GENERALIZED CONVOLUTIONS RELATIVE TO THE HARTLEY TRANSFORMS WITH APPLICATIONS

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Abstract. This paper provides eight new generalized convolutions of the Hartley transforms and considers the applications. In particular, normed ring structures of linear space $L^1(\mathbb{R}^d)$ are constructed, and a necessary and sufficient condition for the solvability of an integral equation of convolution type is obtained with an explicit formula of solutions in $L^1(\mathbb{R}^d)$. The advantages of the Hartley transforms and the convolutions constructed in the paper over that of the Fourier transform are discussed.

1 Introduction

The Hartley transform first proposed in 1942 is defined as

$$(H_1 f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \operatorname{cas}(xy) f(y) dy,$$

where $f(x)$ is a function (real or complex) defined on $\mathbb{R}$, and the integral kernel, known as the cosine-and-sine or cas function, is defined as $\operatorname{cas} xy := \cos xy + \sin xy$ (see [2, 16]). The Hartley transform is a spectral transform closely related to the Fourier transform, as the kernels of the Hartley transform is often written as $\operatorname{cas}(xy) = \frac{1-i}{2} e^{ixy} + \frac{1+i}{2} e^{-ixy}$, and the kernel of the Fourier transform is: $e^{-ixy} = \frac{1-i}{2} \operatorname{cas}(xy) + \frac{1+i}{2} \operatorname{cas}(-xy)$. However, the Hartley transform of a real-valued function is real-valued rather than complex as is the case of the Fourier transform. Therefore, the Hartley transform has some advantages over the Fourier transform in the analysis of real signals as it avoids the use of complex arithmetic. Namely, the use of the Hartley transform for solving numerical solutions of problems also brings about some advantages as computers prefer real numbers. Actually, the Hartley transform is getting of greater importance in telecommunications and radio-science, in signal processing, image reconstruction, pattern recognition, and acoustic signal processing (see [2, 3, 4, 16, 20, 33] and references therein). There are the delightful books [2, 3, 22] involved in the one-dimensional and two-dimensional Hartley transforms and the practical problems. However, there is a profound lack of systematically theoretical studies covering the multi-dimensional Hartley transform, except for the parts in [2, 22] and the interesting book of engineerings [3] that are involved in the one-dimensional and two-dimensional Hartley transforms and the practical problems.

In what follows, the multi-dimensional Hartley transform is defined as

$$(H_1 f)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \operatorname{cas}(xy) f(y) dy,$$

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where $(xy) :=<x,y>$. For the briefness of notations in the paper, we consider additionally the transform

$$(H_2 f)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \text{cas}(-xy) f(y) dy.$$

Obviously,

$$(H_1 f)(x) = (H_2 f)(-x), \quad \text{and} \quad (H_1 f(-y))(x) = (H_2 f(y))(x). \tag{1.1}$$

We therefore may call H_1, H_2 the Hartley transforms.

The main aim of this paper is to obtain generalized convolutions for H_1, H_2 and solve some integral equations of convolution type.

The paper is divided into three sections and organized as follows.

Section 2 is the main aim of this paper. Subsection 2.1 recalls some basic operational properties of the Hartley transforms that are useful for proving the theorems in Sections 2, 3. In Subsection 2.2, Theorem 2.4 provides eight new generalized convolutions for H_1, H_2.

Section 3 considers the applications for constructing normed ring structures of $L^1(\mathbb{R}^d)$, and solving integral equations of convolution type. In particular, Subsection 3.1 shows that the space $L^1(\mathbb{R}^d)$, equipped with each of the constructed convolutions, becomes a normed ring with no unit. Subsection 3.2 investigates the integral equations with the kernel of Gaussian type. Under the normally solvable conditions, Theorem 3.2 gives a necessary and sufficient condition for the solvability of an integral equation of convolution type, and obtain the explicit solutions in $L^1(\mathbb{R}^d)$ of the equation.

2 Generalized convolutions

2.1 Operational properties of the Hartley transforms Let $< x, y >$ denote the scalar product of $x, y \in \mathbb{R}^d$, and $|x|^2 =< x, x >$. Denote by $\alpha = (\alpha_1, \dots, \alpha_d)$ the multi-index, i.e. $\alpha_k \in \mathbb{Z}_+,\ k = 1, \dots, d$, and $|\alpha| = \alpha_1 + \dots + \alpha_d$. Let $\mathcal{S}$ denote the set of all functions infinitely differentiable on $\mathbb{R}^d$ such that

$$\sup_{|\alpha| \le N} \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^N |(D_x^\alpha f)(x)| < \infty$$

for $N = 0, 1, 2, \dots$. (see [24]).

The classical multi-dimensional Hermite function $\Phi_\alpha(x)$ is defined by

$$\Phi_\alpha(x) := (-1)^{|\alpha|} e^{\frac{1}{2}|x|^2} D_x^\alpha e^{-|x|^2} \quad \text{(see [23, 31])}.$$

To begin with, we provide a theorem related to the Hermite functions which is useful for proving the theorems in the paper.

