CONVEXITY OF THE SET OF FIXED POINTS OF A QUASI-PSEUDOCONTRACTIVE TYPE LIPSCHITZ MAPPING AND THE SHRINKING PROJECTION METHOD

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Abstract. We introduce a new definition of nonlinear mapping, which is called a quasi-pseudocontractive type mapping, defined on a nonempty closed convex subset of a real Hilbert space, and we investigate convexity of the set of its fixed points and approximation of its fixed point. We consider an iterative sequence generated by the improved hybrid method, also known as the shrinking projection method, and prove that it converges strongly to a fixed point under some conditions of the constants for quasi-pseudocontractive type Lipschitz mapping.

1. **Introduction.** Let C be a nonempty closed convex subset of a real Hilbert space H and let T be a mapping of C into itself. T is said to be strictly pseudocontractive [4] if there exists a constant $k < 1$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k \|(x - Tx) - (y - Ty)\|^2$$

for all $x, y \in C$. If $k = 0$, T is said to be nonexpansive and it is said to be pseudocontractive when $k = 1$.

As an approximation of a fixed point of nonlinear mappings, Ishikawa [6] proved a weak convergence theorem for Lipschitz and pseudocontractive mappings. On the other hand, Haugazeau [5] proposed a strongly convergent iterative scheme called the hybrid method; see also [2, 11, 12, 13]. Acedo and Xu [1] and Marino and Xu [9] proved strong convergence theorems for strictly pseudocontractive mappings and Zhou [18] proved a strong convergence theorem for Lipschitz and pseudocontractive mappings.

In this paper, we consider the improved hybrid method, also known as the shrinking projection method, introduced by Takahashi, Takeuchi and Kubota [16]. To this end, we introduce a new definition of nonlinear mapping, which is called a quasi-pseudocontractive type mapping, and prove strong convergence for quasi-pseudocontractive type Lipschitz mappings under some conditions by the idea of [7, 8]. We also investigate the convexity of the set of fixed points of quasi-pseudocontractive type Lipschitz mappings and give a sufficient condition using an inequality of the constants.

2. **Preliminaries.** Throughout this paper, we denote by H a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively.

Let C be a nonempty closed convex subset of H and T a mapping of C into itself. The set of all fixed points of T is denoted by $F(T)$, that is, $F(T) = \{z \in C : z = Tz\}$. Suppose

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that $F(T)$ is nonempty. We say that T is quasi-pseudocontractive type if there exists a constant $k > 0$ such that

$$\|Tx - z\|^2 \leq \|x - z\|^2 + k\|x - Tx\|^2$$

for each $x \in C$ and $z \in F(T)$. For a concrete example of this mapping, see Example 3.4.

In a Hilbert space, it is known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$$

for each $x, y \in H$ and $\lambda \in \mathbb{R}$.

Let C be a nonempty closed convex subset of H. We know that for every $x \in H$, there exists a unique element $y \in C$ such that

$$\|x - y\| = \inf_{z \in C} \|x - z\|.$$

The mapping which maps $x \in H$ to $y \in C$ is called the metric projection onto C and, in what follows, we denote this mapping by P_C. For more details, see [14, 15] for example.

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. We define a subset $\text{s-Li}_n C_n$ of H as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset H$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, a subset $\text{w-Ls}_n C_n$ of H is defined by the following: $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H$ such that $\{y_i\}$ converges weakly to y and that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If $C_0 \subset H$ satisfies that $C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n$, it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [10] and we write $C_0 = \text{M-lim}_n C_n$. For more details, see [3].

Tsukada [17] proved the following theorem for the metric projection. We note that the original theorem is proved in the setting of Banach space.

Theorem 2.1 (Tsukada [17]). *Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. If $C_0 = \text{M-lim}_n C_n$ exists and nonempty, then, for each $x \in H$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$.*

3. **Strongly convergent sequence and convexity of the set of fixed points.** In this section, we consider an iterative sequence for a quasi-pseudocontractive type Lipschitz mapping generated by the shrinking projection method. The first result shows strong convergence of this sequence and a condition which guarantees the convexity of the set of fixed points of the mapping.

