BIPOLAR FUZZY STRUCTURES OF SOME TYPES OF IDEALS IN HYPER BCK-ALGEBRAS

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Abstract. Using the notion of bipolar-valued fuzzy sets, the concepts of bipolar fuzzy (weak, s-weak, strong) hyper BCK-ideals are introduced, and their relations are discussed in [3]. Further properties are investigated in this article. We show that if $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy (weak, strong) hyper BCK-ideal of a hyper BCK-algebra H, then the set

$$I := \{x \in H \mid \mu_\Phi^P(x) = \mu_\Phi^P(0), \quad \mu_\Phi^N(x) = \mu_\Phi^N(0)\}.$$

is a (weak, strong) hyper BCK-ideal of H, but not converse by providing examples. Using a collection of ordered pairs of (weak, strong) hyper BCK-ideals of a hyper BCK-algebra H, a bipolar fuzzy (weak, strong) hyper BCK-ideal of H is established.

1 Introduction

Fuzzy set theory is established in the paper [9]. In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0, 1]$. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not belong to the fuzzy set. The membership degrees on the interval $(0, 1)$ indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [1, 10]). In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements. Consider a fuzzy set "*young*" defined on the *age* domain $[0, 100]$. Now consider two ages 50 and 95 with membership degree 0. Although both of them do not satisfy the property "*young*", we may say that age 95 is more apart from the property rather than age 50 (see [7]).

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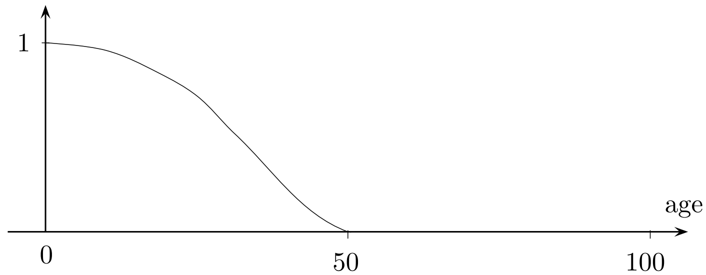

Figure 1. A fuzzy set "*young*"

Only with the membership degrees ranged on the interval [0, 1], it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, Lee [7] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. He gave two kinds of representations of the notion of bipolar-valued fuzzy sets. Using the notion of bipolar-valued fuzzy sets, the concepts of bipolar fuzzy (weak, s-weak, strong) hyper BCK-ideals are introduced, and their relations are discussed by present authors in [3]. The present authors [2] introduced the notion of a bipolar fuzzy implication hyper BCK-ideal in hyper BCK-algebras, and investigated some of their properties. They gave a relationship between bipolar fuzzy implicative hyper BCK-ideal and bipolar fuzzy hyper BCK-ideal. Using the positive (resp. negative) level set, they discussed a characterization of bipolar fuzzy implicative hyper BCK-ideals. They also provided conditions for a bipolar fuzzy hyper BCK-ideal to be a bipolar fuzzy implicative hyper BCK-ideal. Using a collection of implicative hyper BCK-ideals, they established a bipolar fuzzy implicative hyper BCK-ideal.

In this paper, we prove that if $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy (weak, strong) hyper BCK-ideal of a hyper BCK-algebra H, then the set

$$I := \{x \in H \mid \mu_\Phi^P(x) = \mu_\Phi^P(0), \quad \mu_\Phi^N(x) = \mu_\Phi^N(0)\}.$$

is a (weak, strong) hyper BCK-ideal of H, but not converse by providing examples. Using a collection of ordered pairs of (weak, strong) hyper BCK-ideals of a hyper BCK-algebra H, we establish a bipolar fuzzy (weak, strong) hyper BCK-ideal of H.

2 Preliminaries

2.1 Basic results on bipolar valued fuzzy sets Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets etc. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Bipolar-valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter-property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1, 0)$ indicate that elements somewhat satisfy the implicit counter-property (see [7]). Figure 2 shows a bipolar-valued fuzzy set redefined for the fuzzy set "*young*" of Figure 1. The negative membership degrees indicate the satisfaction extent of elements to an implicit counter-property (e.g., old against the property young). This kind of bipolar-valued fuzzy set representation enables the elements with membership degree 0 in traditional fuzzy sets, to be expressed into the elements with membership degree 0 (irrelevant elements) and the elements with negative membership degrees (contrary elements). The age elements 50 and 95, with membership degree 0 in the fuzzy sets of Figure 1, have 0 and a negative membership degree in the bipolar-valued fuzzy set of Figure 2, respectively. Now it is manifested that 50 is an irrelevant age to the property young and 95 is more apart from the property young than 50, i.e., 95 is a contrary age to the property young (see [7]).