Theorem 2.1 ([32]). *Let* $|\alpha| = 4m + k,\ m \in \mathbb{N},\ k = 0, 1, 2, 3.$ *Then*

$$(H_1 \Phi_\alpha)(x) = \begin{cases} \Phi_\alpha(x), & \text{if} \quad k = 0, 1 \\ -\Phi_\alpha(x), & \text{if} \quad k = 2, 3. \end{cases} \tag{2.1}$$

and

$$(H_2 \Phi_\alpha)(x) = \begin{cases} \Phi_\alpha(x), & \text{if} \quad k = 0, 3 \\ -\Phi_\alpha(x), & \text{if} \quad k = 1, 2, \end{cases} \tag{2.2}$$

Proof. Let F, F^{-1} denote the Fourier, and the Fourier inverse transforms

$$(Fg)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i<x,y>} g(y)dy,\ (F^{-1}g)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i<x,y>} g(y)dy,$$

respectively. We first prove a result on the Fourier transform of the Hermite functions similar to (2.1), (2.2). Namely, two following identities hold

$$(F\Phi_\alpha)(x) = (-i)^{|\alpha|}\Phi_\alpha(x), \quad \text{and} \quad (F^{-1}\Phi_\alpha)(x) = (i)^{|\alpha|}\Phi_\alpha(x) \tag{2.3}$$

([31, Theorem 57]). Now let us prove the first identity in (2.3). We have the formula

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\pm i<x,y>-\frac{1}{2}|x|^2} dx = e^{-\frac{1}{2}|y|^2} \tag{2.4}$$

([24, Lemma 7.6]). Obviously,

$$D_x^\alpha e^{\frac{1}{2}|x-iy|^2} = (i)^{|\alpha|} D_y^\alpha e^{\frac{1}{2}|x-iy|^2}. \tag{2.5}$$

Since the function $e^{-\frac{1}{2}|x|^2}$ belongs to $\mathcal{S}$, we can integrate by parts $|\alpha|$ times, and use (2.4), (2.5) to have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \Phi_\alpha(x) e^{-i<x,y>} dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} e^{-i<x,y>} e^{\frac{1}{2}|x|^2} D_x^\alpha e^{-|x|^2} dx = \\
&\int_{\mathbb{R}^d} e^{-|x|^2} D_x^\alpha \left(e^{\frac{1}{2}|x|^2} e^{-i<x,y>} \right) dx = e^{\frac{1}{2}|y|^2} \int_{\mathbb{R}^d} e^{-|x|^2} D_x^\alpha \left(e^{\frac{1}{2}|x-iy|^2} \right) dx = \\
&e^{\frac{1}{2}|y|^2} \int_{\mathbb{R}^d} e^{-|x|^2} (i)^{|\alpha|} D_y^\alpha \left(e^{\frac{1}{2}|x-iy|^2} \right) dx = e^{\frac{1}{2}|y|^2} (i)^{|\alpha|} D_y^\alpha \left(\int_{\mathbb{R}^d} e^{-|x|^2} e^{\frac{1}{2}|x-iy|^2} dx \right) \\
&= e^{\frac{1}{2}|y|^2} (i)^{|\alpha|} D_y^\alpha \left(\int_{\mathbb{R}^d} e^{-i<x,y>-\frac{1}{2}|x|^2} e^{-\frac{1}{2}|y|^2} dx \right) = (2\pi)^{\frac{d}{2}} (i)^{|\alpha|} e^{\frac{1}{2}|y|^2} D_y^\alpha \left(e^{-|y|^2} \right) \\
&= (2\pi)^{\frac{d}{2}} (-i)^{|\alpha|} \left((-1)^{|\alpha|} e^{\frac{1}{2}|y|^2} D_y^\alpha e^{-|y|^2} \right) = (2\pi)^{\frac{d}{2}} (-i)^{|\alpha|} \Phi_\alpha(y).
\end{aligned}$$

The first identity in (2.3) is proved. The second one may be proved in the same way.

We now prove (2.1), (2.2). As the operators are defined on $\mathcal{S}$, we have

$$H_1 = \frac{1}{2}[F + F^{-1}] + \frac{1}{2i}[F^{-1} - F], \quad \text{and } H_2 = \frac{1}{2}[F + F^{-1}] - \frac{1}{2i}[F^{-1} - F]. \tag{2.6}$$

It follows that

$$(H_1\Phi_\alpha)(x) = \frac{1}{2i}\left[(-i)^{|\alpha|} i + (i)^{|\alpha|+1} + (i)^{|\alpha|} - (-i)^{|\alpha|}\right]\Phi_\alpha(x),$$

and

$$(H_2\Phi_\alpha)(x) = \frac{1}{2i}\left[(-i)^{|\alpha|} i + (i)^{|\alpha|+1} - (i)^{|\alpha|} + (-i)^{|\alpha|}\right]\Phi_\alpha(x),$$

Calculating the coefficients in the right sides of two last identities, we get (2.1), and (2.2). The theorem is proved. ☐

Remark 2.1. Different from the Fourier and the Fourier inverse transforms, the Hartley transforms of the Hermite functions are the Hermite functions multiplied by the real constants.

Theorem 2.2 (inversion theorem, [2, 32]). *Assume that* $f \in L^1(\mathbb{R}^d)$ *and* $H_1 f \in L^1(\mathbb{R}^d)$. *Put*

$$f_0(x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (H_1 f)(y) \operatorname{cas}(xy) dy. \tag{2.7}$$

Then $f_0(x) = f(x)$ *for almost every* $x \in \mathbb{R}^d$.

Proof. By using the identities (2.6) and $F^4 = I$ (see [24, Theorem 7.7]), we can easily prove that the Hartley transforms H_1 and H_2 are the continuous, linear, one-to-one mappings of $\mathcal{S}$ onto $\mathcal{S}$, and they are their own inverses, i.e. $H_1^2 = I,\ H_2^2 = I$.