Theorem 3.1. *Let C be a nonempty closed convex subset of H. Let T be a quasi-pseudocontractive type Lipschitz mapping of C into itself with constants $k > 0$ and $L > 0$, respectively, and suppose that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$. For an initial point $x \in C$, let $\{x_n\}$ be a sequence generated by*

$$(1) \quad \begin{cases} x_1 = x \in C, \\ C_1 = C, \\ y_n = \alpha_n Tx_n + (1-\alpha_n)x_n, \\ C_{n+1} = \{z \in C_n : \\ \qquad \|Ty_n - z\|^2 \leq \|x_n - z\|^2 + \beta_n\|x_n - Tx_n\|^2 + \gamma_n\|x_n - Ty_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x, \end{cases}$$

for each $n \in \mathbb{N}$, where

$$\beta_n = \alpha_n(kL^2\alpha_n^2 + (k+1)\alpha_n - 1) \text{ and } \gamma_n = k(1-\alpha_n)$$

for each $n \in \mathbb{N}$. Then the following hold:

(i) $\{x_n\}$ *converges strongly to a point of* $\bigcap_{n=1}^{\infty} C_n$ *for every initial point* $x \in C$*;*
(ii) *if for every initial point* $x \in C$*, the strong limit* x_0 *of* $\{x_n\}$ *belongs to* $F(T)$*, then* $F(T)$ *is a closed convex set and* $x_0 = P_{F(T)}x$ *for each initial point* $x \in C$*.*

Proof. First let us prove that $\{x_n\}$ is well defined. To this end, it is sufficient to show that $D = \{z \in C : \|Ty - z\|^2 \leq \|x - z\|^2 + \beta \|x - Tx\|^2 + \gamma \|x - Ty\|^2\}$ is nonempty, closed and convex for any $x \in C$ and $\alpha \in [0, 1]$, where $\beta = \alpha(kL^2\alpha^2 + (k+1)\alpha - 1)$, $\gamma = k(1 - \alpha)$, and $y = \alpha Tx + (1 - \alpha)x$. Let $x \in C$, $\alpha \in [0, 1]$, and $y = \alpha Tx + (1 - \alpha)x$. Let $z \in F(T)$. Then, since T is quasi-pseudocontractive type and Lipschitz, we have that

$$\begin{aligned}
&\|Ty - z\|^2 \\
&\leq \|y - z\|^2 + k \|y - Ty\|^2 \\
&= \|\alpha Tx + (1 - \alpha)x - z\|^2 + k \|\alpha Tx + (1 - \alpha)x - Ty\|^2 \\
&= \alpha \|Tx - z\|^2 + (1 - \alpha) \|x - z\|^2 - \alpha(1 - \alpha) \|Tx - x\|^2 \\
&\quad + k(\alpha \|Tx - Ty\|^2 + (1 - \alpha) \|x - Ty\|^2 - \alpha(1 - \alpha) \|Tx - x\|^2) \\
&\leq \alpha(\|x - z\|^2 + k \|x - Tx\|^2) + (1 - \alpha) \|x - z\|^2 - \alpha(1 - \alpha) \|Tx - x\|^2 \\
&\quad + k(\alpha L^2 \|x - y\|^2 + (1 - \alpha) \|x - Ty\|^2 - \alpha(1 - \alpha) \|Tx - x\|^2) \\
&= \|x - z\|^2 + \alpha k \|x - Tx\|^2 - \alpha(1 - \alpha) \|x - Tx\|^2 \\
&\quad + k(\alpha^3 L^2 \|x - Tx\|^2 + (1 - \alpha) \|x - Ty\|^2 - \alpha(1 - \alpha) \|x - Tx\|^2) \\
&= \|x - z\|^2 + (\alpha k - \alpha(1 - \alpha) + k(\alpha^3 L^2 - \alpha(1 - \alpha)) \|x - Tx\|^2 \\
&\quad + k(1 - \alpha) \|x - Ty\|^2 \\
&= \|x - z\|^2 + \alpha(kL^2\alpha^2 + (k + 1)\alpha - 1) \|x - Tx\|^2 + k(1 - \alpha) \|x - Ty\|^2 \\
&= \|x - z\|^2 + \beta \|x - Tx\|^2 + \gamma \|x - Ty\|^2 .
\end{aligned}$$

