = (1/2)^i$ for $x \in \Omega_{n(i)}, = 0$ for $x \in \Omega - \cup_{i \in \boldsymbol{N}} \Omega_{\mathrm{n}(i)}$.

$$
\begin{aligned}
\mathrm{u}_*(s) &= 1/2 \text{ for } 0 \leqq s < |\mathrm{I}(1)|, \\
&= (1/2)^n \text{ for } |\cup_{j=1\sim n-1} \mathrm{I}(j)| \leqq s < |\cup_{j=1\sim n} \mathrm{I}(j)|, \\
&= 0 \text{ for } |\cup_{j=1\sim\infty} \mathrm{I}(j)| \leqq s < |\Omega|.
\end{aligned}
$$

This Theorem C can be proved by the extended use of R-formula in (ii). We must use $\mathrm{n}(i)\,\phi(x)$ instead of $\mathrm{n}(i)$ for the E.R. ν. integrable functions $\mathrm{v}(x)$.

2 Sobolev Embeddings, Boundary Value Problems, and Equilibrium Equation.

(I) Sobolev Embeddings.

Sobolev space $\mathrm{W}^{m,p}(\Omega) := \{v \in \mathrm{W}^{m-1,p}(\Omega), D^\alpha v \in \boldsymbol{L}^p(\Omega) \text{ for } \Omega \subset \boldsymbol{R}^N, |\alpha| = m, \alpha \in \boldsymbol{N}^N\}$, Hilbert space $\mathrm{H}^m(\Omega) = \mathrm{W}^{m,2}(\Omega)$, and $\mathrm{W}_0^{m,p}(\Omega)$: closure of $\mathrm{D}(\Omega)$ in $\mathrm{W}^{m,p}(\Omega)$.

Continuous injection: Let us consider the two linear spaces $\boldsymbol{V}, \boldsymbol{H}$, with two norms $\|\cdot\|, |\cdot|$, respectively. The relation "$\boldsymbol{V} \subset \boldsymbol{H}$" means "$\exists c > 0$ such that $|v| \leqq c \parallel v \parallel$ holds for $\forall v \in \boldsymbol{V}$".

Let $\Omega\left(\subseteq \boldsymbol{R}^N\right)$ be an open bounded set with C^1 boundary [1] p.115.

Sobolev Embeddings: (i).

- **(a)** If $1 \leqq p < N$, then $\mathrm{W}^{1,p}(\Omega) \subset \boldsymbol{L}^q(\Omega)$ for $q \in [1, p^*]$ with $p^* = Np/(N-p)$. If $q \neq p^*$, this injection becomes the compact operator "bounded→ compact", [1] p.130.
- **(b)** If $p = N$, then $\mathrm{W}^{1,N}(\Omega) \subset \boldsymbol{L}^q(\Omega)$ for $\forall q \in [1, \infty)$, and these injections are compact.
- **(c)** If $p > N$, then $\mathrm{W}^{1,p}(\Omega) \subset \boldsymbol{C}^{0,\alpha}(\Omega)$ [1] p.10, with $\alpha = 1 - N/p$, and the injection of $\mathrm{W}^{1,p}\left(\bar{\Omega}\right)$ in $\boldsymbol{C}\left(\bar{\Omega}\right)$ is compact.

The Ascoli Theorem, proved by using equicontinuity, is used in the proof of this Proposition.

(ii) Let $W_0^{1,N}(\Omega) \subset \boldsymbol{L}^r(\Omega)$ holds for $1 \leqq \forall r < \infty$. One has $|u|_{\boldsymbol{L}_r(\Omega)} \leqq \left(|\Omega|^{1/r} / N\alpha_N^{1/N}\right) \cdot \left(\int_{[0,1]} (-\mathrm{Log} t)^{r/N'} dt\right)^{1/r} ||\nabla u|_{*u}|_N$, where $N' = N/(N-1)$, [2] p.95.

(II) Boundary Value Problem.

The RA makes it possible to give estimates having various properties. The author of [2] applies them to the boundary value problems of PDEs relating to plasma physics.

(i) Acoordings to G.Talenti [3], when $-\Delta u = \mathrm{f} \in \boldsymbol{L}_+^2(\Omega)$, $u \in \mathrm{H}_0^1(\Omega)$ derives $-\Delta U = \mathrm{f}_\sim$, $U \in \mathrm{H}_0^1(\Omega_\sim)$ by the spherical RA, $\mathrm{u}_*(s) \leqq \mathrm{U}_*(s)$ holds for $\forall s \in \Omega_*$.

