FUZZY TRANSPORTATION PROBLEM WITH RANDOM TRANSPORTATION COSTS

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Abstract. We consider the following transportation problem with both random and fuzzy factors. There exist m supply points and n demand points. For each route between supply point and demand point, unit transportation cost is a random variable according to a normal distribution and existence possibility denoting the preference choosing this route is attached. The probability that the total transportation cost is not greater than the budget F should be not less than the fixed probability level. Under the above setting, we seek transportation pattern minimizing F and maximizing the minimal preference among the routes used in a transportation. Since usually there is no transportation pattern optimizing two objectives at a time, we propose a solution algorithm to find some non-dominated transportation patterns after defining non-domination. Finally we discuss the further research problems.

1. Introduction.

The purpose of traditional transportation problem is to determine the optimal transportation pattern of a certain good from supplies to demand customers so that the total transportation cost becomes minimum. It is investigated by many researchers and known as Hitchcock Koopman transportation problem. Typical solution methods are that of using maximum flow algorithm [3], Hungarian method [5] and combinatorial one [10] etc. This paper extends the classical transportation problem by considering preference of arc in a transportation route and randomness of unit transportation cost of each route. Randomness means that transportation cost may change according to many factors. So bi-criteria are taken into account in this paper. One is to maximize the minimal preference among the routes used in a transportation. Under the condition that the probability such that the total transportation cost is not over the budget F is not below the prescribed level, minimizing the budget F is the other criterion. But usually there exists no solution that optimizes two objectives at a time. So we seek some non-dominated transportation patterns after defining non-domination. Our model is extension of our previous models [2, 7, 9, 11, 12]. As for another fuzzy version, we have considered competitive transportation problem also in order to cope with an actual situation [6]. While Ahuja et al. [1] have considered a random transportation problem and proposed an efficient solution algorithm. Their algorithm is very useful to our problem in order to construct an efficient solution algorithm.

Section 2 formulates our problem and defines non-domination. Section 3 proposes an efficient algorithm to find non-dominated transportation patterns. Finally Sections 4 concludes this paper and discusses further research problems.

2. Problem Formulation.

In this paper, we focus on the following bi-criteria transportation problem on a fuzzy network.

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(1) There exist a set of m supply points $S = \{s_1, s_2, \cdots, s_m\}$ and a set of n demand points $T = \{t_1, t_2, \cdots, t_n\}$.

(2) Edges set A is a set of route connecting each supply point s_i with each demand point t_j denoted by $(i, j),\ i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n$. The total upper limit provided from each supply point s_i is a_i and the total lower limit to each demand point t_j is b_j. Further we assume that these $a_i,\ b_j$ are positive integers and $\sum_{i=1}^m a_i \geq \sum_{j=1}^n b_j$.

(3) For each route from s_i to t_j, positive cost c_{ij} for unit transportation of a good is a random variable according to the normal distribution $N(m_{ij}, \sigma_{ij}^2)$ and they are independent each other. Further preference of the route is also attached and it is denoted by $\mu_{ij} \in (0, 1]$. It reflects on the satisfaction degree using this route. We denote the transportation quantity using the route (i, j) with f_{ij} and assume that these f_{ij} are integer decision variables.

(4) As for total cost $\sum_{i=1}^m \sum_{j=1}^n c_{ij} f_{ij}$, the following chance constraint is attached:

$$\Pr\{\sum_{i=1}^m \sum_{j=1}^n c_{ij} f_{ij} \leq F\} \geq \alpha$$

where $\alpha > 1/2$ and F is also a decision variable denoting the budget to be minimized.

(5) We consider bi-criteria. One is to maximize the minimal preference among the used route. The other is to minimize the budget F. Generally speaking, there exists no transportation pattern optimizing bi-criteria at a time and so we seek non-dominated patterns which definition is given soon after the formulation of our problem.

¿From above setting, the transportation problem is formulated as follows:

$$
\begin{array}{lll}
\text{TP}: & \text{Minimize} & F \\
 & \text{Maximize} & \min_{i,j}\{\mu_{ij} \mid f_{ij} > 0\} \\
 & \text{subject to} & \Pr\{\sum_{i=1}^m \sum_{j=1}^n c_{ij} f_{ij} \leq F\} \geq \alpha \\
 & & \sum_{j=1}^n f_{ij} \leq a_i,\ i = 1, 2, \cdots, m \\
 & & \sum_{i=1}^m f_{ij} \geq b_j,\ j = 1, 2, \cdots, n \\
 & & f_{ij}:\ \text{nonnegative integer},\ i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n
\end{array}
$$

Since

$$\Pr\{\sum_{i=1}^m \sum_{j=1}^n c_{ij} f_{ij} \leq F\} \geq \alpha \Leftrightarrow \Pr\{\frac{\sum_{i=1}^m \sum_{j=1}^n c_{ij} f_{ij} - \sum_{i=1}^m \sum_{j=1}^n m_{ij} f_{ij}}{\sqrt{\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^2 f_{ij}^2}} \leq \frac{F - \sum_{i=1}^m \sum_{j=1}^n m_{ij} f_{ij}}{\sqrt{\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^2 f_{ij}^2}}\} \geq \alpha$$

and $\frac{\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}f_{ij} - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}f_{ij}}{\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} \sigma_{ij}^2 f_{ij}^2}}$ is a random variable according to the standard normal distribution $N(0,1)$, the chance constraint is equivalent to the following deterministic constraint

$$\frac{F - \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}f_{ij}}{\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} \sigma_{ij}^2 f_{ij}^2}} \geq K_\alpha \Leftrightarrow F \geq \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}f_{ij} + K_\alpha \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} \sigma_{ij}^2 f_{ij}^2}$$

where K_α is the α percentile point of the cumulative distribution function of the standard normal distribution and note that $K_\alpha > 0$ since $\alpha > 1/2$.

