MINIMUM TRACKING ERROR PROBLEM FOR JUMP DIFFUSION STOCK-PRICE PROCESS

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Received June 5, 2009

Abstract. This paper is concerned with a problem on optimization in stock price model that allows for jumps in the stochastic processes. We show that the problem is formulated into the optimal stochastic control problem under the criterion of mean square tracking error at the end of the planning horizon. The optimal control and the optimal proportion invested in the index fund which are specified by the product of the market value of risk and the discounted terminal value of bench mark portfolio are derived by the solution to ordinary differential equations associated with an adjoint equation.

1 Introduction. The performance of investment in capital market under uncertainty is often evaluated in comparison with the benchmark portfolio. Some of the most widely known benchmark portfolios are the Standard & Poor's 500 (S & P 500), Financial Times Stock Exchange 100 Share Index, the Tokyo Stock Price Index (TOPIX) and The NIKKEI 225. However, if we actually wish to hold such a benchmark portfolio, we have to make up our minds to hold it at huge transaction, management and information collection costs, since it includes almost all or large number of securities in the market[10]. Thus, it has been emphasized that the portfolio with a small number of securities which closely track the benchmark portfolio is of great concern to risk hedge and to portfolio management. Such a portfolio is called an index fund and in a case in point. So the problem can be regaraded as an optimization problem to minimize the tracking error between the benchmark portfolio and the index fund. In this paper, the tracking error is defined as the mean square error between the returns of benchmark and index fund.

In a discrete time framwork, Green[3], Meada and Salkin[5] developed the relationship between the index fund and the frontier portfolio in the sense of the mean-variance framework. Tabata and Takeda[9] formulated this problem into a quardratic programming problem with 0-1 variables and proposed the efficient method to find an index fund which minimizes the mean square of tracking error in descrete time static model.

On the other hand, much effort has been devoted to the continuous time model in the modern mathematical finance theory. As was pointed out by R. Cont and P. Tankov[1], Lévy processes and other stochastic processes with jumps have become increasingly popular for modelling market fluctuations, both for risk management and option pricing purposes. Applications of the stochastic control to a financial problem, especially to the pricing of derivatives are found in Yong and Zhou[11], Kohlmann and Zhou[4], Framstad [2], Øksendal and Sulem [7] and so on. Mistui and Tabata [4] analized Lévy process by means of the Teugel's martingale and showed the existence of a unique optimal hedging strategy.

In this paper, we derive the optimal index fund based upon the stochastic control technique. In section 2 we give our basic market model with continuous time. Section 3 deals

2000 *Mathematics Subject Classification.* Primary 91B28; Secondary 93E20.
Key words and phrases. tracking error, index fund, stochastic control, jump diffusion process.

with the relationship between our model and the stochastic control problem and developes the method to find the optimal control and the index fund. Section 4 expresses our concluding remarks.

2 Model. Throughout this paper we deal with a financial market consisting of $n+1$ financial assets $S_0, S_1, \cdots, S_n$. Asset S_0 is risk-free called a money market (e.g. bond or bank account). Assets $S_1, S_2, \cdots, S_n$ are risky called stocks. Prices of these risky financial assets are continuously evolved in time and driven by a d-dimensional Brownian motion.

Let us begin with a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with $\mathcal{F}_T = \mathcal{F}$ which is a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$. The price of the money market $S_0(t)$ is gorverned by

$$\mathrm{d}S_0(t) = S_0(t)\rho(t)\mathrm{d}t, \tag{1}$$

where $\rho(t)(>0)$ is an instantaneous risk-free interest rate at time t. Let cádlág (right-continuous with left-hand limit) processes $S_i = \{S_i(t)\}_{0\leq t\leq T},\ i = 1, 2, \cdots, n$ be stochastic processes with representations

$$\begin{aligned} \mathrm{d}S_i(t) = \quad & S_i(t^-)\big[\mu_i(t)\mathrm{d}t + \textstyle\sum_{j=1}^{d} \sigma_{ij}(t)\mathrm{d}B_j \\ & + \textstyle\sum_{k=1}^{l} \int_{\mathbb{R}} \gamma_i(t, z_k)\tilde{N}_k(\mathrm{d}t, \mathrm{d}z_k)\big], \end{aligned} \tag{2}$$