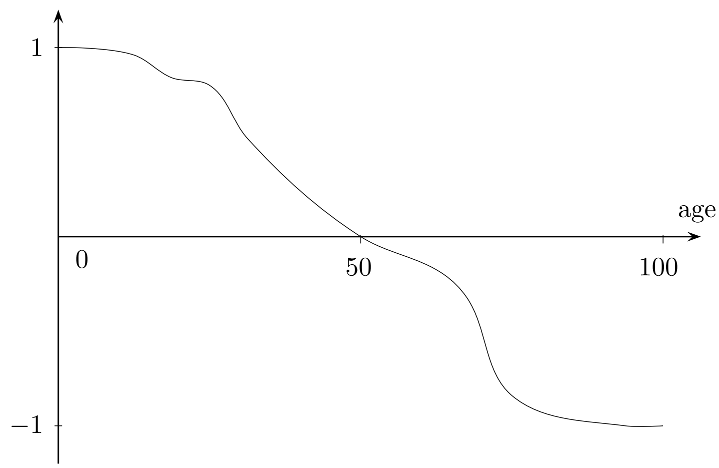

Figure 2. A bipolar fuzzy set "*young*"

In the definition of bipolar-valued fuzzy sets, there are two kinds of representations, so called canonical representation and reduced representation. In this paper, we use the canonical representation of a bipolar-valued fuzzy sets. Let X be the universe of discourse. A *bipolar-valued fuzzy set* Φ in X is an object having the form

$$\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$$

where $\mu_\Phi^P : X \to [0,1]$ and $\mu_\Phi^N : X \to [-1,0]$ are mappings. The positive membership degree $\mu_\Phi^P(x)$ denoted the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$, and the negative membership degree $\mu_\Phi^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$. If $\mu_\Phi^P(x) \neq 0$ and $\mu_\Phi^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$. If $\mu_\Phi^P(x) = 0$ and $\mu_\Phi^N(x) \neq 0$, it is the situation that x does not satisfy the property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$ but somewhat satisfies the counter-property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$. It is possible for an element x to be $\mu_\Phi^P(x) \neq 0$ and $\mu_\Phi^N(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [8]). For the sake of simplicity, we shall use the symbol $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ for the bipolar-valued fuzzy set $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

2.2 Basic results on hyper BCK-algebras We include some elementary aspects of hyper BCK-algebras that are necessary for this paper, and for more details we refer to [4], [5], and [6].

Let H be a nonempty set endowed with a hyperoperation "$\circ$". For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

By a *hyper BCK-algebra* we mean a nonempty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x \circ H \ll \{x\}$,

(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in H.

Note that the condition (HK3) is equivalent to the condition:

$$(\forall x, y \in H)(x \circ y \ll \{x\}). \tag{2.1}$$

In any hyper BCK-algebra H, the following hold:

(a1) $x \circ 0 \ll \{x\},\ 0 \circ x \ll \{0\}\ 0 \circ 0 \ll \{0\}$,

(a2) $(A \circ B) \circ C = (A \circ C) \circ B,\ A \circ B \ll A,\ 0 \circ A \ll \{0\}$,

(a3) $0 \circ 0 = \{0\}$,

(a4) $0 \ll x$,

(a5) $x \ll x$,

(a6) $A \ll A$,

(a7) $A \subseteq B \ \Rightarrow\ A \ll B$,

(a8) $0 \circ x = \{0\}$,

(a9) $0 \circ A = \{0\}$,

(a10) $A \ll \{0\} \ \Rightarrow\ A = \{0\}$,

(a11) $A \circ B \ll A$,

(a12) $x \in x \circ 0$,

(a13) $x \circ 0 \ll \{y\} \ \Rightarrow\ x \ll y$,

(a14) $y \ll z \ \Rightarrow\ x \circ z \ll x \circ y$,

(a15) $x \circ y = \{0\} \ \Rightarrow\ (x \circ z) \circ (y \circ z) = \{0\},\ x \circ z \ll y \circ z$,

(a16) $A \circ \{0\} = \{0\} \ \Rightarrow\ A = \{0\}$

for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H.

A nonempty subset I of a hyper BCK-algebra H is said to be a *hyper BCK-ideal* of H if it satisfies

(I1) $0 \in I$,

(I2) $(\forall x \in H)\ (\forall y \in I)\ (x \circ y \ll I \ \Rightarrow\ x \in I)$.

A nonempty subset I of a hyper BCK-algebra H is called a *strong hyper BCK-ideal* of H if it satisfies (I1) and

(I3) $(\forall x \in H)\ (\forall y \in I)\ ((x \circ y) \cap I \neq \emptyset \ \Rightarrow\ x \in I)$.

Note that every strong hyper BCK-ideal of a hyper BCK-algebra is a hyper BCK-ideal.

A nonempty subset I of a hyper BCK-algebra H is called a *weak hyper BCK-ideal* of H if it satisfies (I1) and

(I4) $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

3 Bipolar fuzzy hyper BCK-ideals In what follows let H denote a hyper BCK-algebra unless otherwise specified.

Definition 3.1. [3] A bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H is called a *bipolar fuzzy hyper BCK-ideal* of H if it satisfies

$$(\forall x, y \in H)(x \ll y \ \Rightarrow\ \mu_\Phi^P(x) \geq \mu_\Phi^P(y),\ \mu_\Phi^N(x) \leq \mu_\Phi^N(y)), \tag{3.1}$$

and

$$\begin{aligned} \mu_\Phi^P(x) &\geq \min\Big\{\inf_{a \in x \circ y} \mu_\Phi^P(a),\ \mu_\Phi^P(y)\Big\}, \\ \mu_\Phi^N(x) &\leq \max\Big\{\sup_{b \in x \circ y} \mu_\Phi^N(b),\ \mu_\Phi^N(y)\Big\} \end{aligned} \tag{3.2}$$

for all $x, y \in H$.