Thus $z \in D$ and hence we obtain that $F(T) \subset D$. Since $F(T)$ is nonempty, so is D. On the other hand, the inequality

$$\|Ty - z\|^2 \leq \|x - z\|^2 + \beta \|x - Tx\|^2 + \gamma \|x - Ty\|^2$$

is equivalent to

$$\|Ty - x\|^2 + 2 \langle x - z, Ty - x \rangle \leq \beta \|x - Tx\|^2 + \gamma \|x - Ty\|^2$$

and therefore D is closed and convex. Consequently we obtain that every C_n is nonempty closed convex subset of C, and hence $\{x_n\}$ is well defined.

Next, let us show (i). From the definition of $\{C_n\}$, one has $C_{n+1} \subset C_n$ for every $n \in \mathbb{N}$. Thus it follows that $\{C_n\}$ converges to $C_0 = \bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. Since $\emptyset \neq F(T) \subset \bigcap_{n=1}^{\infty} C_n$, it follows from Theorem 2.1 that $\{x_n\} = \{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, therefore (i) holds.

To prove (ii), let us suppose that the strong limit x_0 of $\{x_n\}$ belongs to $F(T)$ for every initial point $x \in C$. Let w be an arbitrary point of $F_0 = \bigcap_{u \in C} \bigcap_{n=1}^{\infty} C_n(u)$, where $\{C_n(u)\}$ is a sequence of closed convex subsets defined by (1) with an initial point $u \in C$. Then, since $w \in C_n(w)$ for every $n \in \mathbb{N}$, it follows that $x_n = P_{C_n(w)}w = w$ for every $n \in \mathbb{N}$. This yields that the strong limit of $\{x_n\}$ is w and hence $w \in F(T)$. Therefore we have that $F(T) = F_0$ and thus $F(T)$ is closed and convex. Using (i), we have that the strong limit of $\{x_n\}$ with an initial point $x \in C$ is $P_{C_0}x = P_{F(T)}x$, which completes the proof. □

Now let us consider conditions for the coefficient sequence $\{\alpha_n\}$ and constants k and L which guarantee that the sequence $\{x_n\}$ defined in Theorem 3.1 converges strongly to a fixed point of T.

Theorem 3.2. *Let C be a nonempty closed convex subset of H. Let T be a quasi-pseudocontractive type Lipschitz mapping of C into itself with constants $k > 0$ and $L > 0$, respectively, and suppose that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$. For an initial point $x \in C$, let $\{x_n\}$ be a sequence generated by (1). Suppose that there exists a cluster point α_0 of $\{\alpha_n\}$ which satisfies $\alpha_0 > 0$ and*

$$\frac{k-1}{k} \leq \alpha_0 < \frac{\sqrt{(k+1)^2 + 4kL^2} - (k+1)}{2kL^2}. \tag{2}$$

Then, $F(T)$ is a closed convex set and $\{x_n\}$ converges strongly to $P_{F(T)}x$.