(ii) In [2] (Chapter 5), **the Neumann problem** $(\boldsymbol{P}_N)$ in $\Omega\left(\subset \boldsymbol{R}^N\right)$ is treated: Let $\mathrm{f} \in \boldsymbol{L}^{p'}(\Omega)$; $p' = p/(p-1)$.
$(\boldsymbol{P}_N)$: $u \in \mathrm{W}^{1,p}(\Omega)$, $-\mathrm{div}\,(\hat{\boldsymbol{a}}(x,u,\nabla u)) = f$ in Ω, and $\hat{\boldsymbol{a}}(\nabla u) \cdot \boldsymbol{n}(x) = 0$ on $\partial\Omega$.
$\boldsymbol{n}(x)$: normal unit vector directed to the outside of $\partial\Omega$ at x.

Coercive Condition: $\hat{\boldsymbol{a}}(x,u,\nabla u) \equiv \hat{\boldsymbol{a}}(\nabla u)$ (taking the values in $\boldsymbol{R}^N$) satisfies $\hat{\boldsymbol{a}}(x,\sigma,\xi)\cdot \xi \geqq \alpha|\xi|^p$ for $\forall \xi \in \boldsymbol{R}^N$; $p \in (1,\infty)$.

The corresponding **variational problem** $(\boldsymbol{P}_{vN})$: $\int_\Omega \hat{\boldsymbol{a}}(\nabla u) \cdot \nabla\phi dx = \int_\Omega f\phi dx$ for $\forall \phi \in \mathrm{W}^{1,p}(\Omega)$. $v = |u|$ derives $\mathrm{sign}(u) \cdot (v - v_*(s))_+ \in \mathrm{W}^{1,p}(\Omega)$. $\mathrm{F}(s) \equiv \int_{[0,s]} |\mathrm{f}|_{*v}(\sigma)\, d\sigma$.

Theorem 5.3.1. All variational solution $u \in \mathrm{W}^{1,p}(\Omega)$ of $(\boldsymbol{P}_{\mathrm{vN}})$ satisfies the inequalities **(a)**, **(b)**, and **(c)** for almost all s.

(a) $(\hat{\boldsymbol{a}}(\nabla u) \cdot \nabla u)_{*v}(s) \leqq -\mathrm{v}_*'(s)\,\mathrm{F}(s)$.

(b) $|\nabla \mathrm{v}|_{*v}(s) \leqq \left[(Q/\alpha) \cdot \max(s, |\Omega| - s)^{(1-N)/N} \cdot \mathrm{F}(s)\right]^{1/(p-1)}$.

(c) $-\mathrm{v}_*'(s) \leqq \left[\left(Q/\alpha^{1/p}\right) \cdot \max(s, |\Omega| - s)^{(1-N)/N}\right]^{p'} \cdot \mathrm{F}(s)^{1/(p-1)}$.

$\max(s, |\Omega| - s)^{(1-N)/N} := \max\left(s^{(1-N)/N}, (|\Omega| - s)^{(1-N)/N}\right)$, because of

Corollary 3.3.2. One poses $\mathrm{k}(s) = Q^{-1}\mathrm{Min}\left(s^{1-1/N}, (|\Omega| - s)^{1-1/N}\right)$. If $1 \leqq p \leqq \infty, u_* \in \mathrm{W}_{loc}^{1,p}(\Omega_*)$ for $\forall u \in \mathrm{W}^{1,p}(\Omega)$. Furthermore one has Polya-Szëgo inequalities $|\mathrm{k}(s) \cdot du_*/ds|_{\boldsymbol{L}^p(\Omega_*)} \leqq ||\nabla u|_{*u}|_{\boldsymbol{L}^p(\Omega_*)} \leqq |\nabla u|_{\boldsymbol{L}^p(\Omega)}$, [2] p.70.

Proof of the Theorem.

