ON THE DIEUDONNÉ THEOREM

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Abstract. We obtain for modular measures on lattice ordered effect algebras the classical theorem of Dieudonné related to convergent sequences of regular maps.

1 Introduction In 1933 Nikodým [10] proved the well-known Vitali-Hahn-Saks theorem, namely "If a sequence of Borel measures converges pointwise to a map μ, then μ is a Borel measure."

In 1951 Dieudonné proved the following more general theorem: "If a sequence of regular measures defined on Borel sets of a compact metrizable space converges on every open set, then it converges on every Borel sets. In this case, the sequence is uniformly regular". This theorem generalizes Nikodým' s theorem if one substitutes the pointwise convergence on the Borel σ-algebra for the analogous condition on open sets provided a regularity assumption and a topological condition on the space are satisfied. Brooks in [6] generalizes this theorem to the case the space is either compact or the space is normal and the sequence is uniformly bounded. In this note we furnish a general version of Dieudonné's theorem valid for real-valued modular measures defined on lattice ordered effect algebras. For we use an abstract concept of regularity (see Definition 4.1) where $\mathcal{F}$ and $\mathcal{G}$ play the role of compact sets and open sets, respectively.

The paper, which follows Candeloro and Letta's work, is organized as follows.

After notation and preliminaries we give the exhaustivity condition, the regularity condition and their relationship. Finally, we provide the main result (Theorem 5.1).

We deal with effect algebras. Effect algebras have been introduced by Foulis and Bennett in 1994 [5] for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in quantum physics and in Mathematical Economics, in particular of orthomodular lattices in noncommutative measure theory and MV-algebras in fuzzy measure theory.

2 Preliminaries In this section we shall give some basic definitions and fix some notations.

Definition 2.1 Let $(L, \leq)$ be a poset with a smallest element 0 and a greatest element 1 and let $\ominus$ be a partial operation on L such that $b \ominus a$ is defined if and only if $a \leq b$ and for all a, b, $c \in L$:

If $a \leq b$ then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.

If $a \leq b \leq c$ then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

Then $(L, \leq, \ominus)$ is called a *difference poset (D-poset* for short), or a *difference lattice (D-lattice* for short) if L is a lattice.

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For the rest, let L be a D-lattice.

One defines in L a partial operation $\oplus$ as follows:

$a \oplus b$ is defined and $a \oplus b = c$ if and only if $c \ominus b$ is defined and $c \ominus b = a$.

The operation $\oplus$ is well-defined by the cancellation law [9, page 13] ($a \leq b, c$ and $b \ominus a = c \ominus a$ implies $b = c$), and $(L, \oplus, 0, 1)$ is an effect algebra (see [9, Theorem 1.3.4]), that is the following conditions are satisfied for all $a, b, c \in L$:
If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;
if $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;
there exists a unique $a^\perp \in E$ such that $a \oplus a^\perp$ is defined and $a \oplus a^\perp = 1$;
if $a \oplus 1$ is defined, then $a = 0$.

We say that a and b are *orthogonal* if $a \leq b^\perp$ and we write $a \perp b$. Therefore $a \oplus b$ is defined if and only if $a \perp b$, and in this case $a \oplus b = (a^\perp \ominus b)^\perp$ by [9, Lemma 1.2.5]. If $a_1, \ldots, a_n \in L$ we inductively define $a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ if the right-hand side exists. The sum is independent on any permutation of the elements. We say that a finite family $(a_i)_{i=1}^n$ of (not necessarily different) elements of L is *orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists.

We say that a sequence (a_n) of L is orthogonal if the set $\{a_1, \ldots a_n\}$ is orthogonal, for every $n \in \mathbb{N}$.

A function μ on a D-lattice with values in $\mathbb{R}$ is called a *measure* if for every $a, b \in L$, with $a \perp b$,

$$\mu(a \oplus b) = \mu(a) + \mu(b).$$

A *modular measure* is a measure which also satisfies the modular law, that is for all $a, b \in L$

$$\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b).$$

In the whole paper, we denote by μ a real-valued modular measure on L.

Definition 2.2 [4, 3.2] The total variation $|\mu| : L \to [0, +\infty]$ of μ is defined by

$$|\mu|(a) := \sup\{\sum_{i=1}^{n} |\mu(x_i) - \mu(x_{i-1})| : n \in \mathbb{N}, 0 = x_0 \leq x_1 \leq \cdots \leq x_n = a\}$$

The total variation of a modular measure is a modular measure, too (cf. [4, 3.3]).