Proof. By Theorem 3.1(i), $\{x_n\}$ converges strongly to $x_0 \in C$. Since $x_0 \in \bigcap_{n=1}^{\infty} C_n$, we have that

$$\|Ty_n - x_0\|^2 \leq \|x_n - x_0\|^2 + \beta_n \|x_n - Tx_n\|^2 + \gamma_n \|x_n - Ty_n\|^2$$

for every $n \in \mathbb{N}$, where $y_n = \alpha_n Tx_n + (1 - \alpha_n)x_n$, $\beta_n = \alpha_n(kL^2\alpha_n^2 + (k+1)\alpha_n - 1)$ and $\gamma_n = k(1 - \alpha_n)$ for $n \in \mathbb{N}$. Let $\{\alpha_{n_i}\}$ be a subsequence of $\{\alpha_n\}$ converging to α_0. Then, using the continuity of T, we have that $\{y_{n_i}\}$ converges strongly to $y_0 = \alpha_0 Tx_0 + (1-\alpha_0)x_0$. Also, $\{\beta_{n_i}\}$ and $\{\gamma_{n_i}\}$ converges to $\beta_0 = \alpha_0(kL^2\alpha_0^2 + (k+1)\alpha_0 - 1)$ and $\gamma_0 = k(1 - \alpha_0)$, respectively. Thus we get that

$$\|Ty_0 - x_0\|^2 \leq \|x_0 - x_0\|^2 + \beta_0 \|x_0 - Tx_0\|^2 + \gamma_0 \|x_0 - Ty_0\|^2$$

and hence

$$-\beta_0 \|x_0 - Tx_0\|^2 \leq (\gamma_0 - 1) \|x_0 - Ty_0\|^2.$$

Since $(k-1)/k \leq \alpha_0$, we have that

$$\gamma_0 - 1 = k(1 - \alpha_0) - 1 \leq k\left(1 - \frac{k-1}{k}\right) - 1 = 0.$$

Further, since

$$\frac{-\sqrt{(k+1)^2 + 4kL^2} - (k+1)}{2kL^2} < 0 < \alpha_0 < \frac{\sqrt{(k+1)^2 + 4kL^2} - (k+1)}{2kL^2},$$

we have that $kL^2\alpha_0^2 + (k+1)\alpha_0 - 1 < 0$ and thus we obtain that

$$\beta_0 = \alpha_0(kL^2\alpha_0^2 + (k+1)\alpha_0 - 1) < 0.$$

Hence we have that $\|x_0 - Tx_0\|^2 \leq 0$, which implies $x_0 \in F(T)$. By Theorem 3.1(ii), $F(T)$ is a closed convex set and $x_0 = P_{F(T)}x$, which is the desired result. □

Suppose that $k > 0$ and $L > 0$ satisfy that

$$0 < k < \frac{(1 + 2L^2) + \sqrt{5 + 4L^2}}{2(1 + L^2)}.$$

We note that

$$0 < \frac{\sqrt{(k+1)^2 + 4kL^2} - (k+1)}{2kL^2} < 1$$

for every $k > 0$ and $L > 0$. If $1 < k$, then it holds that

$$\frac{(1 + 2L^2) - \sqrt{5 + 4L^2}}{2(1 + L^2)} < 1 < k < \frac{(1 + 2L^2) + \sqrt{5 + 4L^2}}{2(1 + L^2)}.$$

From the quadratic formula for k, it follows that $(1 + L^2)k^2 - (1 + 2L^2)k - (1 - L^2) < 0$, or equivalently

$$((1 + 2L^2)k + (1 - 2L^2))^2 < (k + 1)^2 + 4kL^2.$$

Thus we have that

$$2L^2(k - 1) < \sqrt{(k + 1)^2 + 4kL^2} - (k + 1)$$

and since both k and L are positive, it follows that

$$\frac{k - 1}{k} < \frac{\sqrt{(k + 1)^2 + 4kL^2} - (k + 1)}{2kL^2}.$$

On the other hand, if $0 < k \leq 1$, then obviously the inequality above holds since $(k-1)/k \leq 0$. Using this inequality, we have the following result.

Theorem 3.3. *Let C be a nonempty closed convex subset of H. Let T be a quasi-pseudocontractive type Lipschitz mapping of C into itself with constants $k > 0$ and $L > 0$, respectively. Suppose that*

$$0 < k < \frac{(1 + 2L^2) + \sqrt{5 + 4L^2}}{2(1 + L^2)}.$$

Then, $F(T)$ is closed and convex.