Now let $g \in \mathcal{S}$, and $f \in L^1(\mathbb{R}^d)$ be given. Using Fubini's theorem, we get

$$\int_{\mathbb{R}^d} f(x)(H_1 g)(x) dx = \int_{\mathbb{R}^d} g(y)(H_1 f)(y) dy. \tag{2.8}$$

Inserting $g = H_1(H_1(g))$ into the right-side of (2.8) and using Fubini's theorem, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x)(H_1 g)(x) dx &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (H_1 g)(x) \operatorname{cas}(xy) dx \right) (H_1 f)(y) dy \\
&= \int_{\mathbb{R}^d} (H_1 g)(x) \left(\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (H_1 f)(y) \operatorname{cas}(xy) dy \right) dx = \int_{\mathbb{R}^d} f_0(x)(H_1 g)(x) dx.
\end{aligned}$$

As it is proved above, the functions $H_1 g$ cover all of $\mathcal{S}$. We then have

$$\int_{\mathbb{R}^d} (f_0(x) - f(x)) \Phi(x) dx = 0 \tag{2.9}$$

for every $\Phi \in \mathcal{S}$. Since $\mathcal{S}$ is dense in $L^1(\mathbb{R}^d)$, we get $f_0(x) - f(x) = 0$ for almost every $x \in \mathbb{R}^d$. The theorem is proved. □

Corollary 2.1 (uniqueness theorem). *If* $f \in L^1(\mathbb{R}^d)$ *and if* $Hf = 0$ *in* $L^1(\mathbb{R}^d)$, *then* $f = 0$ *in* $L^1(\mathbb{R}^d)$.

Let $C_0(\mathbb{R}^d)$ denote the supremum-normed Banach space of all continuous functions on $\mathbb{R}^d$ vanishing at infinity. By using (2.6) and Theorem 7.5 in [24], it is possible to prove the following lemma.

Theorem 2.3 (Riemann-Lebesgue lemma). *Transform* H_1 *is a continuous linear map from* $L^1(\mathbb{R}^d)$ *to* $C_0(\mathbb{R}^d)$.

2.2 Generalized convolutions The theory of convolutions of integral transforms has been developed for a long time, and is applied in many fields of mathematics. In recent years, many papers on the convolutions, generalized convolutions, and polyconvolutions for the well-known transforms, most notably those by Fourier, Mellin, Laplace, Hankel, and their applications have been published (see [1, 5, 6, 7, 8, 12, 13, 14, 25, 26, 27, 28, 29, 30, 32]). This subsection provides eight new generalized convolutions for the Hartley transforms.

The nice idea of generalized convolution focuses on the factorization identity. We now deal with the concept of convolutions.

Let U_1, U_2, U_3 be the linear spaces on the field of scalars $\mathcal{K}$, and let V be a commutative algebra on $\mathcal{K}$. Suppose that $K_1 \in L(U_1, V)$, $K_2 \in L(U_2, V)$, $K_3 \in L(U_3, V)$ are the linear operators from U_1, U_2, U_3 to V respectively. Let δ denote an element in algebra V.

Definition 2.1 ([7, 18, 19]). A bilinear map $* : \ U_1 \times U_2 :\longrightarrow U_3$ is called the convolution with weight-element δ for K_3, K_1, K_2 (that in order) if the following identity holds $K_3(*(f,g)) = \delta K_1(f)K_2(g)$, for any $f \in U_1, g \in U_2$. The above identity is called the factorization identity of the convolution.

The image $*(f,g)$ is denoted by $f \underset{K_3,K_1,K_2}{\overset{\delta}{*}} g$. If δ is the unit of V, we say briefly the convolution for K_3, K_1, K_2. In the case of $U_1 = U_2 = U_3$ and $K_1 = K_2 = K_3$, the convolution is denoted simply by $f \underset{K_1}{\overset{\delta}{*}} g$, and by $f \underset{K_1}{*} g$ if δ is the unit of V. The factorization identities play a key role of many applications.

In what follows, we consider $U_1 = U_2 = U_3 = L^1(\mathbb{R}^d)$ with the integral in Lebesgue's sense, and V is the algebra of all measurable functions (real or complex) on $\mathbb{R}^d$.

Put $\gamma(x) := e^{-\frac{1}{2}|x|^2}$. By using $\gamma(x) = \gamma(-x)$, we have

$$\int_{\mathbb{R}^d} \sin(xy)\gamma(y)dy = 0, \quad \text{and} \quad (F\gamma)(x) = (F^{-1}\gamma)(x) = \gamma(x)$$

(see [24, Lemma 7.6]). It is easy to prove that

$$(H_1\gamma)(x) = \gamma(x), \quad \text{and} \quad (H_2\gamma)(x) = \gamma(x). \tag{2.10}$$

The following lemma is useful for proving the proceeding theorem in this subsection.

Lemma 2.1. *The following identity holds:*

$$\frac{e^{-\frac{1}{2}|x|^2}}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)[\cos x(u+v) + \sin x(u+v)]dudv =$$
$$\frac{1}{(2\pi)^{\frac{3d}{2}}} \int_{\mathbb{R}^d} \operatorname{cas}(xy)dy \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)\ e^{-\frac{|y-u-v|^2}{2}}dudv.$$

Proof. Using the identities (2.10), we have

$$\begin{aligned}
&\frac{e^{-\frac{1}{2}|x|^2}}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)[\cos x(u+v) + \sin x(u+v)]dudv \\
&= \frac{1}{(2\pi)^{\frac{3d}{2}}} \int_{\mathbb{R}^d} \operatorname{cas}(xt)e^{\frac{-|t|^2}{2}}dt \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \cos x(u+v)f(u)g(v)dudv \\
&+ \frac{1}{(2\pi)^{\frac{3d}{2}}} \int_{\mathbb{R}^d} \operatorname{cas}(-xt)e^{\frac{-|t|^2}{2}}dt \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sin x(u+v)f(u)g(v)dudv = \\
&\frac{1}{(2\pi)^{\frac{3d}{2}}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big[\operatorname{cas}(xt)\cos x(u+v) + \operatorname{cas}(-xt)\sin x(u+v)\Big]f(u)g(v) \times \\
&e^{-\frac{|t|^2}{2}}dtdudv = \frac{1}{(2\pi)^{\frac{3d}{2}}} \int_{\mathbb{R}^d} \operatorname{cas} x(t+u+v)e^{-\frac{|t|^2}{2}}dt \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)dudv \\
&= \frac{1}{(2\pi)^{\frac{3d}{2}}} \int_{\mathbb{R}^d} \operatorname{cas}(xy)dy \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)e^{-\frac{|y-u-v|^2}{2}}dudv.
\end{aligned}$$

The lemma is proved. $\square$

Theorem 2.4 below presents four generalized convolutions with the weight-function γ for the transforms H_1, H_2.