Notation 2.3 *For $a \in L$, we put $\tilde{\mu}(a) := \sup\{|\mu(b)| : b \in L, b \leq a\}$.*

Remark 2.4 Observe that, thanks to the representation of the total variation as

$$|\mu|(a) = \sup\{\sum_{i=1}^{n} |\mu(a_i)| : n \in \mathbb{N} \oplus_{i=1}^{n} a_i = a\}$$

(cf. [4, 3.3]) and the Hahn decomposition theorem (cf. [3, 2.5]), we get

$$\tilde{\mu}(a) \leq |\mu|(a) \leq 2\tilde{\mu}(a).$$

For the rest of the paper let $\mathcal{F}, \mathcal{G} \subseteq L$ be two lattices closed under the sum and $g \ominus f \in \mathcal{G}$ for each $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

3 Exhaustivity

Definition 3.1 A modular measure μ is called $\mathcal{G}$-exhaustive if for every orthogonal sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ we have $\lim_n \mu(g_n) = 0$. A sequence $(\mu_i)_{i\in\mathbb{N}}$ is called uniformly $\mathcal{G}$-exhaustive if for every orthogonal sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{G}$ we have $\lim_n \mu_i(g_n) = 0$ uniformly with respect to $i \in \mathbb{N}$. If $\mathcal{G} = L$ we use the terms exhaustive and uniform exhaustive.

Remark 3.2 [1, 2.2] A measure on L is exhaustive if and only if every monotone sequence is a Cauchy sequence.

Remark 3.3 A real-valued modular function is exhaustive if and ony if so is its total variation.

Proof. Apply [11, 3.6] and [13, 1.3.11].QED

4 Regularity and exhaustivity

Definition 4.1 A modular measure μ is regular if it satisfies the following properties:
(i) For every $a \in L$ there exists a couple of sequences $g_n \downarrow$ in $\mathcal{G}$ and $f_n \uparrow$ in $\mathcal{F}$ with

$$f_n < f_{n+1} < a < g_{n+1} < g_n \qquad (1)$$

for every $n \in \mathbb{N}$ and

$$\lim_n \tilde{\mu}(g_n \ominus f_n) = 0 \qquad (2)$$

(ii) For every $b \in \mathcal{F}$, there exists a couple of sequences $g_n \downarrow$ in $\mathcal{G}$ and $f_n \downarrow$ in $\mathcal{F}$ with

$$b < f_{n+1} < g_n < f_n \qquad (3)$$

for every $n \in \mathbb{N}$ and

$$\lim_n \tilde{\mu}(g_n \ominus b) = 0 \qquad (4).$$

Let (μ_i) be a sequence of real-valued modular measures. Then it is uniformly regular if it satisfies the following properties:
(i) For every $a \in L$ there exists a couple of sequences $g_n \downarrow$ in $\mathcal{G}$ and $f_n \uparrow$ in $\mathcal{F}$ with

$$f_n < f_{n+1} < a < g_{n+1} < g_n$$

for each $n \in \mathbb{N}$ and

$$\lim_n \sup_i \tilde{\mu}_i(g_n \ominus f_n) = 0 \qquad (5)$$

(ii) For every $b \in \mathcal{F}$, there exists a couple of sequences $g_n \downarrow$ in $\mathcal{G}$ and $f_n \downarrow$ in $\mathcal{F}$ with

$$b < f_{n+1} < g_n < f_n$$

for each $n \in \mathbb{N}$ and

$$\lim_n \sup_i \tilde{\mu}_i(g_n \ominus b) = 0 \quad (6)$$

Theorem 4.2 *Suppose that $\mathcal{G}$ is closed under countable sums. Let (μ_i) be a sequence of pointwise convergent $\mathcal{G}$-exhaustive real-valued modular measures. Then (μ_i) is uniformly $\mathcal{G}$-exhaustive.*

Proof. By way of contradiction there exists $\varepsilon > 0$, a strictly increasing sequence $i_n \in \mathbb{N}$ and an orthogonal sequence $(g_n) \subseteq \mathcal{G}$ such that $|\mu_{i_n}(g_n)| \geq \varepsilon$ for each $n \in \mathbb{N}$. Define $\nu_n(A) = \mu_{i_n}(\oplus_{h\in A} g_h)$. With the aid of [2, 2.5], one can check that they form a sequence of finitely additive measures on the power set of $\mathbb{N}$.