Definition 3.2. [3] A bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H is called a *bipolar fuzzy strong hyper BCK-ideal* of H if it satisfies

$$\begin{aligned} \inf_{a \in x \circ x} \mu_\Phi^P(a) \geq \mu_\Phi^P(x) &\geq \min\{\sup_{b \in\, x \circ y} \mu_\Phi^P(b),\ \mu_\Phi^P(y)\}, \\ \sup_{c \in x \circ x} \mu_\Phi^N(c) \leq \mu_\Phi^N(x) &\leq \max\{\inf_{d \in\, x \circ y} \mu_\Phi^N(d),\ \mu_\Phi^N(y)\} \end{aligned} \tag{3.3}$$

for all $x, y \in H$.

Definition 3.3. [3] A bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H is called a *bipolar fuzzy weak hyper BCK-ideal* of H if it satisfies

$$\begin{aligned} \mu_\Phi^P(0) \geq \mu_\Phi^P(x) &\geq \min\Big\{\inf_{a \in x \circ y} \mu_\Phi^P(a),\ \mu_\Phi^P(y)\Big\}, \\ \mu_\Phi^N(0) \leq \mu_\Phi^N(x) &\leq \max\Big\{\sup_{b \in x \circ y} \mu_\Phi^N(b),\ \mu_\Phi^N(y)\Big\} \end{aligned} \tag{3.4}$$

for all $x, y \in H$.

A bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H is said to satisfy the **inf-sup** *property* [3] if for any nonempty subset T of H there exist $x_0, y_0 \in T$ such that $\mu_\Phi^P(x_0) = \inf_{x \in T} \mu_\Phi^P(x)$ and $\mu_\Phi^N(y_0) = \sup_{y \in T} \mu_\Phi^N(y)$.

Note that, in a finite hyper BCK-algebra, every bipolar fuzzy set satisfies the **inf-sup** property.

Lemma 3.4. [3] *Let* $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ *be a bipolar fuzzy hyper* BCK*-ideal of* H *and let* $x, y \in H$. *Then*

1. $\mu_\Phi^P(0) \geq \mu_\Phi^P(x)$ *and* $\mu_\Phi^N(0) \leq \mu_\Phi^N(x)$.
2. *If* $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ *satisfies the* **inf-sup** *property, then*

$$\begin{aligned} &\mu_\Phi^P(x) \geq \min\{\mu_\Phi^P(a),\, \mu_\Phi^P(y)\}, \\ &\mu_\Phi^N(x) \leq \max\{\mu_\Phi^N(b),\, \mu_\Phi^N(y)\} \end{aligned} \tag{3.5}$$

for some $a, b \in x \circ y$.

Theorem 3.5. *If* $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ *is a bipolar fuzzy hyper* BCK*-ideal of a hyper* BCK*-algebra* H, *then the set*

$$I := \{x \in H \mid \mu_\Phi^P(x) = \mu_\Phi^P(0), \quad \mu_\Phi^N(x) = \mu_\Phi^N(0)\}$$

is a hyper BCK*-ideal of* H.

Proof. Obviously $0 \in I$. Let $x, y \in H$ be such that $x \circ y \ll I$ and $y \in I$. Then $\mu_\Phi^P(y) = \mu_\Phi^P(0)$, $\mu_\Phi^N(y) = \mu_\Phi^N(0)$, and for each $a \in x \circ y$ there exists $z \in I$ such that $a \ll z$. Thus $\mu_\Phi^P(a) \geq \mu_\Phi^P(z) = \mu_\Phi^P(0)$ and $\mu_\Phi^N(a) \leq \mu_\Phi^N(z) = \mu_\Phi^N(0)$ by (3.1). It follows from (3.2) that

$$\mu_\Phi^P(x) \geq \min\{\inf_{a \in x \circ y} \mu_\Phi^P(a),\, \mu_\Phi^P(y)\} = \mu_\Phi^P(0)$$

and

$$\mu_\Phi^N(x) \leq \max\{\sup_{a \in x \circ y} \mu_\Phi^N(a),\, \mu_\Phi^N(y)\} = \mu_\Phi^N(0)$$

so that $\mu_\Phi^P(x) = \mu_\Phi^P(0)$ and $\mu_\Phi^N(x) = \mu_\Phi^N(0)$. Hence $x \in I$, which shows that I is a hyper BCK-ideal of H. □

The converse of Theorem 3.5 is not true as seen in the following example.

Example 3.6. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

	0	a	b
μ_Φ^P	0.5	0.5	0.8
μ_Φ^N	−0.4	−0.4	−0.7

Then

$$I := \{x \in H \mid \mu_\Phi^P(x) = \mu_\Phi^P(0), \quad \mu_\Phi^N(x) = \mu_\Phi^N(0)\} = \{0, a\}$$

which is a hyper BCK-ideal of H. But $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is not a bipolar fuzzy hyper BCK-ideal of H.