(a) The inequality is derived from $\int_{v > \mathrm{v}_*(s)} (\hat{\boldsymbol{a}}(\nabla u) \cdot \nabla u)\, dx = \int_\Omega \mathrm{f}(x) \cdot \mathrm{sign}(u) \cdot (v - \mathrm{v}_*(s))_+\, dx$.
"Let $\mathrm{G}(s) = \int_\Omega \mathrm{g}(x)(\mathrm{v}(x) - \mathrm{v}_*(s))_+\, dx$, $g \in \boldsymbol{L}^{p'}(\Omega)$.
$\mathrm{G}'(s) = -\mathrm{v}_*'(s) \int_{\{v > \mathrm{v}_*(s)\}} \mathrm{g}(x)\, dx$ holds." (cf. R-formula)

(b) $[\hat{\boldsymbol{a}}(\nabla u) \cdot \nabla u]_{*v}(s) \geqq \alpha[|\nabla \mathrm{v}|^p]_{*v}(s)$ is derived from the Coercive Condition. Next "The **Poincaré-Sobolev inequality** $-\mathrm{u}_*'(s) \leqq \mathrm{K}(s, \Omega, V) \cdot |\nabla u|_{*u}(s)$ with $V = \mathrm{W}^{1,p}(\Omega)$ [2] p.85" is used .

One has $(|\nabla v|^p)_{*v}(s) \geqq (|\nabla v|_{*v}(s))^p$, by using the Hörder inequality.

Höder inequality: $\left(F_1 \cdot F_2 \chi_{\Omega - \mathrm{P}(u)}\right)_{*u} \leqq \left[|F_1|^p \chi_{\Omega - \mathrm{P}(u)}\right]_{*u}^{1/p} \cdot \left[|F_2|^{p'} \chi_{\Omega - \mathrm{P}(u)}\right]_{*u}^{1/p'}$,
[1]p.39, [2] p.116.

(**c**) One must use the Poincaré-Sobolev inequality with $V = \mathrm{W}^{1,p}(\Omega)$ one more time.

(iii) Theorem 5.4.1. Let $V = \{v \in \mathrm{H}^1(\Omega),\ v = \text{constant on } \partial\Omega\}$. There exists a solution $u \in \mathrm{H}^2(\Omega) \cap V, \Omega \subset \boldsymbol{R}^2$, satisfying [**T**]: $-\Delta u + \lambda u_- = 0\,(\lambda > 0)$ in Ω, $u = \gamma =$ constant on $\partial\Omega = \Gamma$, $\int_{\partial\Omega} \nabla u \cdot \boldsymbol{n}(x)\,dl = I > 0$ (for given I). $u_- = u_+ - u$. (due to R.Sermange.)

Theorem 5.4.2. Suppose that λ_1: the first eigenvalue of the Dirichlet problem: $-\Delta\phi_1 = \lambda_1\phi_1$, $\phi_1 = 0$ on $\partial\Omega, \int_\Omega \phi_1 dx = \lambda_1^{-1}$.

(a) $\lambda > \lambda_1 \rightleftarrows \mathrm{u}|_\Gamma > 0$, **(b)** $\lambda = \lambda_1 \rightleftarrows \mathrm{u}|_\Gamma = 0$, **(c)** $\lambda < \lambda_1 \rightleftarrows \mathrm{u}|_\Gamma < 0$.

The Harnack inequality and the Green formula are used in the proof.

(III) Equiliblium Non-Local Equation.

Theorem 8.2.4. $\mathrm{b}_1(x), \mathrm{b}_2(x) \in \boldsymbol{L}^\infty(\Omega)$. There exists $u \in \mathrm{H}_0^1(\Omega) \cap \mathrm{W}^{2,p}(\Omega)$, $p \in [1,\infty)$ such that $-\Delta u = \mathrm{F}\left(x; [b_1\Phi_1(\nabla u)]_{*u}(|u > \mathrm{u}(\cdot)|), u_*'(|u > \mathrm{u}(\cdot)|), u_*', [b_2\Phi_2(\nabla u)]_{*u}\right)$.

Here $|\Phi_j(\xi)| \leqq c_1|\xi|$, for $\forall\xi \in \boldsymbol{R}^N, j = 1, 2$.

$\mathrm{F}(x; X) : \Omega \times \boldsymbol{L}^1(\Omega)^2 \times \boldsymbol{L}^1(\Omega_*)^2 \to (\varepsilon, \infty)$, $\varepsilon > 0$, bounded.

$x \in \Omega \to \mathrm{F}(x; X)$ measurable, $\mathrm{F}(x; X_n) \to \mathrm{F}(x; X)$ as $X_n \to X$.

$(\mathrm{G}(v), \phi) = \int_\Omega \phi(x)\,\mathrm{F}(x; \mathrm{X}(v))\,dx$, for $\forall\phi \in \boldsymbol{L}^2(\Omega)$. $\mathrm{G}(v) \in \boldsymbol{C}_\varepsilon = \{f \in \boldsymbol{L}^2(\Omega); f \geqq \varepsilon\}$ is used in the Proof of this Theorem 8.2.4.