Theorem 2.4. *If $f, g \in L^1(\mathbb{R}^d)$, then each of the integral transforms* (2.11), (2.12), (2.13), (2.14) *below is the generalized convolution:*

$$(f \overset{\gamma}{\underset{H_1}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)\big[-e^{-\frac{|x+u+v|^2}{2}} + e^{-\frac{|x+u-v|^2}{2}} + e^{-\frac{|x-u+v|^2}{2}} + e^{-\frac{|x-u-v|^2}{2}}\big]dudv, \tag{2.11}$$

$$(f \overset{\gamma}{\underset{H_1,H_2,H_2}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)\big[e^{-\frac{|x+u+v|^2}{2}} + e^{-\frac{|x+u-v|^2}{2}} + e^{-\frac{|x-u+v|^2}{2}} - e^{-\frac{|x-u-v|^2}{2}}\big]dudv, \tag{2.12}$$

$$(f \overset{\gamma}{\underset{H_1,H_2,H_1}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)\big[e^{-\frac{|x+u+v|^2}{2}} + e^{-\frac{|x+u-v|^2}{2}} - e^{-\frac{|x-u+v|^2}{2}} + e^{-\frac{|x-u-v|^2}{2}}\big]dudv, \tag{2.13}$$

$$(f \overset{\gamma}{\underset{H_1,H_1,H_2}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)\big[e^{-\frac{|x+u+v|^2}{2}} - e^{-\frac{|x+u-v|^2}{2}} + e^{-\frac{|x-u+v|^2}{2}} + e^{-\frac{|x-u-v|^2}{2}}\big]dudv. \tag{2.14}$$

Proof. Let us first prove $(f \overset{\gamma}{\underset{H_1}{*}} g) \in L^1(\mathbb{R}^d)$. Indeed, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} |(f \overset{\gamma}{\underset{H_1}{*}} g)|(x)dx \le \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(u)||g(v)|e^{-\frac{|x+u+v|^2}{2}}dudvdx\\
&+\frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(u)||g(v)|e^{-\frac{|x+u-v|^2}{2}}dudvdx\\
&+\frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(u)||g(v)|e^{-\frac{|x-u+v|^2}{2}}dudvdx\\
&+\frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(u)||g(v)|e^{-\frac{|x-u-v|^2}{2}}dudvdx < +\infty.
\end{aligned}$$

The same line of proof works for the integral transforms (2.12), (2.13), (2.14). Therefore, it suffices to prove the factorization identities for these transforms.

We now prove the factorization identity of the convolution (2.11). Using Lemma 2.1 and replacing u with $-u$, and v with $-v$, when it is necessary, we have

$$\begin{aligned}
\gamma(x)(H_1f)(x)(H_1g)(x) &= \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)\operatorname{cas}(xu)\operatorname{cas}(xv)dudv\\
&= -\frac{e^{-\frac{|x|^2}{2}}}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)[\cos x(u+v) - \sin x(u+v)]dudv\\
&+\frac{e^{-\frac{|x|^2}{2}}}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)[\cos x(u-v) - \sin x(u-v)]dudv\\
&+\frac{e^{-\frac{|x|^2}{2}}}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)[\cos x(u-v) + \sin x(u-v)]dudv\\
&+\frac{e^{-\frac{|x|^2}{2}}}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)[\cos x(u+v) + \sin x(u+v)]dudv\\
&= \frac{1}{2(2\pi)^{3d/2}}\int_{\mathbb{R}^d}\operatorname{cas}(xy)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)\big[-e^{-\frac{|y+u+v|^2}{2}} + e^{-\frac{|y+u-v|^2}{2}}
\end{aligned}$$

$$+ e^{-\frac{|y-u+v|^2}{2}} + e^{-\frac{|y-u-v|^2}{2}}\big]dudvdy = H_1(f \underset{H_1}{\overset{\gamma}{*}} g)(x),$$

as desired.

Proof of the factorization identities for the convolutions (2.12), (2.13), (2.14). We write $\check{f}(x) := f(-x)$, $\check{g}(x) := g(-x)$. Using the factorization identity of the convolution (2.11) and replacing u with $-u$, v with $-v$, we obtain

$$\gamma(x)(H_2f)(x)(H_2g)(x) = \gamma(x)(H_1\check{f})(x)(H_1\check{g})(x) = H_1\big(\check{f} \underset{H_1}{\overset{\gamma}{*}} \check{g}\big)(x) =$$
$$\frac{1}{2(2\pi)^{3d/2}}\int_{\mathbb{R}^d} \operatorname{cas}(xy)dy \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(-u)g(-v)\big[-e^{-\frac{|y+u+v|^2}{2}} + e^{-\frac{|y+u-v|^2}{2}}$$
$$+ e^{-\frac{|y-u+v|^2}{2}} + e^{-\frac{|y-u-v|^2}{2}}\big]dudv = H_1(f \underset{H_1,H_2,H_2}{\overset{\gamma}{*}} g)(x).$$