Theorem 3.7. *If $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy strong hyper BCK-ideal of a hyper BCK-algebra H, then the set*

$$I := \{x \in H \mid \mu_\Phi^P(x) = \mu_\Phi^P(0),\ \mu_\Phi^N(x) = \mu_\Phi^N(0)\}$$

is a strong hyper BCK-ideal of H.

Proof. Clearly $0 \in I$. Let $x \in H$ and $y \in I$ be such that $(x \circ y) \cap I \neq \emptyset$. Then $\mu_\Phi^P(y) = \mu_\Phi^P(0)$ and $\mu_\Phi^N(y) = \mu_\Phi^N(0)$. Since $(x \circ y) \cap I \neq \emptyset$, there exists $b \in (x \circ y) \cap I$. Thus

$$\begin{aligned} &\mu_\Phi^P(x) \geq \min\{\sup_{c \in x \circ y} \mu_\Phi^P(c),\ \mu_\Phi^P(y)\} \geq \min\{\mu_\Phi^P(b), \mu_\Phi^P(y)\} = \mu_\Phi^P(0), \\ &\mu_\Phi^N(x) \leq \max\{\inf_{d \in x \circ y} \mu_\Phi^N(d),\ \mu_\Phi^N(y)\} \leq \max\{\mu_\Phi^N(b), \mu_\Phi^N(y)\} = \mu_\Phi^N(0). \end{aligned} \tag{3.6}$$

Since $0 \in x \circ x$, it follows from (3.3) that

$$\begin{aligned} \mu_\Phi^P(0) &\geq \inf_{a \in x \circ x} \mu_\Phi^P(a) \geq \mu_\Phi^P(x), \\ \mu_\Phi^N(0) &\leq \sup_{c \in x \circ x} \mu_\Phi^N(c) \leq \mu_\Phi^P(x). \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we have $\mu_\Phi^P(x) = \mu_\Phi^P(0)$ and $\mu_\Phi^N(x) = \mu_\Phi^N(0)$, and so $x \in I$. Therefore I is a strong hyper BCK-ideal of H. □

The converse of Theorem 3.7 is not true as seen in the following example.

Example 3.8. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

	0	a	b
μ_Φ^P	0.7	0.7	0.8
μ_Φ^N	−0.4	−0.4	−0.7

Then

$$I := \{x \in H \mid \mu_\Phi^P(x) = \mu_\Phi^P(0),\ \mu_\Phi^N(x) = \mu_\Phi^N(0)\} = \{0, a\}$$

which is a strong hyper BCK-ideal of H. But $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is not a bipolar fuzzy strong hyper BCK-ideal of H.

Theorem 3.9. *If $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy weak hyper BCK-ideal of a hyper BCK-algebra H, then the set*

$$I := \{x \in H \mid \mu_\Phi^P(x) = \mu_\Phi^P(0),\ \mu_\Phi^N(x) = \mu_\Phi^N(0)\}$$

is a weak hyper BCK-ideal of H.

Proof. Obviously $0 \in I$. Let $x \in H$ and $y \in I$ be such that $x \circ y \subseteq I$. Then $\mu_\Phi^P(y) = \mu_\Phi^P(0)$ and $\mu_\Phi^N(y) = \mu_\Phi^N(0)$. Using (3.4), we have

$$\mu_\Phi^P(0) \geq \mu_\Phi^P(x) \geq \min\{\inf_{a \in x \circ y} \mu_\Phi^P(a), \mu_\Phi^P(y)\} = \mu_\Phi^P(0),$$

$$\mu_\Phi^N(0) \leq \mu_\Phi^N(x) \leq \max\{\sup_{b \in x \circ y} \mu_\Phi^N(b), \mu_\Phi^N(y)\} = \mu_\Phi^N(0).$$

Thus $\mu_\Phi^P(0) = \mu_\Phi^P(x)$ and $\mu_\Phi^N(0) = \mu_\Phi^N(x)$, that is, $x \in I$. Hence I is a weak hyper BCK-ideal of H. □

The converse of Theorem 3.9 is not true as seen in the following example.

Example 3.10. Consider a hyper BCK-algebra $H = \mathbb{N} \cup \{0\}$ in which the hyperoperation "$\circ$" is defined as follows:

$$x \circ y := \begin{cases} \{0, x\} & \text{if} \quad x \leq y, \\ \{x\} & \text{if} \quad x > y. \end{cases}$$

Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in H where

$$\mu_\Phi^P(n) := \begin{cases} 0 & \text{if} \quad n \leq 3, \\ 2 - 1.7 & \text{if} \quad n = 4, \\ 2 - 1.73 & \text{if} \quad n = 5, \\ 2 - 1.732 & \text{if} \quad n = 6, \\ \cdots & \end{cases}$$

$$\mu_\Phi^N(n) := \begin{cases} 0 & \text{if} \quad n \leq 4, \\ -2 + 1.7 & \text{if} \quad n = 5, \\ -2 + 1.73 & \text{if} \quad n = 6, \\ -2 + 1.732 & \text{if} \quad n = 7, \\ \cdots & \end{cases}$$

Then

$$I := \{x \in H \mid \mu_\Phi^P(x) = \mu_\Phi^P(0),\ \mu_\Phi^N(x) = \mu_\Phi^N(0)\} = \{0, 1, 2, 3\}$$

which is a weak hyper BCK-ideal of H. But $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is not a bipolar fuzzy weak hyper BCK-ideal of H.