Let $0 < c_3 < \inf\{\int_\Omega |\nabla\phi|^2 dx; |\phi|_2 = 1, \phi \in \mathrm{H}_0^1(\Omega)\}$, $\mathrm{p}(t) \in \boldsymbol{C}^1(\boldsymbol{R}), \mathrm{p}'(t) \geqq 0$, and $|p'(t)| \leqq c_3|t| + c_4$. Let $\mathrm{a}(x) \in \boldsymbol{L}^\infty(\Omega)$, $\mathrm{b}(x) \geqq 0$, and $v(x) \in \mathrm{W}^{1,q}(\Omega)$ for $q > N$ and $\Omega\left(\subseteq \boldsymbol{R}^N\right)$. $\mathrm{F}(v)(x)$ is given as $\mathrm{F}(v)(x) = \mathrm{a}(x)\left[F_0^2 - \int_{\mathrm{I}_v(x)} \left([\mathrm{p}(v_*)]' b_{*v}\right)(\sigma)\,d\sigma\right]_+^{1/2}$, where $\mathrm{I}_v(x) = [\mathrm{m}_v(0), \mathrm{m}_v(v_+(x))]$.

Theorem 8.2.5. Let N=2, and $\Phi_\varepsilon(\xi) = |\xi|/(\varepsilon + |\xi|)$, $\varepsilon > 0$.
There exists a solution $u \in \mathrm{W}^{2,p}(\Omega) \cap \mathrm{H}_0^1(\Omega)$, $p > 1$, of the equation $-\Delta \mathrm{u}(x) = \mathrm{F}(u)(x) + \mathrm{p}'(u)(x)\left[\mathrm{b}(x) - [b\Phi_\varepsilon(\nabla\ u)]_{*u}(|u > \mathrm{u}(x)|)\right]$.

3 Chemotaxis System.

Let $\mathrm{u}(t,x) : Q_T \equiv (0,T) \times \Omega \to \boldsymbol{R}$ measurable, and $\mathrm{u}_*(t,s) = \mathrm{u}(t)_*(s)$. Let $b, u \in \boldsymbol{L}^1(Q_T)$, and $\mathrm{b}_{*u}(t,s) = \mathrm{b}(t)_{*\mathrm{u}(t)}(s)$. In [2] (Chapter 9), **the Chemotaxis system (Ch)** is studied by using the RA.

(**Ch**): $\partial_t u = \mathrm{div}(\nabla u - \bar{\chi} u \nabla v)$ and $0 = \Delta v - \gamma v + \alpha u$ in Q_T, with $u(0,x) = u_0 \geqq 0$ in $\Omega \subset \boldsymbol{R}^N$. $\bar{\chi}, \gamma, \alpha$: positive constants.

Theorem 9.5.1. Suppose that $u_0 \in \mathrm{W}_0^{1,p}(\Omega)$, $p > N$. There exist a time $T_{\max} > 0$, and an unique solution (u,v) of the (**Ch**), satisfying **(1)** and **(2)**.

(1) $u > 0, v > 0$, in $(0, T_{\max}) \times \Omega$,

(2) $u \in \boldsymbol{C}\left([0,T_{\max}); \mathrm{W}_0^{1,p}(\Omega)\right) \cap \boldsymbol{C}^1([0,T_{\max}); \mathrm{L}^p(\Omega)), \mathrm{u}(t) \in \mathrm{W}^{2,p}(\Omega)$ for $0 < t < T_{\max}$, and $v \in \boldsymbol{C}\left((0,T_{\max}); \mathrm{W}^{2,p}(\Omega) \cap \mathrm{W}_0^{1,p}(\Omega)\right)$.

This Theorem is not proved in [2]. Next let $Q_{T*} \equiv (0,T) \times \Omega_*$.

Comparison Theorem Two functions f and g are defined on Q_{T*}.

(i) $f, g \in \boldsymbol{L}^\infty(Q_{T*}) \cap \mathrm{H}^1\left(0,T; \boldsymbol{L}^2(\Omega_*)\right) \cap \left(\cap_{\delta>0} \boldsymbol{L}^2\left(0,T; \mathrm{W}^{2,2}(\delta, |\Omega|)\right)\right)$.