where $\mu_i(t), \sigma_{ij}(t)$ and $\gamma_i(t, z)$ are given bounded deterministic functions. $\gamma; [0,T)\times\mathbb{R}^l\times\Omega \to \mathbb{R}^{n\times l}$ is adapted process such that the integral exists. l denotes the number of discontinuities. To ensure that $S_i(t) \geq 0$ for all t and $i = 1, 2, \cdots, n$, we assume that $\gamma_i(t, z) \geq -1$ a.s. Lévy measure ν. Note that

$$\tilde{N}_k(\mathrm{d}t, \mathrm{d}z_k) = N_k(\mathrm{d}t, \mathrm{d}z_k) - \nu_k(\mathrm{d}z_k)\mathrm{d}t, \quad 1 \leq k \leq l$$

is a compensated Poisson random measure of Lévy process and $N_k(\mathrm{d}t, \mathrm{d}z_k)$ shows the number of jumps of size $\mathrm{d}z_k$ during an infinitesimal time interval $\mathrm{d}t$. $\nu(\mathrm{d}z) = \mathbb{E}N(1, \mathrm{d}z)$ is called Lévy measure and represents the expected number of jumps of size $\mathrm{d}z$. Here $B_j,\ j = 1, \cdots, d$ denotes $d(\leq n)$ independent standard Brownian motions.

Since cádlág functions $S(t)$ can have at most a countable number of l discontinuities, the set $\{t \in [0,T]; S(t) \neq S(t-)\}$ is finite or countable. Also, for any $\epsilon > 0$, the number of discontinuities("jumps") on $[0,T]$ larger than ϵ should be finite. So a cádlág function on $[0,T]$ has a finite number of small jumps (larger than ϵ) and possibly infinite, but countable number of small jumps. If t is a discontinuity point in time we denote by

$$\Delta S(t) = S(t) - S(t-)$$

the "jump" of S at t. We may regard this market as a jump diffusion extension of the classical Black-Scholes market.

A portfolio in this market is defined as an $(n+1)$-dimensional cádlág and adapted process. Let $Y(t)$ be the benchmark or target portfolio process defined on $\mathcal{L}^2(\Omega, \mathcal{F}, P)$ and represented by the linear combination of $S_k(t)$ as

$$Y(t) = \sum_{k=1}^{n} c_k(t)S_k(t), \tag{3}$$

where $\mathcal{F}_t$-mesurable adaptd function $c_k(t)$ expresses the proportion of stock k at time t and satisfies

$$\sum_{k=1}^{n} c_k(t) = 1, \quad c_k(t) \geq 0. \tag{4}$$

Let $x(t)$ be the stochastic process called index fund which consists of m risky assets and the riskfree asset $\rho(t)$. To avoid the difficulty such as NP-completeness for the combinatorial problem and for simplicity, we assume that the index fund $x(t)$ is consist of first m risky securities and riskfree asset $\rho(t)$, whre $m \leq n$. That is

$$x(t) = \sum_{i=0}^{m} W_i(t) S_i(t), \tag{5}$$

where $W_i(t),\ i = 1, \cdots, m$ is unknown proportion invested in asset i and $W_0(t)$ denotes the proportion invested in the riskfree asset $S_0(t)$ at time t.

From equations (1), (2) and (5) it follows that $\mathrm{d}x(t)$ should satisfy the following equation