We can apply the classical Vitali-Hahn-Saks theorem. So these restrictions form a uniformly exhaustive sequence. This contradicts the assumptions and completes the proof. QED

Lemma 4.3 *Suppose that (μ_i) is a sequence of regular uniformly $\mathcal{G}$-exhaustive real-valued modular measures. Let $b \in \mathcal{F}$ such that (3) and (4) hold for μ_i ($i \in \mathbb{N}$). Then (6) holds.*

Proof. Let $a \in L$ with $a \leq g_n \ominus b$.

Then $|\mu_i|(a) = \lim_m |\mu_i|(a \ominus ((f_m \ominus b) \wedge a))$. Indeed: For every $m > n$

$$|\mu_i|(a) - |\mu_i|(a \ominus ((f_m \ominus b) \wedge a)) = |\mu_i|((f_m \ominus b) \wedge a) \leq |\mu_i|(g_{m-1} \ominus b) \leq 2\tilde{\mu}_i(g_{m-1} \ominus b).$$

It suffices to show that $\lim_n \sup_i \tilde{\lambda}_i(g_n \ominus b) = 0$, where λ_i is the restriction of μ_i to $\mathcal{G}$.

By way of contradiction, there exists $\varepsilon > 0$ such that:

For every p, there exists $n > p$, $i \in \mathbb{N}$ and $a \in \mathcal{G}$ such that $a \leq g_n \ominus b$, $|\mu_i|(a) > \varepsilon$ and so $|\mu_i|(a \ominus ((f_m \ominus b) \wedge a)) > \varepsilon$ for a sufficiently large m. So, we can construct by induction four sequences (n_k), (i_k), (m_k) and $(a_k) \in \mathcal{G}$ with $a_k \leq g_{n_k} \ominus b \leq g_{n_{k-1}} \ominus b$ and $|\mu_i|(a_k \ominus ((f_{m_k} \ominus b) \wedge a_k)) > \varepsilon$. Since

$$g_{n_{k-1}} \ominus g_{m_k} = (g_{n_{k-1}} \ominus b) \ominus (g_{m_k} \ominus b) \geq ((f_{m_k} \ominus b) \vee a_k) \ominus (f_{m_k} \ominus b),$$

we have $|\mu_{i_k}|(g_{n_{k-1}}) - |\mu_{i_k}|(g_{m_k}) \geq |\mu_{i_k}|(a_k \ominus ((f_{m_k} \ominus b) \vee a_k)) - |\mu_{i_k}|(f_{m_k} \ominus b) = |\mu_i|(a_k \ominus ((f_{m_k} \ominus b) \wedge a_k)) > \varepsilon$, with $n_{k-1} < n_k < m_k$. Being g_k a monotone sequence which is not a Cauchy sequence, this contradicts the uniform $\mathcal{G}$-exhaustivity of the sequence (μ_i).QED

Applying Lemma 4.3 for $\mathcal{F} = \mathcal{G} = L$ and $b = 0$ we get:

Corollary 4.4 *Let (μ_i) be a sequence of uniformly exhaustive real-valued modular measures and let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence of elements of L such that $\lim_n \tilde{\mu}_i(a_n) = 0$ for each $i \in \mathbb{N}$. Then the limit is 0 uniformly with respect to $i \in \mathbb{N}$.*

Theorem 4.5 *Let μ_i be a sequence of regular uniformly $\mathcal{G}$-exhaustive real-valued modular measures. Then it is uniformly exhaustive and uniformly regular.*

Proof. We have to show that μ_i is uniformly regular. The last item defining the uniform regularity is fulfilled by Lemma 4.3. For the first: Let $a \in L$ and f_n, g_n satisfying (1) and (2). To prove (5), apply Corollary 4.4 to $a_n := g_n \ominus f_n$.

We continue proving that $(\mu_i)_{i\in\mathbb{N}}$ is uniformly exhaustive.

