N PERSON SILENT GAMES ON SALE IN WHICH THE PRICE IS A UNIMODAL FUNCTION WITH TIME

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Abstract. We consider a class of games which is suggested from the timing problems for putting some kind of farm products on the market. N players take possession of the rights to put some kind of farm products on the market with even ratio. Each of the n players can put the farm products at any time in $[0,1]$. The price of them increases over $[0,m] \subset [0,1]$ and decreases over $(m,1]$ with pass time t as long as none of the n players takes his action, however if one of the n players put his farm products on the market the price falls discontinuously and then fluctuates analogously as before. All players have to put their farm products on the market within the unit interval $[0,1]$. In such a situation, each player wishes to put at the optimal time which gives him the highest price, considering opponents' action time with each other. This model yields us a certain class of n person non-zero sum infinite games.

1. Introduction We consider a class of games which is suggested from the correlative phenomena between the price fluctuations and supply in a market on farm products. N players, Player $1,\cdots,n$ take possession of the right to put some kind of farm products on the market with even ratio. We call such kind of products product A in this paper. We can harvest product A at a specific season every year periodically. Each of the n players wants to decide the optimal time to put his product A on the market until the next harvest season. We consider one time period where the harvest time in each year is the beginning and the next harvest time is the end. The price of product A increases smoothly until some point and then decreases with time as long as none of the n players puts on the market and keeps his own product. But, when one of the n players puts his product A on the market, the price of product A possessed by other players falls discontinuously and then fluctuates with time analogously as before until all of n players put the rest on the market. In such a situation, each player has to decide the optimal action time considering the current price and his opponent's action time, with each other.

This problem is applicable to the correlation phenomena between the price and supply on land, not only to the problem of farm products. As well as the usual games of timing [1, 2], we have to introduce two patterns of information available to the players. If a player is informed of his opponent's action time as soon as his opponent put product A on the market, we say they are in a noisy version. If neither player learns when nor whether his opponent has put product A on the market, we say they are in a silent version. We consider the case that all of the n players are in a silent version, that is, n person silent game, in this paper. Related to our models, there are two works on the competition for a territory [3, 4]. Our previous work deals with two person games for our model in this paper, which include not only silent version also noisy version [5]. In the noisy version, we have to employ ϵ-equilibrium mixed strategy but not Nash equilibrium point.

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2. Notations and Assumptions Since we consider one period game, we express the period as the unit interval $[0,1]$. Throughout of this paper, we use the following notations:

$\nu(t)$ is the price of product A at time $t \in [0,1]$, when none of the n players put their product A in their market. We assume that $\nu(t)$ is differentiable and

$$\nu'(t) \left\{ \begin{array}{c} \geq \\ < \end{array} \right\} 0 \text{ for } \left\{ \begin{array}{c} 0 \leq t \leq m \\ m < t \leq 1 \end{array} \right\}.$$

r is the discount factor after one of the n players has already put product A on the market and is assumed $0 < r < 1$. That is, if one of the n players sells his product A at time $t \in [0,1]$, the price of his opponents' falls down from $\nu(t)$ to $r\nu(t)$ immediately. In this situation, it is natural to assume $0 < \nu(0) < \infty$.

Here we also assume the following. If k of the n players put their product A at a same time $t \in [0,1]$ and the just before price is $\hat{\nu}(t)$, each of the k players has to sell his product A at the price after fall, that is $r^k \hat{\nu}(t)$.