$$\begin{aligned} \mathrm{d}x(t) &= \sum_{i=0}^{m} W_i(t)\mathrm{d}S_i(t) = W_0(t)\mathrm{d}S_0(t) + \sum_{i=1}^{m} W_i(t)\mathrm{d}S_i(t) \\ &= \rho(t)W_0(t)S_0(t)\mathrm{d}t + \sum_{i=1}^{m} S_i(t^-)W_i(t)\Big\{\mu_i(t)\mathrm{d}t \\ &\quad + \sum_{j=1}^{d} \sigma_{ij}(t)\mathrm{d}B_j(t) + \sum_{k=1}^{l} \int_{\mathbb{R}} \gamma_i(t, z_k)\tilde{N}_k(\mathrm{d}t, \mathrm{d}z_k)\Big\} \\ &= \Big\{\rho(t)x(t) + \sum_{i=1}^{m}(\mu_i(t) - \rho(t))W_i(t)S_i(t^-)\Big\}\mathrm{d}t \\ &\quad + \sum_{i=1}^{m}\Big\{\sum_{j=1}^{d}\big(\sigma_{ij}(t)W_i(t)S_i(t^-)\big)\mathrm{d}B_j(t) \\ &\quad + W_i(t)S_i(t^-)\sum_{k=1}^{l}\int_{\mathbb{R}} \gamma_i(t, z_k)\tilde{N}_k(\mathrm{d}t, \mathrm{d}z_k)\Big\}. \end{aligned} \tag{6}$$

Now, we introduce the m-dimensional excess rate of return vector $\tilde{\boldsymbol{\mu}}(t)$ as

$$\tilde{\boldsymbol{\mu}}(t) = (\mu_1(t) - \rho(t), \ldots, \mu_m(t) - \rho(t))^\top = (\boldsymbol{\mu}(t) - \rho(t)\mathbf{1})^\top,$$

where

$$\boldsymbol{\mu}(t) = (\mu_1, \cdots, \mu_m)^\top, \quad \mathbf{1} = (1, 1, \ldots, 1)^\top.$$

Here '$\top$' denotes transposition. If we define the control vector $\boldsymbol{u}(t) = (u_1, u_2, \cdots, u_m)^\top$ as

$$\boldsymbol{u}(t) = (W_1(t)S_1(t^-), \cdots, W_m(t)S_m(t^-))^\top, \quad W_0(t) + \sum_{i=1}^{m} W_i(t) = 1$$

then equation(6) is rewritten as

$$\begin{aligned} \mathrm{d}x(t) &= \big[\rho(t)x(t) + \tilde{\boldsymbol{\mu}}(t)^\top \boldsymbol{u}(t)\big]\mathrm{d}t + \sum_{j=1}^{d}\big[\boldsymbol{\sigma}(j)^\top \boldsymbol{u}(t)\big]\mathrm{d}B_j(t) \\ &\quad + \int_{\mathbb{R}} \boldsymbol{u}(t)^\top \Gamma(t, z)\tilde{\boldsymbol{N}}(\mathrm{d}t, \mathrm{d}z) \\ &= \big[\rho(t)x(t) + \tilde{\boldsymbol{\mu}}(t)^\top \boldsymbol{u}(t)\big]\mathrm{d}t + \boldsymbol{u}(t)^\top V(t)\mathrm{d}\boldsymbol{B}(t) \\ &\quad + \int_{\mathbb{R}} \boldsymbol{u}(t)^\top \Gamma(t, z)\tilde{\boldsymbol{N}}(\mathrm{d}t, \mathrm{d}z), \end{aligned} \tag{7}$$

where

$$\boldsymbol{\sigma}(j)^\top = (\sigma_{1j}, \sigma_{2j}, \cdots, \sigma_{mj}), \quad j = 1, 2, \cdots, d$$

is an m-dimensional vector and V denotes $m \times d$ matrix with the (i,j)-th element σ_{ij}. $\boldsymbol{B}$ is a d-dimensional vector of the j-th element B_j and $\Gamma(t,z)$ denotes $m \times l$ matrix with the (i,k) element $\gamma_i(z_k)$. The vector $\tilde{\boldsymbol{N}}$ is l-dimensional as

$$\tilde{\boldsymbol{N}}(\mathrm{d}t, \mathrm{d}z)^\top = \big(\tilde{N}_1(\mathrm{d}t, \mathrm{d}z_1), \cdots, \tilde{N}_l(\mathrm{d}t, \mathrm{d}z_l)\big)$$