Similarly, the factorization identities for the convolutions (2.13), (2.14) can be proved. The theorem is proved. □

Corollary 2.2. *If $f, g \in L^1(\mathbb{R}^d)$, then each of the integral transforms* (2.4a), (2.5a), (2.6a), (2.7a) *below defines the generalized convolution:*

$$(f \underset{H_2}{\overset{\gamma}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)\big[-e^{-\frac{|x+u+v|^2}{2}}$$
$$+ e^{-\frac{|x+u-v|^2}{2}} + e^{-\frac{|x-u+v|^2}{2}} + e^{-\frac{|x-u-v|^2}{2}}\big]dudv, \tag{2.4a}$$

$$(f \underset{H_2,H_1,H_1}{\overset{\gamma}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)\big[e^{-\frac{|x+u+v|^2}{2}}$$
$$+ e^{-\frac{|x+u-v|^2}{2}} + e^{-\frac{|x-u+v|^2}{2}} - e^{-\frac{|x-u-v|^2}{2}}\big]dudv, \tag{2.5a}$$

$$(f \underset{H_2,H_1,H_2}{\overset{\gamma}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)\big[e^{-\frac{|x+u+v|^2}{2}}$$
$$+ e^{-\frac{|x+u-v|^2}{2}} - e^{-\frac{|x-u+v|^2}{2}} + e^{-\frac{|x-u-v|^2}{2}}\big]dudv, \tag{2.6a}$$

$$(f \underset{H_2,H_2,H_1}{\overset{\gamma}{*}} g)(x) = \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(u)g(v)\big[e^{-\frac{|x+u+v|^2}{2}}$$
$$- e^{-\frac{|x+u-v|^2}{2}} + e^{-\frac{|x-u+v|^2}{2}} + e^{-\frac{|x-u-v|^2}{2}}\big]dudv. \tag{2.7a}$$

Proof. By (2.11), we have $\Big(H_1(f \underset{H_1}{\overset{\gamma}{*}} g)\Big)(x) = \gamma(x)(H_1f)(x)(H_1g)(x)$. Replacing x with $-x$ in this identity and using (1.1), we obtain (2.4a). In the same way as above, the convolutions (2.5a), (2.6a), (2.7a) can be proved. □

3 Applications

3.1 Normed ring structures on $L^1(\mathbb{R}^d)$ In the theory of normed rings, the multiplication of two elements can be a convolution. This section proves that $L^1(\mathbb{R}^d)$, equipped with each of the convolution multiplications in Section 2 and an appropriate norm, becomes a normed ring. Some of them are commutative. Also, the space $L^1(\mathbb{R}^d)$ could be a commutative Banach algebra.

Definition 3.1 (see Naimark [21]). A vector space V with a ring structure and a vector norm is called the normed ring if $\|vw\| \le \|v\|\|w\|$, for all $v, w \in V$.
If V has a multiplicative unit element e, it is also required that $\|e\| = 1$.

Let X denote the linear space $L^1(\mathbb{R}^d)$. For the convolutions in Theorem 2.4 the norm for $f \in X$ is chosen as

$$\|f\| = \frac{2}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(x)| dx.$$

Theorem 3.1. X, *equipped with each of the convolution multiplications in Theorem* 2.4, *becomes a normed ring having no unit. Moreover,*

(a) *The convolution multiplications* (2.11) *and* (2.12) *are commutative.*

(b) *The convolution multiplications* (2.13) *and* (2.14) *are non-commutative.*

Proof. The proof for the first statement is divided into two steps.

Step 1. X has a normed ring structure. It is clear that X, equipped with each of those convolution multiplications, has a ring structure. We have to prove the multiplicative inequality. We now prove the inequality for (2.11). The other cases can be proved similarly. Using the following formula

$$\int_{\mathbb{R}^d} e^{-\frac{|x\pm u\pm v|^2}{2}} dx = (2\pi)^{\frac{d}{2}} \quad (u, v \in \mathbb{R}^d),$$

we have

$$\begin{aligned} \frac{2}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f \overset{\gamma}{\underset{H_1}{*}} g|(x) dx \leq \frac{1}{(2\pi)^{\frac{3d}{2}}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(u)||g(v)| \Big[e^{-\frac{|x+u+v|^2}{2}} + e^{-\frac{|x+u-v|^2}{2}} \\ + e^{-\frac{|x-u+v|^2}{2}} + e^{-\frac{|x-u-v|^2}{2}} \Big] dudvdx \leq \frac{4}{(2\pi)^d} \Big(\int_{\mathbb{R}^d} |f(u)| du \Big) \Big(\int_{\mathbb{R}^d} |g(v)| dv \Big) \\ = \Big(\frac{2}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(u)| du \Big) \Big(\frac{2}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |g(v)| dv \Big). \end{aligned}$$

Thus, $\|f \overset{\gamma}{\underset{H_1}{*}} g\| \leq \|f\|.\|g\|$.

Step 2. X has no unit. For briefness of our proof, let us use the abbreviation $f*g$ for $f \overset{\gamma}{\underset{H_1}{*}} g$, $f \overset{\gamma}{\underset{H_1,H_2,H_2}{*}} g$, $f \overset{\gamma}{\underset{H_1,H_2,H_1}{*}} g$, or $f \overset{\gamma}{\underset{H_1,H_1,H_2}{*}} g$. Suppose that there exists an $e \in X$ such that $f = f * e = e * f$ for every $f \in X$. We then have $\Phi_0 = \Phi_0 * e = e * \Phi_0$. By the factorization identity of convolutions, we get $\mathcal{H}_j \Phi_0 = \gamma \mathcal{H}_k \Phi_0 \mathcal{H}_l e$, where $\mathcal{H}_j, \mathcal{H}_k, \mathcal{H}_l \in \{H_1, H_2\}$ (e.g. $\mathcal{H}_j = \mathcal{H}_k = H_1$, etc). By using Theorem 2.1 and $\Phi_0(x) \neq 0$ for every $x \in \mathbb{R}^d$, $\gamma(x)(\mathcal{H}_l e)(x) = 1$ for every $x \in \mathbb{R}^d$. The last identity fails, as $\lim\limits_{x\to\infty} \big[\gamma(x)(H_l e)(x)\big] = 0$. Hence, X has no unit.