Throughout this paper, we use notations on the expectation for real valued function $M_i(x_1, \cdots, x_n)$ defined on $[0,1] \times \cdots \times [0,1]$ when Player i employs his mixed strategy ($cdfs$) $F_i(x)$ on $[0,1]$ as follows:

$$M_i(F_1, \cdots, F_n) = \int_0^1 \cdots \int_0^1 M_i(x_1, \cdots, x_n) dF_1(x_1) \cdots dF_n(x_n)$$

and

$$M_1(x_1, F_2, \cdots, F_n) = \int_0^1 \cdots \int_0^1 M_1(x_1, \cdots, x_n) dF_2(x_2) \cdots dF_n(x_n)$$

3. The Formulation and The Main Results We assume that all of the n players are silent players in our model, that is, each player is able to put his product A in the market secretly, then none of the other $n-1$ players learn the exact current price $\hat{\nu}(t)$ at time t and is informed of the exact current price immediately after he has sold his product A. In such a situation, it is natural to define the pure strategy of Player i as $x_i \in [0,1]$. And we find all of the n players are in the same situation. Therefore, it is not necessary for each player to know which players have already put their products before his planed time, but enough to learn the number of players that have already acted till that time. However, each of the n players cannot learn the number, since they are silent players. Hence we evaluate the expected payoff to Player 1 when Player 1 takes his pure strategy $x_1 \in [0,1]$. We let $y_{(j)}$ denote the jth smallest of pure strategies taken by other $n-1$ players, $x_2, \cdots x_n$. That is $y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n-1)}$. Then the expected payoff to Player 1 $M_1(x_1, \cdots, x_n)$ is given as follows:

$$M_1(x_1, \cdots, x_n) = \begin{cases} \nu(x_1), & 0 \leq x_1 < y_{(1)} \\ r\nu(x_1), & y_{(1)} \leq x_1 < y_{(2)} \\ \cdots, & \cdots \\ r^{n-1}\nu(x_1), & y_{(n-1)} \leq x_1 \leq 1. \end{cases} \tag{1}$$

Observing the above expected payoff function and $\nu(x_1)$ is an unimodal function with respect to x_1 and has the maximum value at $x_1 = m \in [0,1]$, we consider the following mixed strategy (cdf) $F(x)$: We choose some point a in the interval $[0,m]$ and then define as

$$F(x) = \begin{cases} 0, & 0 \leq x < a \\ \int_0^x f(t)dt, & a \leq x < m \\ 1, & m \leq x \leq 1. \end{cases} \tag{2}$$

That is, we suppose $F(x)$ is a *cdf* which has *pdf* $f(x) > 0$ only over the interval (a, m). Therefore, if Player 1 uses pure the strategy x and each of other $n-1$ players chooses the mixed strategy $F(x_i)$ given by equation (2), we have the expected payoff to Player 1 $M_1(x, F, \cdots, F)$ as follows:

$$M_1(x,F,\cdots,F) = \begin{cases} \nu(x), & 0 \le x < a \\ \nu(x)[\sum_{k=0}^{n-1} \binom{n-1}{k} \{rF(x)\}^k \{1-F(x)\}^{n-k-1}], & a \le x < m \\ r^{n-1}\nu(x), & m \le x \le 1. \end{cases}$$

which leads to

$$M_1(x,F,\cdots,F) = \begin{cases} \nu(x), & 0 \le x < a \\ \nu(x)[1-(1-r)F(x)]^{n-1}, & a \le x < m \\ r^{n-1}\nu(x), & m \le x \le 1. \end{cases} \tag{3}$$

Putting

$$M_1(x, F, \cdots, F) = \text{const for } x \in (a, m)$$

we obtain

$$\nu(x)[1-(1-r)F(x)] = (n-1)(1-r)f(x)\nu(x) > 0, \quad a < x < m.$$

Hence we get

$$F(x) = \{1/(1-r)\}[1 - \{c/\nu(x)\}^{1/(n-1)}\}], \quad a < x < m \tag{4}$$

where c is an integration constant. The boundary value conditions

$$F(a) = 0 \quad \text{and} \quad F(m) = 1$$

give

$$c = r^{n-1}\nu(m) \;;\; \nu(a) = r^{n-1}\nu(m),$$

however, these conditions are satisfied only when the inequality $\nu(0) \le r^{n-1}\nu(m)$ holds.