3 Tracking Error and Optimal Stochastic Control Problem. Our object is to find the optimal control $\boldsymbol{u}(t)$ which minimizes the mean square tracking error between x and Y at the end of planning horizen T,

$$P(1) : \min_{\boldsymbol{u} \in \mathcal{A}} \mathbb{E}[\frac{1}{2}(x(T) - Y(T))^2].$$

Assume that $\boldsymbol{u}$ is adapted and càdlàg and equation (7) has unique strong solution. And the admissible set $\mathcal{A}(t)$ of $\boldsymbol{u}$ is given by

$$\mathcal{A}(t) = \{u_i(t) : \sum_{i=0}^{m} \frac{u_i(t)}{S_i(t)} = 1, \ u_i(t) > 0, \quad i = 0, 1, \ldots, m\}.$$

It should be noted that the objective function of contorl problem $P(1)$ is equivalent to the following maximum problem $P(2)$:

$$P(2) : \max_{\boldsymbol{u} \in \mathcal{A}} \mathbb{E}\big\{ -\frac{1}{2}(x(T) - Y(T))^2 \big\}.$$

Here, we introduce the Hamiltonian $H : [0,T] \times \mathbb{R} \times \mathcal{A} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{R} \to \mathbb{R}$ as

$$\begin{aligned} H(t, x, \boldsymbol{u}, p, \boldsymbol{q}, \boldsymbol{r}) &= \big[\rho(t)x(t) + \tilde{\boldsymbol{\mu}}^\top \boldsymbol{u}\big]p + \boldsymbol{u}^\top V \boldsymbol{q} \\ &\quad + \sum_{k=1}^{l} \sum_{i=1}^{m} \int_{\mathbb{R}} u_i(t)\gamma_i(t, z_k) r_k(t^-, z)\nu_k(\mathrm{d}z_k) \\ &= \big[\rho(t)x(t) + \tilde{\boldsymbol{\mu}}^\top \boldsymbol{u}\big]p + \boldsymbol{u}^\top V \boldsymbol{q} \\ &\quad + \int_{\mathbb{R}} \boldsymbol{u}^\top \Gamma(t, z)\boldsymbol{r}(t^-, z)\nu(\mathrm{d}z), \end{aligned} \tag{8}$$

where $\mathcal{R}$ is a set of functions $r : \mathbb{R}^{l+1} \to \mathbb{R}^l$ such that the integrand of Hamiltonian exists. Hence the adjoint equation for Hamiltonian (8) with the unknown processes $p(t) \in \mathbb{R}$, $\boldsymbol{q} \in \mathbb{R}^d$ and $\boldsymbol{r}(t,z) \in \mathbb{R}^l$ is given by

$$\begin{aligned} \mathrm{d}p(t) &= -\nabla_x H \mathrm{d}t + \boldsymbol{q}(t)^\top \mathrm{d}\boldsymbol{B}(t) + \int_{\mathbb{R}} \boldsymbol{r}(t^-, z)^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z) \\ &= -\rho(t)p(t)\mathrm{d}t + \boldsymbol{q}(t)^\top \mathrm{d}\boldsymbol{B}(t) + \int_{\mathbb{R}} \boldsymbol{r}(t^-, z)^\top \tilde{N}(\mathrm{d}t, \mathrm{d}z) \qquad (9) \\ p(T) &= \nabla_x\big(-\frac{1}{2}(x(T) - Y(T))^2\big) = -(x(T) - Y(T)). \qquad (10) \end{aligned}$$

Here we try a solution of the form

$$p(t) = f(t)x(t) + g(t), \tag{11}$$

where $f(t)$ and $g(t)$ are deterministic C^1 functions. Equation (10) is a boundary condition at time T. Thus, equation (9) is a backward stochastic differential equation. Substituting this equation in equation (9) and using equation (6), we get