By way of contradiction: Let $\varepsilon > 0$. Then there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of orthogonal elements of L, a sequence $(i_n)_{n\in\mathbb{N}}$ of positive integers such that

$$|\mu_{i_n}(a_n)| > \varepsilon \quad \text{for every} \quad n \in \mathbb{N} \quad (*).$$

Thanks to the regularity we may suppose that $a_n \in \mathcal{F}$. Put

$$b_n := a_1 \oplus \cdots \oplus a_n.$$

We claim that there exists a couple of sequences $g_n \uparrow$ in $\mathcal{G}$, $f_n \uparrow$ in $\mathcal{F}$ such that

$$b_n \leq g_n \leq f_n \leq g_{n+1}, \qquad \sup_{i \in \mathbb{N}} \tilde{\mu}_i(f_n \ominus b_n) \leq \frac{\varepsilon}{2} - \frac{\varepsilon}{2^{n+1}}$$

Proceed by induction:
(a) By uniformly regularity, pick

$$g_1 \in \mathcal{G},\ f_1 \in \mathcal{F},\ b_1 \leq g_1 \leq f_1,\ \sup_{i \in \mathbb{N}} \tilde{\mu}_i(f_1 \ominus b_1) \leq \frac{\varepsilon}{2} - \frac{\varepsilon}{4}$$

(b) Again by uniformly regularity, we can construct the $(n+1)$-th step in such a way

$$g_{n+1} \in \mathcal{G},\ f_{n+1} \in \mathcal{F},\ b_{n+1} \vee f_n \leq g_{n+1} \leq f_{n+1}; \quad \sup_{i \in \mathbb{N}} \tilde{\mu}_i(f_{n+1} \ominus (b_{n+1} \vee f_n)) \leq \frac{\varepsilon}{2^{n+2}}$$

Hence, for each $i \in \mathbb{N}$, as $f_{n+1} \ominus b_{n+1} \leq (f_{n+1} \ominus (b_{n+1} \vee f_n)) \oplus ((b_{n+1} \vee f_n) \ominus b_{n+1})$ and $\tilde{\mu}_i((b_{n+1} \vee f_n) \ominus b_{n+1}) \leq \tilde{\mu}_i(f_n \ominus b_n)$, we get

$$\tilde{\mu}_i(f_{n+1} \ominus b_{n+1}) \leq \tilde{\mu}_i(f_{n+1} \ominus (b_{n+1} \vee f_n)) + \tilde{\mu}_i(f_n \ominus b_n) \leq \frac{\varepsilon}{2^{n+2}} + \frac{\varepsilon}{2} - \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2} - \frac{\varepsilon}{2^{n+2}} < \frac{\varepsilon}{2}$$

Then, we get

$$\tilde{\mu}_i(a_{n+1}) = \tilde{\mu}_i(b_{n+1} \ominus b_n) \leq \tilde{\mu}_i(g_{n+1} \ominus f_n) + \tilde{\mu}_i(f_n \ominus b_n)$$

as $b_{n+1} \ominus b_n \leq (g_{n+1} \ominus f_n) \oplus (f_n \ominus b_n)$. Since $\lim_n \tilde{\lambda}_i(g_{n+1} \ominus f_n) = 0$ (where λ_i is the restriction of μ_i to $\mathcal{G}$) as $g_{n+1} \ominus f_n$ are orthogonal elements of $\mathcal{G}$, and by regularity $\lim_n \tilde{\mu}_i(g_{n+1} \ominus f_n) = 0$. Therefore, $\tilde{\mu}_i(a_{n+1}) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, which contradicts (*).

We have proved that the sequence is uniformly exhaustive. QED

5 The theorem

Theorem 5.1 *Let $\mathcal{G}$ be closed under countable sums. Let (μ_i) be a sequence of $\mathcal{G}$-exhaustive regular real-valued modular measures converging on every element of $\mathcal{G}$. Then the sequence is converging on every element of L and it is uniformly exhaustive. Therefore its limit is exhaustive and regular.*

Proof. Thanks to Theorem 4.2, the sequence is uniformly $\mathcal{G}$-exhaustive.

We shall show that for each element $a \in L$, the sequence $(\mu_i(a))_{i \in N}$ is a Cauchy sequence; that's enough: apply Theorem 4.5 for completing the proof.

For, let $\varepsilon > 0$. Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$ satisfying $f \leq a \leq g$ and with $\sup_i \tilde{\mu}_i(g \ominus f) < \frac{\varepsilon}{3}$. Since $(\mu_i(g))_{i \in \mathbb{N}}$ is a Cauchy sequence, there exists $k \in \mathbb{N}$ such that $|\mu_i(g) - \mu_j(g)| < \frac{\varepsilon}{3}$ for every i, $j \geq k$. Then for such integers

$|\mu_i(a) - \mu_j(a)| = |\mu_i(g) - \mu_j(g) + \mu_j(g \ominus a) - \mu_i(g \ominus a)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. This proves that $(\mu_i(a))_{i \in \mathbb{N}}$ is a Cauchy sequence. QED

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