Therefore, we consider the case where $\nu(0) \le r^{n-1}\nu(m)$ first. Then there exist the unique root a which satisfies equation $\nu(a) = r^{n-1}\nu(m)$ in the interval $[0, m]$. Thus we denote the unique root by a^0. So we have

$$\nu(a^0) = r^{n-1}\nu(m), \quad a^0 \le x \le m$$

and then

$$M_1(x,F,\cdots,F) = \begin{cases} \nu(x) < \nu(a^0) = r^{n-1}\nu(m), & 0 \le x < a^0 \\ \nu(a^0) = r^{n-1}\nu(m), & a^0 \le x \le m \\ r^{n-1}\nu(x) < r^{n-1}\nu(m), & m \le x \le 1. \end{cases} \tag{5}$$

After all, we obtain Theorem 1.

Theorem 1. Assume that $\nu(0) \le r^{n-1}\nu(m)$ and then let a^0 be the unique root of equation $\nu(a) = r^{n-1}\nu(m)$ in the interval $[0, m]$. We consider the mixed strategy given by the following *cdf* :

$$F^0(x) = \begin{cases} 0, & 0 \le x < a \\ \{1/(1-r)\}[1-\{\nu(a^0)/\nu(x)\}^{1/(n-1)}], & a \le x < m \\ 1, & m \le x \le 1 \end{cases}$$

Then, the n-tuple of mixed strategies $(F^0, \cdots, F^0)$ is a Nash equilibrium point of n-person non-zero sum game given by (1). And the corresponding expected payoff ν_i to Player i is given as follows:

$$\nu_i = M_i(F^0, \cdots, F^0) = r^{n-1}\nu(m), \quad i = 1, \cdots, n.$$

According to Theorem 1, each of the n players is forced to concentrate his all probability for putting his product A in the market over the interval where the price of product A is increasing if all players employ Nash equilibrium strategy.

Here, we consider the case where $\nu(0) > r^{n-1}\nu(m)$. As we assumed if k of the n players have acted at a same time $t \in [0, 1]$ each of the k players gets the fallen price $r^k\hat{\nu}(t)$ but not the price $\hat{\nu}(t)$ immediate before the action time, we have

$$M_1(x, 0, \cdots, 0) = r^{n-1}\nu(x) \leq r^{n-1}\nu(m) < \nu(0), \quad 0 \leq x \leq m.$$

We also have

$$M_1(0, x_2, \cdots, x_n) = \begin{cases} \nu(0) > r^{n-1}\nu(m), \\ \\ r^k\nu(0), \end{cases} \quad \text{for} \quad \begin{cases} 0 < y_{(1)} \leq \cdots \leq y_{(n-1)} \leq 1 \\ \\ 0 = y_{(k-1)} < y_{(k)} \leq \cdots \leq y_{(n-1)} \leq 1. \end{cases}$$

Hence, all players want to put their product A in the market at time sufficiently close to time 0, however they intend to avoid the same time as other players with each other.

Observing

$$F(x) = \begin{cases} \{1/(1-r)\}[1 - \{\nu(0)/\nu(x)\}^{1/(n-1)}], & 0 \leq x < m \\ 1, & m \leq x \leq 1, \end{cases}$$

we find

$$F(0) = (1/(1-r))[1 - \{\nu(0)/\nu(x)\}^{1/(n-1)}] = 0$$

and

$$\begin{aligned} F(m) &= \{1/(1-r)\}[1 - \{\nu(0)/\nu(m)\}^{1/(n-1)}] \\ &< \{1/(1-r)\}[1 - \{r^{n-1}\nu(m)/\nu(m)\}^{1/(n-1)}] = 1. \end{aligned}$$

Thus putting

$$\alpha = 1 - \{1/(1-r)\}[1 - \{\nu(0)/\nu(m)\}^{1/(n-1)}] \geq 0 \tag{6}$$

and considering the following *cdf* given by

$$F(x) = \begin{cases} \{1/(1-r)\}[1 - \{\nu(0)/\nu(x)\}^{1/(n-1)}], & 0 \leq x < m \\ F(m) + \alpha = 1, & m \leq x \leq 1 \end{cases}$$

we also obtain

$$M_1(x, F, \cdots, F) = \begin{cases} \nu(0) > r^{n-1}\nu(m), & 0 \leq x < m \\ \nu(a) = r^{n-1}\nu(m) < \nu(0), & x = m \\ r^{n-1}\nu(x) < r^{n-1}\nu(m) < \nu(0), & m < x \leq 1. \end{cases} \tag{7}$$