$$\begin{aligned} \mathrm{d}p(t) &= f(t)\big[\rho(t)x(t) + \tilde{\boldsymbol{\mu}}(t)^\top \boldsymbol{u}(t) + \boldsymbol{u}(t)^\top V(t)\mathrm{d}\boldsymbol{B}(t) \\ &\quad + \boldsymbol{u}(t^-)^\top \int_{\mathbb{R}} \Gamma(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z)\big] + x(t)f'(t)\mathrm{d}t + g'(t)\mathrm{d}t \\ &= \big[f(t)\rho(t)x(t) + f(t)\tilde{\boldsymbol{\mu}}(t)^\top \boldsymbol{u}(t) + x(t)f'(t) + g'(t)\big]\mathrm{d}t \\ &\quad + f(t)\boldsymbol{u}(t)^\top V(t)\mathrm{d}\boldsymbol{B}(t) + f(t)\boldsymbol{u}(t^-)^\top \int_{\mathbb{R}} \Gamma(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z) \end{aligned} \tag{12}$$

Comparing each coefficient of equation (9) with equation (12), and using equation (11), we obtain the following relations :

$$\begin{aligned} f(t)\rho(t)x(t) &+ f(t)\tilde{\boldsymbol{\mu}}(t)^\top \boldsymbol{u}(t) + x(t)f'(t) + g'(t) \\ &= -\rho(t)(f(t)x(t) + g(t)) \\ \boldsymbol{q}(t)^\top &= f(t)\boldsymbol{u}(t)^\top V(t) \\ \boldsymbol{r}(t,z)^\top &= f(t)\boldsymbol{u}(t)^\top \Gamma(t,z) \end{aligned} \tag{13}$$

Now let $\hat{\boldsymbol{u}} \in \mathcal{A}$ be a candidate for the optimal control with corresponding $\hat{x}, \hat{p}, \hat{\boldsymbol{q}}, \hat{\boldsymbol{r}}(t,\cdot)$. Then

$$\begin{aligned} H(t,\hat{x},\boldsymbol{u},\hat{p},\hat{\boldsymbol{q}},\hat{\boldsymbol{r}}(t,\cdot)) &= \rho(t)\hat{x}(t)\hat{p}(t) + \boldsymbol{u}(t)^\top \Big[\tilde{\boldsymbol{\mu}}(t)\hat{p}(t) \\ &\quad + V(t)\hat{\boldsymbol{q}}(t) + \int_{\mathbb{R}} \Gamma(t,z)\hat{\boldsymbol{r}}(t,z)\nu(\mathrm{d}z)\Big] \end{aligned} \tag{14}$$

Since the righthand side of this equation is a linear expression in $u_i > 0$, H is increasing in u_i if the coefficient of u_i is positive. And H is a decreasing function of u_i if the coefficient of u_i is negative. So, it is natural to guess that each coefficinet of u_i should vanishes in order that H has the optimal value, i.e.

$$\tilde{\boldsymbol{\mu}}(t)\hat{p}(t) + V(t)\hat{\boldsymbol{q}}(t) + \int_{\mathbb{R}} \Gamma(t,z)\hat{\boldsymbol{r}}(t,z)\nu(\mathrm{d}z) = \boldsymbol{0} \tag{15}$$

Using equations (12) and (13), we get

$$\hat{\boldsymbol{q}}(t) = f(t)V(t)^\top \hat{\boldsymbol{u}}(t), \quad \hat{\boldsymbol{r}}(t,x)^\top = f(t)\hat{\boldsymbol{u}}(t)^\top \Gamma(t,x)$$

From this and equation (5),

$$\begin{aligned} V(t)\hat{\boldsymbol{q}}(t) &= f(t)V(t)V(t)^\top \hat{\boldsymbol{u}}(t) \\ &= -\big\{\tilde{\boldsymbol{\mu}}(t)\hat{p}(t) + f(t)\int_{\mathbb{R}} \Gamma(t,z)\Gamma(t,z)^\top \nu(\mathrm{d}z)\hat{\boldsymbol{u}}(t)\big\} \\ &= -\big\{\tilde{\boldsymbol{\mu}}(t)(f(t)\hat{x}(t) + g(t)) \\ &\quad + f(t)\int_{\mathbb{R}} \Gamma(t,z)\Gamma(t,z)^\top \nu(\mathrm{d}z)\hat{\boldsymbol{u}}(t)\big\} \end{aligned} \tag{16}$$