(ii) $|\partial \mathrm{f}(t,s)/\partial s| \leqq \mathrm{c}(t)$ and $|\partial \mathrm{g}(t,s)/\partial s| \leqq c(t)\max\left(\mathrm{s}^{-l},1\right)$, where $0 \leqq l < 1, \mathrm{c}(t) \in \boldsymbol{L}^2(0,T)$

Suppose that f and g satisfy the system: $\partial_t f - \left(N\alpha_N^{1/N}\right)^2 s^{2-2/N}\partial^2 f/\partial s^2 - \alpha\bar{\chi} f\partial f/\partial s \leqq \partial_t g - \left(N\alpha_N^{1/N}\right)^2 s^{2-2/N}\partial^2 g/\partial s^2 - \alpha\bar{\chi} g\partial g/\partial s$,
$0 = \mathrm{f}(t,0) \leqq \mathrm{g}(t,0)$ and $\partial \mathrm{f}(t,|\Omega|))/\partial s \leqq \partial g(t,|\Omega|)/\partial s$ for $\forall t \in [0,T]$. $\mathrm{f}(0,s) \leqq \mathrm{g}(0,s)$ for $s \in \Omega_*$, and $\mathrm{g}(t,s) \geqq 0$.

Then $f \leqq g$ holds in Q_{T*}.

Proof. $w = f - g$, then $w_+ = \mathrm{Max}(w,0) = 0$ and $f \leqq g$.

Theorem 9.5.2. $T_{\max} = \infty$ holds in the Case **(1)** $\sim$ **(3)**.
(1) $N = 1$.
(2) $N = 2, \alpha\bar{\chi}|u_0|_1 < 8\pi$.
(3) $N \geqq 3, \alpha\bar{\chi}|u_0|_{L_N} < N\alpha_N^{2/N}|\Omega|^{1/N}$.

Theorem 9.5.2. is proved in [2] p.240 by using the above Comparison Theorem.

Proof $\mathrm{f}(t,s) = \mathrm{k}(k,s) \equiv \int_{[0,s]} u_*(t,\sigma)\,d\sigma$ is used as f. Then $\partial_t f - \left(N\alpha_N^{1/N}\right)^2 s^{2-2/N}\partial^2 f/\partial s^2 - \alpha\bar{\chi}\partial f/\partial s \leqq 0$ holds in p.p. $Q_{T\max *} = (0,T_{\max}) \times \Omega_*$. $\mathrm{g}(t,s) = pe^{-\lambda t}\tanh(\alpha\bar{\chi}\mathrm{ps}/4) := \mathrm{pe}^{-\lambda t}\mathrm{h}_p(s)$ for $N = 1, \mathrm{g}(t,s) = e^{-\lambda t}\mathrm{aqs}/(1+\mathrm{qs})$ for $N = 2$, and $\mathrm{g}(t,s) = e^{-\lambda t}qs^{1-1/N}, q = (N/\beta)\alpha_N^{2/N}|\Omega|^{-1/N}$ for $N \geqq 3$ are used.

Then $|\mathrm{u}(t)|_{\mathrm{L}^\infty(\Omega)} \leqq Ce^{-\lambda t}$ is derived.

Conclusion and Postscript. We investigate the relation between the RA and A-integral (by A.N.Kolmogorov etc.) and derive the following **(i)** $\sim$ **(iii)**:
(i) $\int_\Omega \mathrm{u}(x)\,dx = \int_{\Omega*} \mathrm{u}_*(s)\,ds$, holds for step function u taking finite values and defined on a bounded set Ω.
(ii) $v_{u*} = v_*$ holds under the conditions such that u is step function, $v \in \boldsymbol{L}^1(\Omega)$, and $\mathrm{u}(x_1) \leqq \mathrm{u}(x_2) \rightleftarrows \mathrm{v}(x_1) \leqq \mathrm{v}(x_2)$.
(iii) u in v_{*u} gives a measure preserving transform in Ω_*.
In plasma physics we find the equation $\partial_t \mathrm{H}(u) - \Delta u = \mathrm{G}(t,x,u_*'(t,\cdot),\mathrm{b}_{*u}(t,\cdot))$,
$\mathrm{H}(u)$: a monotone function. If $\mathrm{u}(t,\cdot)$ is stationary, it becomes the solution of the equation $-\Delta u = \mathrm{G}(x,u_*',b_{*u})$.
Stellarator(USA) and Tokomak(USSR) are machines, which aim to realize the nuclear fusion. The Chemotaxis system is the one appearing in the biological classification related to Chemistry.

Acknowledgement. On 9.January '09, I received the post card from Mrs.R.Ishihara, which tells the death of Prof.T.Ishihara. When I received it, I was typing the manuscript of the review of the work [2]. I have a feeling that the late Prof. asks me to write and show the outline of the contents of the work [2]. Then I would like to dedicate this work to him.

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