For a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in a set H, the *positive level set* and *negative level set* are denoted by $P(\mu_\Phi^P; \alpha)$ and $N(\mu_\Phi^N; \beta)$, and are defined as follows:

$$P(\mu_\Phi^P; \alpha) := \{x \in H \mid \mu_\Phi^P(x) \geq \alpha\},\ \alpha \in [0, 1],$$

$$N(\mu_\Phi^N; \beta) := \{x \in H \mid \mu_\Phi^N(x) \leq \beta\},\ \beta \in [-1, 0],$$

respectively.

Lemma 3.11. [3] *Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in H. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H if and only if for every $(\alpha, \beta) \in [0, 1] \times [-1, 0]$, the nonempty positive level set $P(\mu_\Phi^P; \alpha)$ and the nonempty negative level set $N(\mu_\Phi^N; \beta)$ are hyper BCK-ideals of H.*

Lemma 3.12. *Let I be a hyper BCK-ideal of H. Let $x, y \in H$ be such that $x \ll y$. If $y \in I$, then $x \in I$.*

Proof. Straightforward. $\square$

Let Λ_P (resp. Λ_N) be a subset of $[0,1]$ (resp. $[-1,0]$) such that every subset of Λ_P (resp. Λ_N) contains both its supremum and its infimum.

Theorem 3.13. *Let $\{(I_\alpha, J_\beta) \mid (\alpha, \beta) \in \Lambda_P \times \Lambda_N\}$ be a collection of ordered pairs of hyper BCK-ideals of H such that*

1. $(\forall \alpha_1, \alpha_2 \in \Lambda_P)\ (\alpha_1 > \alpha_2 \ \Rightarrow\ I_{\alpha_1} \subseteq I_{\alpha_2})$,
2. $(\forall \beta_1, \beta_2 \in \Lambda_N)\ (\beta_1 > \beta_2 \ \Rightarrow\ J_{\beta_2} \subseteq J_{\beta_1})$.

Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

$$\mu_\Phi^P(x) := \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\},\ \mu_\Phi^N(x) := \inf\{\beta \in \Lambda_N \mid x \in J_\beta\}$$

for all $x \in H$. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H.

Proof. Let $x, y \in H$ be such that $x \ll y$. Using Lemma 3.12, we have

$$\mu_\Phi^P(y) = \sup\{\alpha \in \Lambda_P \mid y \in I_\alpha\} \le \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\} = \mu_\Phi^P(x),$$

$$\mu_\Phi^N(y) = \inf\{\beta \in \Lambda_N \mid y \in J_\beta\} \ge \inf\{\beta \in \Lambda_N \mid x \in J_\beta\} = \mu_\Phi^N(x).$$

For any $x, y \in H$, let

$$\alpha_1 = \inf_{a \in x \circ y} \mu_\Phi^P(a) = \inf_{a \in x \circ y} [\sup\{\alpha \in \Lambda_P \mid a \in I_\alpha\}],$$

$$\alpha_2 = \mu_\Phi^P(y) = \sup\{\alpha \in \Lambda_P \mid y \in I_\alpha\}.$$

Then $y \in I_{\alpha_2}$ and $a \in I_{\alpha_1}$ for all $a \in x \circ y$. If $\alpha_1 \le \alpha_2$, then $y \in I_{\alpha_2} \subseteq I_{\alpha_1}$. It follows that $x \circ y \subseteq I_{\alpha_1}$ so that $x \circ y \ll I_{\alpha_1}$. Using (I2), we get $x \in I_{\alpha_1}$, and so

$$\mu_\Phi^P(x) = \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\} \ge \alpha_1 = \min\{\alpha_1, \alpha_2\} = \min\{\inf_{a \in x \circ y} \mu_\Phi^P(a), \mu_\Phi^P(y)\}.$$

If $\alpha_1 > \alpha_2$, then $I_{\alpha_1} \subseteq I_{\alpha_2}$. Since $a \in I_{\alpha_1}$ for all $a \in x \circ y$, it follows that $a \in I_{\alpha_2}$ for all $a \in x \circ y$ so that $x \circ y \subseteq I_{\alpha_2}$. Hence $x \circ y \ll I_{\alpha_2}$, and since $y \in I_{\alpha_2}$, we get $x \in I_{\alpha_2}$ by (I2). Therefore

$$\mu_\Phi^P(x) = \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\} \ge \alpha_2 = \min\{\alpha_1, \alpha_2\} = \min\{\inf_{a \in x \circ y} \mu_\Phi^P(a), \mu_\Phi^P(y)\}.$$