Proof. If $F(T) = \emptyset$, then the result is trivial. Thus we may assume that $F(T) \neq \emptyset$. Define a sequence $\{\alpha_n\}$ by

$$\alpha_n = \max\left\{\frac{k - 1}{k}, \frac{\sqrt{(k + 1)^2 + 4kL^2} - (k + 1)}{4kL^2}\right\}$$

for every $n \in \mathbb{N}$. Then, from the calculation above, the cluster point α_0 of $\{\alpha_n\}$ satisfies that

$$\frac{k - 1}{k} \leq \alpha_0 < \frac{\sqrt{(k + 1)^2 + 4kL^2} - (k + 1)}{2kL^2}.$$

Applying Theorem 3.2, we obtain the result. □

Let us show an example of the mapping which is neither pseudocontractive nor strictly pseudocontractive, but satisfies the assumption of this theorem.

Example 3.4. Let U be a mapping of $\mathbb{R}^2$ into itself defined by

$$U(x) = U(x_1, x_2) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta)$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, where $\pi/2 < \theta < \pi$ with $\cos\theta > -1/5$. We have $\langle x - y, Ux - Uy\rangle = \cos\theta \|Ux - Uy\|^2$ for all $x, y \in \mathbb{R}^2$. Let $T = I - U$, where I is the identity mapping on $\mathbb{R}^2$. Then, we get that

$$\|Tx - z\|^2 = \|x - z\|^2 + (1 - 2\cos\theta) \|x - Tx\|^2$$

for every $x \in \mathbb{R}^2$ and $z \in F(T)$. We also have that

$$\|Tx - Ty\| = \sqrt{2(1 - \cos\theta)} \|x - y\|$$

for all $x, y \in \mathbb{R}^2$. Let $k = 1 - 2\cos\theta$ and $L = \sqrt{2(1 - \cos\theta)}$. Then, T is a quasi-pseudocontractive type Lipschitz mapping with constants $L > 0$ and $k > 0$ satisfying that

$$1 < k < \frac{(1 + 2L^2) + \sqrt{5 + 4L^2}}{2(1 + L^2)}.$$

4. **A condition for the sequence of coefficients for convex combination.** Let us consider conditions for $\{\alpha_n\}$ and its cluster point α_0. In Theorem 3.2, we suppose that α_0 satisfies $(k-1)/k \leq \alpha_0 < (\sqrt{(k+1)^2+4kL^2}-(k+1))/(2kL^2)$. For the existence of such α_0, it is sufficient that $k>0$ and $L>0$ satisfy

$$0 < k < \frac{(1+2L^2)+\sqrt{5+4L^2}}{2(1+L^2)}, \tag{3}$$

as we have already seen before. On the other hand, it holds that

$$0 < \frac{\sqrt{(k+1)^2+4kL^2}-(k+1)}{2kL^2} < 1$$

for any $k>0$ and $L>0$. Thus we have that

$$[0,1] \cap \left[\frac{k-1}{k}, \frac{\sqrt{(k+1)^2+4kL^2}-(k+1)}{2kL^2}\right[\neq \emptyset$$

whenever k and L satisfy the inequality (3). Suppose that $\{\alpha_n\} \subset [0,1]$ is dense in $[0,1]$. Then, we may find a cluster point α_0 of $\{\alpha_n\}$ satisfying (2). Namely, such a sequence $\{\alpha_n\}$ always satisfies the condition in Theorem 3.2 whenever k and L satisfy (3). Consequently, we obtain the following result.

Theorem 4.1. *Let C be a nonempty closed convex subset of H. Let T be a quasi-pseudocontractive type Lipschitz mapping of C into itself with constants $k>0$ and $L>0$, respectively, and suppose that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ which is dense in $[0,1]$. For an initial point $x \in C$, let $\{x_n\}$ be a sequence generated by (1). Suppose that*

$$0 < k < \frac{(1+2L^2)+\sqrt{5+4L^2}}{2(1+L^2)}$$

Then, $F(T)$ is a closed convex set and $\{x_n\}$ converges strongly to $P_{F(T)}x$.