We now prove the last statements of the theorem.

(a) Obviously, convolution multiplications (2.11) and (2.12) are commutative.

(b) Consider the convolution multiplication (2.13). Choose the multi-indexes α, β so that $|\alpha| = 4m, |\beta| = 4n + 1$. Using the factorization identity and Theorem 2.1, we get $H_1(\Phi_\alpha \overset{\gamma}{\underset{H_1,H_2,H_1}{*}} \Phi_\beta) = \gamma \Phi_\alpha \Phi_\beta$, and $H_1(\Phi_\beta \overset{\gamma}{\underset{H_1,H_2,H_1}{*}} \Phi_\alpha) = -\gamma \Phi_\alpha \Phi_\beta$. It follows that $(\Phi_\alpha \overset{\gamma}{\underset{H_1,H_2,H_1}{*}} \Phi_\beta)(x) \not\equiv 0$, $(\Phi_\beta \overset{\gamma}{\underset{H_1,H_2,H_1}{*}} \Phi_\alpha)(x) \not\equiv 0$, and $H_1(\Phi_\alpha \overset{\gamma}{\underset{H_1,H_2,H_1}{*}} \Phi_\beta) = -H_1(\Phi_\beta \overset{\gamma}{\underset{H_1,H_2,H_1}{*}} \Phi_\alpha)$. By Corollary 2.1, $\Phi_\alpha \overset{\gamma}{\underset{H_1,H_2,H_1}{*}} \Phi_\beta \neq \Phi_\beta \overset{\gamma}{\underset{H_1,H_2,H_1}{*}} \Phi_\alpha$. Thus, the convolution multiplication (2.13) is not commutative.

The non-commutativity of the convolution multiplication (2.14) is proved in the same way. The theorem is proved. ∎

3.2 Integral equations with the kernel of Gaussian type Consider the equation

$$\lambda\varphi(x) + \frac{2}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big[k_1(u)e^{\frac{-|x+u+v|^2}{2}} + k_2(u)e^{\frac{-|x-u-v|^2}{2}}\Big]\varphi(v)dudv = p(x). \tag{3.1}$$

where $\lambda \in \mathbb{C}$ is predetermined, $k_1(x), k_2(x), p(x)$ are given, and $\varphi(x)$ is to be determined. In what follows, given functions are assumed to belong to $L^1(\mathbb{R}^d)$, and unknown function will be determined there; the functional identity $f(x) = g(x)$ means that it is valid for almost every $x \in \mathbb{R}^d$. However, if the functions f, g are continuous, there should be emphasis that the identity $f(x) = g(x)$ is true for every $x \in \mathbb{R}^d$.

In equation (3.1), the function

$$K(x,v) = \frac{2}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\Big[k_1(u)e^{\frac{-|x-u+v|^2}{2}} + k_2(u)e^{\frac{-|x-u-v|^2}{2}}\Big]du \tag{3.2}$$

is considered as the kernel. It is easily seen that if the functions $k_1(u), k_2(u)$ in (3.2) are of the Gaussian type, so is $K(x,v)$.
Convolution integral equations with Gaussian kernels have some applications in Physics, Medicine and Biology (see [9, 10, 11]).

Write:

$$\begin{aligned}
\boldsymbol{A}(x) &:= \lambda - \gamma(x)(H_1k_1)(x) + \gamma(x)(H_2k_1)(x) + \gamma(x)(H_1k_2)(x) + \gamma(x)(H_2k_2)(x);\\
\boldsymbol{B}(x) &:= \gamma(x)(H_2k_1)(x) + \gamma(x)(H_1k_1)(x) - \gamma(x)(H_2k_2)(x) + \gamma(x)(H_1k_2)(x);\\
\boldsymbol{D_{H_1,H_2}}(x) &:= \boldsymbol{A}(x)\boldsymbol{A}(-x) - \boldsymbol{B}(x)\boldsymbol{B}(-x);\\
\boldsymbol{D_{H_1}}(x) &:= \boldsymbol{A}(-x)(H_1p)(x) - \boldsymbol{B}(x)(H_2p)(x);\\
\boldsymbol{D_{H_2}}(x) &:= \boldsymbol{A}(x)(H_2p)(x) - \boldsymbol{B}(-x)(H_1p)(x).
\end{aligned} \tag{3.3}$$

Theorem 3.2. *Assume that* $\boldsymbol{D_{H_1,H_2}}(x) \neq 0$ *for every* $x \in \mathbb{R}^d$, *and* $\dfrac{\boldsymbol{D_{H_1}}}{\boldsymbol{D_{H_1,H_2}}} \in L^1(\mathbb{R}^d)$. *Equation* (3.1) *has solution in* $L^1(\mathbb{R}^d)$ *if and only if*

$$H_1\left(\frac{\boldsymbol{D_{H_1}}}{\boldsymbol{D_{H_1,H_2}}}\right) \in L^1(\mathbb{R}^d). \tag{3.4}$$

If condition (3.4) *is satisfied, then the solution of* (3.1) *is given in an explicit form* $\varphi(x) = H_1\left(\frac{\boldsymbol{D_{H_1}}}{\boldsymbol{D_{H_1,H_2}}}\right)$.