Now, let

$$\beta_1 = \sup_{b \in x \circ y} \mu_\Phi^N(b) = \sup_{b \in x \circ y} [\inf\{\beta \in \Lambda_N \mid b \in J_\beta\}],$$

$$\beta_2 = \mu_\Phi^N(y) = \inf\{\beta \in \Lambda_N \mid y \in J_\beta\}.$$

Then $y \in J_{\beta_2}$ and $b \in J_{\beta_1}$ for all $b \in x \circ y$. Assume that $\beta_1 \leq \beta_2$. Then $b \in J_{\beta_1} \subseteq J_{\beta_2}$, and so $x \circ y \subseteq J_{\beta_2}$. Thus $x \circ y \ll J_{\beta_2}$. Since $y \in J_{\beta_2}$ and J_{β_2} is a hyper BCK-ideal, it follows from (I2) that $x \in J_{\beta_2}$ so that

$$\mu_\Phi^N(x) = \inf\{\beta \in \Lambda_N \mid x \in J_\beta\} \leq \beta_2 = \max\{\beta_1, \beta_2\} = \max\{\sup_{b \in x \circ y} \mu_\Phi^N(b), \mu_\Phi^N(y)\}.$$

Finally, if $\beta_1 > \beta_2$, then $J_{\beta_2} \subseteq J_{\beta_1}$. Thus $x \circ y \subseteq J_{\beta_1}$ and so $x \circ y \ll J_{\beta_1}$. Since $y \in J_{\beta_2} \subseteq J_{\beta_1}$ and J_{β_1} is a hyper BCK-ideal, we have $x \in J_{\beta_1}$ by (I2). Therefore

$$\mu_\Phi^N(x) = \inf\{\beta \in \Lambda_N \mid x \in J_\beta\} \leq \beta_1 = \max\{\beta_1, \beta_2\} = \max\{\sup_{b \in x \circ y} \mu_\Phi^N(b), \mu_\Phi^N(y)\}.$$

Consequently, $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H. □

Corollary 3.14. *Let $\{(I_\alpha, J_\beta) \mid (\alpha, \beta) \in \Lambda_P \times \Lambda_N\}$ be a collection of ordered pairs of strong hyper BCK-ideals of H such that*

1. $(\forall \alpha_1, \alpha_2 \in \Lambda_P)\ (\alpha_1 > \alpha_2 \ \Rightarrow\ I_{\alpha_1} \subseteq I_{\alpha_2})$,
2. $(\forall \beta_1, \beta_2 \in \Lambda_N)\ (\beta_1 > \beta_2 \ \Rightarrow\ J_{\beta_2} \subseteq J_{\beta_1})$.

Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

$$\mu_\Phi^P(x) := \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\},\ \mu_\Phi^N(x) := \inf\{\beta \in \Lambda_N \mid x \in J_\beta\}$$

for all $x \in H$. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H, and hence a bipolar fuzzy weak hyper BCK-ideal of H.

Corollary 3.15. *Let $\{(I_\alpha, J_\beta) \mid (\alpha, \beta) \in \Lambda_P \times \Lambda_N\}$ be a collection of ordered pairs of hyper BCK-ideals of H such that*

1. $(\forall \alpha_1, \alpha_2 \in \Lambda_P)\ (\alpha_1 > \alpha_2 \ \Rightarrow\ I_{\alpha_1} \subseteq I_{\alpha_2})$,
2. $(\forall \beta_1, \beta_2 \in \Lambda_N)\ (\beta_1 > \beta_2 \ \Rightarrow\ J_{\beta_2} \subseteq J_{\beta_1})$.

Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

$$\mu_\Phi^P(x) := \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\},\ \mu_\Phi^N(x) := \inf\{\beta \in \Lambda_N \mid x \in J_\beta\}$$

for all $x \in H$. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy weak hyper BCK-ideal of H.

Theorem 3.16. *Let $\{(I_\alpha, J_\beta) \mid (\alpha, \beta) \in \Lambda_P \times \Lambda_N\}$ be a collection of ordered pairs of strong hyper BCK-ideals of H such that*

1. $(\forall \alpha_1, \alpha_2 \in \Lambda_P)\ (\alpha_1 > \alpha_2 \ \Rightarrow\ I_{\alpha_1} \subseteq I_{\alpha_2})$,
2. $(\forall \beta_1, \beta_2 \in \Lambda_N)\ (\beta_1 > \beta_2 \ \Rightarrow\ J_{\beta_2} \subseteq J_{\beta_1})$.

Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

$$\mu_\Phi^P(x) := \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\},\ \mu_\Phi^N(x) := \inf\{\beta \in \Lambda_N \mid x \in J_\beta\}$$

for all $x \in H$. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy strong hyper BCK-ideal of H.