We shall show examples satisfying the assumption in this result.

Example 4.2. Let us define a real sequence $\{\alpha_n\}$ by

$$\alpha_n = \frac{1+\cos(\pi r n)}{2}$$

for every $n \in \mathbb{N}$, where r is an irrational number. Then $\{\alpha_n\}$ is a sequence in $[0,1]$ and we may show that it is dense in $[0,1]$. Therefore this sequence satisfies the assumption in Theorem 4.1, that is, for any $k>0$ and $L>0$ satisfying (3), the inequality (2) holds for some cluster point of $\{\alpha_n\}$.

Example 4.3. Let us define a sequence $\{\alpha_n\}$ as follows:

$$\begin{aligned}
&\alpha_1 = 1, \\
&\alpha_2 = \frac{1}{2}, \quad \alpha_3 = \frac{2}{2}, \\
&\alpha_4 = \frac{1}{3}, \quad \alpha_5 = \frac{2}{3}, \quad \alpha_6 = \frac{3}{3}, \\
&\alpha_7 = \frac{1}{4}, \quad \alpha_8 = \frac{2}{4}, \quad \alpha_9 = \frac{3}{4}, \quad \alpha_{10} = \frac{4}{4}, \ldots .
\end{aligned}$$

More precisely, we define this sequence by using a sequence $\{s_k\}$. Let

$$s_k = 0+1+2+\cdots+j = \sum_{j=0}^{k} j$$

for every $k \in \mathbb{N} \cup \{0\}$. Then, for each $i \in \mathbb{N}$, there exists unique $k(i) \in \mathbb{N}$ such that $s_{k(i)-1} < i \leq s_{k(i)}$. Using this mapping $k : \mathbb{N} \to \mathbb{N}$, we define α_n by

$$\alpha_n = \frac{n - s_{k(n)-1}}{k(n)}$$

for each $n \in \mathbb{N}$. Then, the sequence $\{\alpha_n\} \subset [0,1]$ is dense in $[0,1]$ and hence it satisfies the assumption in Theorem 4.1.

For every $L > 0$, we have

$$1 < \frac{(1 + 2L^2) + \sqrt{5 + 4L^2}}{2(1 + L^2)}.$$

Therefore, if $0 < k \leq 1$, then

$$0 < k < \frac{(1 + 2L^2) + \sqrt{5 + 4L^2}}{2(1 + L^2)}$$

holds. Thus we have the following result.

Theorem 4.4. *Let C be a nonempty closed convex subset of H. Let T be a quasi-pseudocontractive type Lipschitz mapping of C into itself with constants $k > 0$ and $L > 0$, respectively, and suppose that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ which is dense in $[0,1]$. For an initial point $x \in C$, let $\{x_n\}$ be a sequence generated by (1). If $0 < k \leq 1$, then $F(T)$ is a closed convex set and $\{x_n\}$ converges strongly to $P_{F(T)}x$.*

Suppose that a mapping $T : C \to C$ has a fixed point. We say that T is quasi-pseudocontractive if

$$\|Tx - z\|^2 \leq \|x - z\|^2 + \|x - Tx\|^2$$

for every $x \in C$ and $z \in F(T)$. Namely, a quasi-pseudocontractive mapping is a quasi-pseudocontractive type mapping with constant 1. As a special case of the previous theorem, we also get the following result for a quasi-pseudocontractive mapping.

Theorem 4.5. *Let C be a nonempty closed convex subset of H. Let T be a quasi-pseudocontractive Lipschitz mapping of C into itself with Lipschitz constant $L > 0$ and suppose that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence in $[0,1]$ which is dense in $[0,1]$. For an initial point $x \in C$, let $\{x_n\}$ be a sequence generated by (1). Then $F(T)$ is a closed convex set and $\{x_n\}$ converges strongly to $P_{F(T)}x$.*

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