Thus,

$$\begin{aligned} \hat{\boldsymbol{u}}(t) &= -\frac{\big(V(t)V(t)^\top + \int_{\mathbb{R}} \Gamma(t,z)\Gamma(t,z)^\top \nu(\mathrm{d}z)\big)^{-1}}{f(t)}\big(f\hat{x}(t) + g\big)\tilde{\boldsymbol{\mu}}(t) \\ &= -\frac{(\Lambda(t))^{-1}}{f(t)}\big(f\hat{x}(t) + g\big)\tilde{\boldsymbol{\mu}}(t), \end{aligned} \tag{17}$$

where $\Lambda(t)$ is $m \times m$ matrix given by

$$\Lambda(t) = V(t)V(t)^\top + \int_{\mathbb{R}} \Gamma(t,z)\Gamma(t,z)^\top \nu(\mathrm{d}z).$$

So we have

$$\tilde{\boldsymbol{\mu}}(t)^\top \hat{\boldsymbol{u}}(t) = -\frac{\tilde{\boldsymbol{\mu}}(t)^\top (\Lambda(t))^{-1}}{f(t)} \big(f\hat{x}(t) + g\big)\tilde{\boldsymbol{\mu}}(t) \tag{18}$$

On the other hand, by equation (13) $\tilde{\boldsymbol{\mu}}(t)^\top \hat{\boldsymbol{u}}(t)$ is written as

$$\tilde{\boldsymbol{\mu}}(t)^\top \hat{\boldsymbol{u}}(t) = -\frac{(2f\rho + f')\hat{x}(t) + \rho g + g'}{f(t)} \tag{19}$$

Combining equation (18) and (19), we get the following oridinary differential equations :

$$\left.\begin{array}{c} f(t)\tilde{\boldsymbol{\mu}}(t)^\top (\Lambda(t))^{-1}\tilde{\boldsymbol{\mu}}(t) - 2f(t)\rho(t) - f'(t) = 0 \\ (\tilde{\boldsymbol{\mu}}(t)^\top (\Lambda(t))^{-1}\tilde{\boldsymbol{\mu}}(t) - \rho(t))g(t) - g'(t) = 0 \end{array}\right\} \tag{20}$$

$$f(T) = -1, \quad g(T) = Y(T)$$

It is easily verified that the solutions to the ordinary differential equations (20) are given by

$$\left.\begin{array}{c} f(t) = -e^{-\int_t^T (\tilde{\boldsymbol{\mu}}(s)^\top (\Lambda(s))^{-1}\tilde{\boldsymbol{\mu}}(s) - 2\rho(s))\mathrm{d}s} \\ g(t) = Y(T)e^{-\int_t^T (\tilde{\boldsymbol{\mu}}(s)^\top (\Lambda(s))^{-1}\tilde{\boldsymbol{\mu}}(s) - \rho(s))\mathrm{d}s} \end{array}\right\} \tag{21}$$

Note that the pair of processes

$$\hat{p}(t) = f(t)\hat{x}(t) + g(t), \ \hat{\boldsymbol{q}}(t) = f(t)V(t)^\top \hat{\boldsymbol{u}}(t)$$

is the solution to the ajdoint equation and satisfies the sufficient condition of maximum principleiTheorem 3.4 in Øksendal and Sulem[7]).

Using equation (17) the optimal control is given by a feedback control as follows:

$$\begin{aligned} \hat{\boldsymbol{u}}(t,\hat{x}) &= -\frac{(f(t)\hat{x}(t) + g(t))}{f(t)} (\Lambda(t))^{-1}\tilde{\boldsymbol{\mu}}(t) \\ &= -\big[\hat{x}(t)(\Lambda(t))^{-1}\tilde{\boldsymbol{\mu}}(t) + \frac{g(t)}{f(t)}(\Lambda(t))^{-1}\tilde{\boldsymbol{\mu}}(t)\big] \end{aligned} \tag{22}$$