Proof. For any $x \in H$, we have

$$\inf_{a \in x \circ x} \mu_\Phi^P(a) = \inf_{a \in x \circ x}[\sup\{\alpha \in \Lambda_P \mid a \in I_\alpha\}] \geq \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\} = \mu_\Phi^P(x),$$

$$\sup_{b \in x \circ x} \mu_\Phi^N(b) = \sup_{b \in x \circ x}[\inf\{\beta \in \Lambda_N \mid b \in J_\beta\}] \leq \inf\{\beta \in \Lambda_N \mid x \in J_\beta\} = \mu_\Phi^N(x),$$

For any $x, y \in H$, let

$$\alpha_1 := \sup_{b \in x \circ y} \mu_\Phi^P(b) = \sup_{b \in x \circ y}[\sup\{\alpha \in \Lambda_P \mid b \in I_\alpha\}].$$

Then there exists $d \in x \circ y$ such that $d \in I_{\alpha_1}$, i.e., $(x \circ y) \cap I_{\alpha_1} \neq \emptyset$. If $\alpha_2 := \mu_\Phi^P(y) = \sup\{\alpha \in \Lambda_P \mid y \in I_\alpha\}$, then $y \in I_{\alpha_2}$. Assume that $\alpha_1 \leq \alpha_2$. Then $I_{\alpha_2} \subseteq I_{\alpha_1}$, and so $y \in I_{\alpha_1}$. Since I_{α_1} is a strong hyper BCK-ideal of H, it follows that $x \in I_{\alpha_1}$ so that

$$\mu_\Phi^P(x) = \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\} \geq \alpha_1 = \min\{\alpha_1, \alpha_2\} = \min\{\sup_{b \in x \circ y} \mu_\Phi^P(b), \mu_\Phi^P(y)\}.$$

If $\alpha_1 > \alpha_2$, then $I_{\alpha_1} \subseteq I_{\alpha_2}$, and so $\emptyset \neq (x \circ y) \cap I_{\alpha_1} \subseteq (x \circ y) \cap I_{\alpha_2}$. Since $y \in I_{\alpha_2}$ and I_{α_2} is a strong hyper BCK-ideal, we have $x \in I_{\alpha_2}$. Hence

$$\mu_\Phi^P(x) = \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\} \geq \alpha_2 = \min\{\alpha_1, \alpha_2\} = \min\{\sup_{b \in x \circ y} \mu_\Phi^P(b), \mu_\Phi^P(y)\}.$$

Now for any $x, y \in H$ if we take

$$\beta_1 := \inf_{d \in x \circ y} \mu_\Phi^N(d) = \inf_{d \in x \circ y}[\inf\{\beta \in \Lambda_N \mid d \in J_\beta\}],$$

then there exists $u \in x \circ y$ such that $u \in J_{\beta_1}$, that is, $(x \circ y) \cap J_{\beta_1} \neq \emptyset$. Let

$$\beta_2 := \mu_\Phi^N(y) = \inf\{\beta \in \Lambda_N \mid y \in J_\beta\}.$$

Then $y \in J_{\beta_2}$. If $\beta_1 \leq \beta_2$, then $J_{\beta_1} \subseteq J_{\beta_2}$, and so $(x \circ y) \cap J_{\beta_2} \neq \emptyset$. Since $y \in J_{\beta_2}$ and J_{β_2} is a strong hyper BCK-ideal, it follows that $x \in J_{\beta_2}$ so that

$$\mu_\Phi^N(x) = \inf\{\beta \in \Lambda_N \mid x \in J_\beta\} \leq \beta_2 = \max\{\beta_1, \beta_2\} = \max\{\inf_{d \in x \circ y} \mu_\Phi^N(d), \mu_\Phi^N(y)\}.$$

If $\beta_1 > \beta_2$, then $J_{\beta_2} \subseteq J_{\beta_1}$. Thus $y \in J_{\beta_1}$, and hence $x \in J_{\beta_1}$. Therefore

$$\mu_\Phi^N(x) = \inf\{\beta \in \Lambda_N \mid x \in J_\beta\} \leq \beta_1 = \max\{\beta_1, \beta_2\} = \max\{\inf_{d \in x \circ y} \mu_\Phi^N(d), \mu_\Phi^N(y)\}.$$

Consequently, $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy strong hyper BCK-ideal of H. □

Theorem 3.17. *Let $\{(I_\alpha, J_\beta) \mid (\alpha, \beta) \in \Lambda_P \times \Lambda_N\}$ be a collection of ordered pairs of weak hyper BCK-ideals of H such that*

1. $(\forall \alpha_1, \alpha_2 \in \Lambda_P)$ $(\alpha_1 > \alpha_2 \Rightarrow I_{\alpha_1} \subseteq I_{\alpha_2})$,
2. $(\forall \beta_1, \beta_2 \in \Lambda_N)$ $(\beta_1 > \beta_2 \Rightarrow J_{\beta_2} \subseteq J_{\beta_1})$.

Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

$$\mu_\Phi^P(x) := \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\},\ \mu_\Phi^N(x) := \inf\{\beta \in \Lambda_N \mid x \in J_\beta\}$$

for all $x \in H$. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy weak hyper BCK-ideal of H.