Proof. From convolutions (2.11), (2.12), (2.13), (2.14) it follows that

$$\begin{aligned}
\frac{2}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{-|x+u+v|^2}{2}} f(u)g(v)dudv &= -(f \overset{\gamma}{\underset{H_1}{*}} g)(x)\\
&\quad + (f \overset{\gamma}{\underset{H_1,H_2,H_2}{*}} g)(x) + (f \overset{\gamma}{\underset{H_1,H_1,H_2}{*}} g)(x) + (f \overset{\gamma}{\underset{H_1,H_2,H_1}{*}} g)(x),
\end{aligned}$$

and

$$\begin{aligned}
\frac{2}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{-|x-u-v|^2}{2}} f(u)g(v)dudv &= (f \overset{\gamma}{\underset{H_1}{*}} g)(x)\\
&\quad - (f \overset{\gamma}{\underset{H_1,H_2,H_2}{*}} g)(x) + (f \overset{\gamma}{\underset{H_1,H_1,H_2}{*}} g)(x) + (f \overset{\gamma}{\underset{H_1,H_2,H_1}{*}} g)(x).
\end{aligned}$$

Using the factorization identities of those convolutions, we get

$$(3.5)\quad H_1\left(\frac{2}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{-|x+u+v|^2}{2}} f(u)g(v)dudv\right)(x) = \gamma(x)\Big[-(H_1f)(x)(H_1g)(x) + (H_2f)(x)(H_2g)(x) + (H_1f)(x)(H_2g)(x) + (H_2f)(x)(H_1g)(x)\Big],$$

and

$$(3.6)\quad H_1\left(\frac{2}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{\frac{-|x-u-v|^2}{2}} f(u)g(v)dudv\right)(x) = \gamma(x)\Big[(H_1f)(x)(H_1g)(x) - (H_2f)(x)(H_2g)(x) + (H_1f)(x)(H_2g)(x) + (H_2f)(x)(H_1g)(x)\Big].$$

Necessity. Suppose that equation (3.1) has a solution $\varphi \in L^1(\mathbb{R}^d)$. Applying H_1 to both sides of (3.1) and using (3.5), (3.6), we obtain

$$(3.7)\qquad \boldsymbol{A}(x)(H_1\varphi)(x) + \boldsymbol{B}(x)(H_2\varphi)(x) = (H_1p)(x),$$

where $\boldsymbol{A}(x), \boldsymbol{B}(x)$ are defined as in (3.3). In equation (3.7), replacing x with $-x$, we get the system of two linear equations

$$(3.8)\qquad \begin{cases} \boldsymbol{A}(x)(H_1\varphi)(x) + \boldsymbol{B}(x)(H_2\varphi)(x) = (H_1p)(x) \\ \boldsymbol{B}(-x)(H_1\varphi)(x) + \boldsymbol{A}(-x)(H_2\varphi)(x) = (H_2p)(x), \end{cases}$$

where $(H_1\varphi)(x)$, $(H_2\varphi)(x)$ are the unknown functions. The determinants of (3.8) are defined as in (3.3). By $\boldsymbol{D_{H_1,H_2}}(x) \neq 0$ for every $x \in \mathbb{R}^d$, we get $(H_1\varphi)(x) = \frac{\boldsymbol{D_{H_1}}(x)}{\boldsymbol{D_{H_1,H_2}}(x)}$. We now can apply Theorem 2.2 to obtain $\varphi(x) = H_1\Big(\dfrac{\boldsymbol{D_{H_1}}}{\boldsymbol{D_{H_1,H_2}}}\Big)(x)$. Thus, $H_1\Big(\dfrac{\boldsymbol{D_{H_1}}}{\boldsymbol{D_{H_1,H_2}}}\Big) \in L^1(\mathbb{R}^d)$. The necessity is proved.

Sufficiency. Obviously, $\frac{\boldsymbol{D_{H_2}}(x)}{\boldsymbol{D_{H_1,H_2}}(x)} = \frac{\boldsymbol{D_{H_1}}(-x)}{\boldsymbol{D_{H_1,H_2}}(-x)}$. It follows that $\frac{\boldsymbol{D_{H_2}}(x)}{\boldsymbol{D_{H_1,H_2}}(x)} \in L^1(\mathbb{R}^d)$. It is easy to prove that $H_1\left(\dfrac{\boldsymbol{D_{H_1}}}{\boldsymbol{D_{H_1,H_2}}}\right)(x) = H_2\left(\dfrac{\boldsymbol{D_{H_2}}}{\boldsymbol{D_{H_1,H_2}}}\right)(x)$. Consider the function

$$\varphi(x) = H_1\left(\frac{\boldsymbol{D_{H_1}}}{\boldsymbol{D_{H_1,H_2}}}\right)(x) = H_2\left(\frac{\boldsymbol{D_{H_2}}}{\boldsymbol{D_{H_1,H_2}}}\right)(x).$$

This implies $\varphi \in L^1(\mathbb{R}^d)$. By Theorem 2.2,

$$(H_1\varphi)(x) = \frac{\boldsymbol{D_{H_1}}(x)}{\boldsymbol{D_{H_1,H_2}}(x)}, \quad \text{and} \quad (H_2\varphi)(x) = \frac{\boldsymbol{D_{H_2}}(x)}{\boldsymbol{D_{H_1,H_2}}(x)}.$$

Hence, two functions $(H_1\varphi)(x), (H_2\varphi)(x)$ together fulfill (3.8). We thus have

$$\boldsymbol{A}(x)(H_1\varphi)(x) + \boldsymbol{B}(x)(H_2\varphi)(x) = (H_1p)(x).$$

This equation coincides with exactly the equation

$$H_1\left(\lambda\varphi(x) + \frac{2}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big[k_1(u)e^{\frac{-|x+u+v|^2}{2}} + k_2(u)e^{\frac{-|x-u-v|^2}{2}}\Big]\varphi(v)dudv\right)(x) = (H_1p)(x).$$

By Theorem 2.2, $\varphi(x)$ fulfills equation (3.1) for almost every $x \in \mathbb{R}^d$. The theorem is proved. $\square$

In the general theory of integral equations, the requirement that $D_{H_1,H_2}(x) \neq 0$ for every $x \in \mathbb{R}^d$ as in Theorem 3.2 is the normally solvable condition of the equation.