By equation (11) and equation (14), Hamiltonian

$$H(t,\hat{x},\hat{\boldsymbol{u}},\hat{p},\hat{\boldsymbol{q}},\hat{\boldsymbol{r}}) = \rho(t)\hat{x}(t)\hat{p}(t) = \rho(t)f(t)\big(\hat{x}(t) + \frac{g(t)}{2f(t)}\big)^2 - \frac{\rho(t)}{4f(t)}(g(t))^2$$

is maximized at

$$\hat{x}(t) = -\frac{g(t)}{2f(t)}$$

since $f(t) < 0$ for all $t \in [0,T]$. Then the optimal control (22) is expressed as

$$\hat{\boldsymbol{u}}(t) = -\frac{1}{2}\frac{g(t)}{f(t)}(\Lambda(t))^{-1}\tilde{\boldsymbol{\mu}}(t) = \frac{Y(T)}{2}e^{-\int_t^T \rho(s)\mathrm{d}s}(\Lambda(t))^{-1}\tilde{\boldsymbol{\mu}}(t) \tag{23}$$

The $m \times m$ matrix $\Lambda(t)$ works on, unlike Brownian motion, the stock price processes undergoing several abrupt upward jumps. Once the optimal control $\hat{\boldsymbol{u}}$ is derived as equation (23), the optimal proportion $W_i(t)$ invested in the index fund is evaluated by

$$W_i(t) = \frac{\tilde{W}_i(t)}{\sum_{j=0}^{m} \tilde{W}_j(t)}$$

where

$$\tilde{W}_i(t) = \hat{u}_i(t)/S_i(t).$$

Note that for the simplest case of $m = 1$

$$\Lambda(t) = \sigma^2(t) + \int_{\mathbb{R}} \gamma^2(t,z)\nu(\mathrm{d}z),$$

and if there is no jump $\nu = 0$, the above equation is reduced to

$$\Lambda(t) = \sigma^2(t), \Lambda(t)^{-1} = 1/\sigma^2(t).$$

In particular, if $\rho(t), \boldsymbol{\mu}(t)$ and $\Lambda(t)$ are constants in time t, that is, the case of Black-Scholes model, we have

$$\frac{g(t)}{f(t)} = -Y(T)e^{-\rho(T-t)}, \quad \hat{x}(t) = \frac{Y(T)}{2}e^{-\rho(T-t)}$$

and

$$\begin{aligned} \hat{\boldsymbol{u}}(t,x) &= (\Lambda)^{-1}(\boldsymbol{\mu} - \rho\mathbf{1})\big(Y(T)e^{-\rho(T-t)} - \hat{x}(t)\big) \\ &= \frac{Y(T)}{2}(\Lambda)^{-1}(\boldsymbol{\mu} - \rho\mathbf{1})e^{-\rho(T-t)} \end{aligned}$$

This result shows that the optimal control is given by the product of the market value of risk (excess rate of return) $\frac{(\Lambda)^{-1}}{2}(\boldsymbol{\mu} - \rho\mathbf{1})$ and the discounted terminal value of benchmark portfolio $Y(T)e^{-\rho(T-t)}$. Furthermore, we can observe that the optimal index fund is determined only by the terminal value of benchmark portfolio $Y(T)$.

4 Concluding Remarks. The stochastic optimization problem to find the index fund was considered in the framework of the continuous time model with jumps. After the problem was formulated into the optimal stochastic control problem, the optimal index fund was derived based upon the solution of the ordinary differential equations associated with the stochastic control problem. The solutions $f(t)$ and $g(t)$ of the ordinary differential equations (20) were explicitly provided as equation (21).

Generally, the problem of minimizing the tracking error under the given number m of risky assets included in the index fund belongs to a class of combinatorial stochastic control problem which will be formulated as a stochastic quardratic NP-complete problem with zero-one variables. Since it will be very difficult to solve in practice, we supposed that the assets included in the index fund were priviously fixed to avoid such difficulty. This important and difficult problem is left for future research.

Acknowledgement
This research is supported in part by Grant-in-Aid for Scientific Research (C) 20500262 of Japan Society for the Promotion of Science.

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