Proof. Obviously, $\mu_\Phi^P(0) \geq \mu_\Phi^P(x)$ and $\mu_\Phi^N(0) \leq \mu_\Phi^N(x)$ for all $x \in H$ since every weak hyper BCK-ideal contains 0. Let $x, y \in H$. Take

$$\alpha_1 := \inf_{a \in x \circ y} \mu_\Phi^P(a) = \inf_{a \in x \circ y} [\sup\{\alpha \in \Lambda_P \mid a \in I_\alpha\}],$$

$$\alpha_2 := \mu_\Phi^P(y) = \sup\{\alpha \in \Lambda_P \mid y \in I_\alpha\}.$$

Then $a \in I_{\alpha_1}$ for all $a \in x \circ y$, and so $x \circ y \subseteq I_{\alpha_1}$ and $y \in I_{\alpha_2}$. Assume that $\alpha_1 \leq \alpha_2$. Then $I_{\alpha_2} \subseteq I_{\alpha_1}$, and thus $x \circ y \subseteq I_{\alpha_1}$ and $y \in I_{\alpha_1}$. Since I_{α_1} is a weak hyper BCK-ideal, it follows that $x \in I_{\alpha_1}$ so that

$$\mu_\Phi^P(x) = \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\} \geq \alpha_1 = \min\{\alpha_1, \alpha_2\} = \min\{\inf_{a \in x \circ y} \mu_\Phi^P(a), \mu_\Phi^P(y)\}.$$

If $\alpha_1 > \alpha_2$, then $I_{\alpha_1} \subseteq I_{\alpha_2}$, and so $x \circ y \subseteq I_{\alpha_1} \subseteq I_{\alpha_2}$ and $y \in I_{\alpha_2}$. Since I_{α_2} is a weak hyper BCK-ideal, we have $x \in I_{\alpha_2}$. Hence

$$\mu_\Phi^P(x) = \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\} \geq \alpha_2 = \min\{\alpha_1, \alpha_2\} = \min\{\inf_{a \in x \circ y} \mu_\Phi^P(a), \mu_\Phi^P(y)\}.$$

Let

$$\beta_1 := \sup_{b \in x \circ y} \mu_\Phi^N(b) = \sup_{b \in x \circ y} [\inf\{\beta \in \Lambda_N \mid b \in J_\beta\}],$$

$$\beta_2 := \mu_\Phi^N(y) = \inf\{\beta \in \Lambda_N \mid y \in J_\beta\}.$$

Then $b \in J_{\beta_1}$ for all $b \in x \circ y$, and so $x \circ y \subseteq J_{\beta_1}$, and $y \in J_{\beta_2}$. Suppose that $\beta_1 \leq \beta_2$. Then $J_{\beta_1} \subseteq J_{\beta_2}$, and hence $x \circ y \subseteq J_{\beta_2}$. Since $y \in J_{\beta_2}$ and J_{β_2} is a weak hyper BCK-ideal, it follows from (I4) that $x \in J_{\beta_2}$ so that

$$\mu_\Phi^N(x) = \inf\{\beta \in \Lambda_N \mid x \in J_\beta\} \leq \beta_2 = \max\{\beta_1, \beta_2\} = \max\{\sup_{b \in x \circ y} \mu_\Phi^N(b), \mu_\Phi^N(y)\}.$$

Now, if $\beta_1 > \beta_2$ then $J_{\beta_2} \subseteq J_{\beta_1}$. Note that $x \circ y \subseteq J_{\beta_1}$ and $y \in J_{\beta_2} \subseteq J_{\beta_1}$. Since J_{β_1} is a weak hyper BCK-ideal, we have $x \in J_{\beta_1}$ and so

$$\mu_\Phi^N(x) = \inf\{\beta \in \Lambda_N \mid x \in J_\beta\} \leq \beta_1 = \max\{\beta_1, \beta_2\} = \max\{\sup_{b \in x \circ y} \mu_\Phi^N(b), \mu_\Phi^N(y)\}.$$

Therefore $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy weak hyper BCK-ideal of H. □

References

[1] D. Dubois and H. Prade, *Fuzzy sets and systems: Theory and Applications,* Academic Press, 1980.

[2] Y. B. Jun, M. S. Kang and H. S. Kim, *Bipolar fuzzy implicative hyper BCK-ideals in hyper BCK-algerbas,* Sci. Math. Jpn. Online **e-2009** (2009), 93–104.

[3] Y. B. Jun, M. S. Kang and H. S. Kim, *Bipolar fuzzy hyper BCK-ideals in hyper BCK-algerbas,* Internat. J. Fuzzy Systems (submitted).

[4] Y. B. Jun and X. L. Xin, *Fuzzy hyper BCK-ideals of hyper BCK-algebras,* Sci. Math. Jpn. **53** (2001), no. 2, 353–360.

[5] Y. B. Jun, X. L. Xin, E. H. Roh and M. M. Zahedi, *Strong hyper BCK-ideals of hyper BCK-algebras,* Math. Japonica **51** (2000), no. 3, 493–498.

[6] Y. B. Jun, M. M. Zahedi, X. L. Xin and R. A. Borzooei, *On hyper BCK-algebras,* Italian J. of Pure and Appl. Math. **8** (2000), 127-136.

[7] K. M. Lee, *Bipolar-valued fuzzy sets and their operations,* Proc. Int. Conf. on Intelligent Technologies, Bangkok, Thailand (2000), 307–312.

[8] K. M. Lee, *Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets,* J. Fuzzy Logic Intelligent Systems **14** (2004), no. 2, 125–129.

[9] L. A. Zadeh, *Fuzzy sets,* Information and Control **8** (1965), 338–353.

[10] H. -J. Zimmermann, *Fuzzy Set Theory and its Applications,* Kluwer-Nijhoff Publishing, 1985.

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