It is known that (3.1) is a Fredholm integral equation of first kind if $\lambda = 0$, and that of second kind if $\lambda \neq 0$. For the second kind, Proposition 3.1 below is the illustration of the conditions appearing in Theorem 3.2.

Proposition 3.1. *Let $\lambda \neq 0$.*

(i) $\boldsymbol{D_{H_1,H_2}}(x) \neq 0$ *for every x outside a ball with a finite radius.*

(ii) *Suppose that $k_1, k_2, p \in L^1(\mathbb{R}^d)$. If $\boldsymbol{D_{H_1,H_2}}(x) \neq 0$ for every $x \in \mathbb{R}^d$, and if $H_1p \in L^1(\mathbb{R}^d)$, then $\dfrac{\boldsymbol{D_{H_1}}}{\boldsymbol{D_{H_1,H_2}}} \in L^1(\mathbb{R}^d)$.*

Proof. (i) By the Riemann-Lebesgue lemma for the Hartley integral transforms, it is easily seen that the function $\boldsymbol{D_{H_1,H_2}}(x)$ is continuous on $\mathbb{R}^d$ and $\lim\limits_{|x|\to\infty} \boldsymbol{D_{H_1,H_2}}(x) = \lambda^2$. Now the part (i) follows from $\lambda \neq 0$ and the continuity of $\boldsymbol{D_{H_1,H_2}}$.
(ii) By the continuity of $\boldsymbol{D_{H_1,H_2}}$ and $\lim\limits_{|x|\to\infty} \boldsymbol{D_{H_1,H_2}}(x) = \lambda^2 \neq 0$, there exist $R > 0, \epsilon_1 > 0$ so that $\inf\limits_{|x|>R} |\boldsymbol{D_{H_1,H_2}}(x)| > \epsilon_1$. Since $\boldsymbol{D_{H_1,H_2}}$ does not vanish in the compact set $S(0,R) = \{x \in \mathbb{R}^d : |x| \leq R\}$, there exists $\epsilon_2 > 0$ so that $\inf\limits_{|x|\leq R} |\boldsymbol{D_{H_1,H_2}}(x)| > \epsilon_2$. We then have $\sup\limits_{x\in\mathbb{R}^d} \dfrac{1}{|\boldsymbol{D_{H_1,H_2}}(x)|} \leq \max\{\dfrac{1}{\epsilon_1}, \dfrac{1}{\epsilon_2}\} < \infty$. It follows that the function $\dfrac{1}{|\boldsymbol{D_{H_1,H_2}}(x)|}$ is continuous and bounded on $\mathbb{R}^d$. Therefore, $\dfrac{\boldsymbol{D_{H_1}}}{\boldsymbol{D_{H_1,H_2}}} \in L^1(\mathbb{R}^d)$, provided $\boldsymbol{D_{H_1}} \in L^1(\mathbb{R}^d)$. We shall prove that if $H_1p \in L^1(\mathbb{R}^d)$, then $\boldsymbol{D_{H_1}} \in L^1(\mathbb{R}^d)$. Indeed, as $(H_2p)(x) = (H_1p)(-x)$, $H_2p \in L^1(\mathbb{R}^d)$. Since the functions $\boldsymbol{A}(x), \boldsymbol{B}(x)$ are continuous and bounded on $\mathbb{R}^d$ and $H_1p, H_2p \in L^1(\mathbb{R}^d)$, we have $\boldsymbol{D_{H_1}} \in L^1(\mathbb{R}^d)$. The proposition is proved. □

Remark 3.1. The equation with four terms in kernel

$$\lambda\varphi(x) + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Big[k_1(u) e^{\frac{-|x+u+v|^2}{2}} + k_2(u) e^{\frac{-|x-u+v|^2}{2}} + k_3(u) e^{\frac{-|x+u-v|^2}{2}} + k_4(u) e^{\frac{-|x-u-v|^2}{2}} \Big] \varphi(v) du dv = p(x)$$

can be reduced to an equation of the form (3.1) by changing variable u by $-u$ in the second and third terms of the inner integral functions, and grouping $k_2(-u), k_3(-u)$ with $k_1(u), k_4(u)$, respectively.

Comparison. a) By constructing some generalized convolutions, the papers [19, 25, 26, 27, 28, 29, 30] solved their integral equations. By using the Wiener-Lèvy theorem, those papers provided the sufficient conditions for the solvability and obtained the implicit solutions of those equations (see ones more [15, 17]).
By means of the normally solvable conditions of system of functional equations, the generalized convolutions for H_1, H_2 in Theorem 2.4 work out the necessary and sufficient condition and the explicit solutions of the equations.

b) The Hartley transforms have the additional advantage of being their own inverses. The convolution transforms in Theorem 2.4 and their corollaries do not contain any complex coefficient. Therefore, if the objects in integral equations are real-valued, then the use of generalized convolutions in those theorems and the inverse Hartley transforms brings about the remarkable advantage computationally over that of Fourier's.

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