Hence we define *cdf* $F^*(x)$ as

$$F^*(x) = \begin{cases} \{1/(1-\alpha)\}\{1/(1-r)\}[1 - \{\nu(0)/\nu(x)\}^{1/(n-1)}], & 0 \leq x < m \\ 1, & m \leq x \leq 1 \end{cases} \tag{8}$$

Since $F^*(x)$ satisfies

$$F^*(0) = 0 \quad : \quad F^*(m) = 1 \tag{9}$$

and

$$\begin{aligned} \nu(x)\,&[1-(1-r)F^*(x)]^{n-1} \\ &= \nu(x)\{1/(1-\alpha)\}[\{\nu(0)/\nu(x)\}^{1/(n-1)}]^{n-1} \\ &= \{1/(1-\alpha)\}\nu(0), \qquad 0 \le x < m, \end{aligned} \tag{10}$$

we have the following relations:

$$M_1(x, F^*, \cdots, F^*) = \begin{cases} \{1/(1-\alpha)\}\nu(0), & 0 \le x < m \\ r^{n-1}\nu(m), & x = m \\ r^{n-1}\nu(x) < r^{n-1}\nu(m) < \nu(0) & m < x \le 1. \end{cases} \tag{11}$$

After all we get Theorem 2.

Theorem 2. Assume that $\nu(0) > r^{n-1}\nu(m)$. And then we put α as

$$\alpha = 1 - \{1/(1-r)\}[1 - \{\nu(0)/\nu(m)\}^{1/(n-1)}] > 0.$$

Consider the following *cdf* $F^*(x)$ defined by

$$F^*(x) = \begin{cases} \{1/(1-\alpha)\}\{1/(1-r)\}[1 - \{\nu(0)/\nu(x)\}^{1/(n-1)}], & 0 \le x < m \\ 1, & m \le x \le 1. \end{cases}$$

Then, the n-tuple of mixed strategies $(F^*, \cdots, F^*)$ is a Nash equilibrium point of n-person non-zero sum game given by (1). And the corresponding expected payoff ν_i to Player i is given as follows:

$$\nu_i = M_i(F^*, \cdots, F^*) = \{1/(1-\alpha)\}\nu(0), \quad i = 1, \cdots, n.$$

Theorem 2 gives us the following suggestion. It is natural to suppose that $\nu(0) > r^{n-1}\nu(m)$ when the price of product A is not so high at time m or many players participate in this game. In such situations, Nash equilibrium strategy forces each of the all participants to concentrate the chances for his action on the interval $(0, m)$ by distributing his probability in order to be proportional to the density over this narrow interval.

Furthermore, Theorem 1 and Theorem 2 teach us an interesting feature on the Nash equilibrium for our model. The corresponding equilibrium mixed strategy (*cdf*) of each player consists only of the density function on interval (a, m) where $\nu(x)$ is increasing in $[0, 1]$, while has no probability over the interval $[m, 1]$ where $\nu(x)$ is decreasing in $[0, 1]$, whatever the concrete shape of the price function $\nu(x)$ may be.

4. The Concluding Remarks We assumed that the discount factor r is constant over the all interval $[0, 1]$ in our model, However, it may be natural to assume that r verifies depending on the pass time t in $[0, 1]$ and the value of the price function $\nu(x)$, in the real world. We have formulated and analyzed the silent version in this paper. On the other hand it is important to deal with the noisy version from the view point of the real timing problem on sale. Since each player can learn which player has already acted at any time t in $[0, 1]$, we have to formulate our problem from the view point of Dynamic Programming.

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