Therefore since F should be minimized, this problem is transformed into the following deterministic equivalent problem P.

$$\begin{array}{lll} \mathrm{P:} & \text{Minimize} & \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}f_{ij} + K_\alpha \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} \sigma_{ij}^2 f_{ij}^2} \\ & \text{Maximize} & \min_{i,j}\{\mu_{ij} \mid f_{ij} > 0\} \\ & \text{subject to} & \sum_{j=1}^{n} f_{ij} \leq a_i,\ i = 1, 2, \cdots, m \\ & & \sum_{i=1}^{m} f_{ij} \geq b_j,\ j = 1, 2, \cdots, n \\ & & f_{ij}:\ \text{nonnegative integer},\ i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n \end{array}$$

Next we define the transportation pattern vector $\mathrm{v}(\mathbf{f}) = (v(\mathbf{f})_1,\ v(\mathbf{f})_2)$ corresponding to a transportation pattern $\mathbf{f} = (f_{ij})$ as follows

$$v(\mathbf{f})_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}f_{ij} + K_\alpha \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} \sigma_{ij}^2 f_{ij}^2}, \quad v(\mathbf{f})_2 = \min_{i,j}\{\mu_{ij} \mid f_{ij} > 0\}.$$

(**Non-dominated Transportation Pattern**)

For two transportation patterns $\mathbf{f}^a,\ \mathbf{f}^b$, if $v(\mathbf{f}^a)_1 \leq v(\mathbf{f}^b)_1,\ v(\mathbf{f}^a)_2 \geq v(\mathbf{f}^b)_2$ and at least one inequality holds as a strict inequality, we call $\mathbf{f}^a$ dominates $\mathbf{f}^b$. If there exists no transportation pattern dominating $\mathbf{f}$, $\mathbf{f}$ is called non-dominated transportation pattern.

Now sorting $\mu_{ij},\ i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n$ and let the result be

$$1 \geq \mu^1 > \mu^2 > \cdots > \mu^g > 0$$

where g is the number of different μ_{ij}. Define a route set

$$A_k = \{(i, j) \mid \mu_{ij} \geq \mu^k,\ i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n\},\ k = 1, 2, \cdots, g$$

that is, a set of routes that their preference are not less than μ^k.

3. Solution Procedure. We solve following sub-problem P_k in order to find non-dominated transportation patterns.

$$\begin{aligned}
\mathrm{P}_k: \quad & \text{Minimize} && \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}f_{ij} + K_\alpha\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}^2 f_{ij}^2} \\
& \text{subject to} && \sum_{j=1}^{n} f_{ij} \le a_i,\ i=1,2,\cdots,m \\
& && \sum_{i=1}^{m} f_{ij} \ge b_j,\ j=1,2,\cdots,n \\
& && f_{ij}:\ \text{nonnegative integer for } (i,j)\in A_k,\ f_{ij}=0 \text{ for } (i,j)\notin A_k
\end{aligned}$$

Now we introduce the following sub-problem P_k^R with positive parameter R in order to solve P_k.

$$\begin{aligned}
\mathrm{P}_k^R: \quad & \text{Minimize} && R\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}f_{ij} + K_\alpha\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}^2 f_{ij}^2 \\
& \text{subject to} && \sum_{j=1}^{n} f_{ij} \le a_i,\ i=1,2,\cdots,m \\
& && \sum_{i=1}^{m} f_{ij} \ge b_j,\ j=1,2,\cdots,n \\
& && f_{ij}:\ \text{nonnegative integer for } (i,j)\in A_k,\ f_{ij}=0 \text{ for } (i,j)\notin A_k
\end{aligned}$$

Note that feasible regions of transportation pattern are same for both problems P_k and P_k^R. According to the very same manner as [8], we have the following relation between P_k and P_k^R.

Theorem 1 *An optimal transportation pattern* $\mathbf{f}^R(k) = (f_{ij}^R(k))$ *for* P_k^R *is also optimal transportation pattern for* P_k *if* $R = 2\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}^2(f_{ij}^R(k))^2}$.

So in order to find an optimal transportation pattern for P_k, we consider optimal transportation patterns of problem P_k^R by changing R. P_k^R is equivalent to the following problem $\overline{\mathrm{P}}_k^R$.

$$\begin{aligned}
\overline{\mathrm{P}}_k^R: \quad & \text{Minimize} && \sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}^R(f_{ij}) \\
& \text{subject to} && \sum_{j=1}^{n} f_{ij} \le a_i,\ i=1,2,\cdots,m \\
& && \sum_{i=1}^{m} f_{ij} \ge b_j,\ j=1,2,\cdots,n \\
& && f_{ij}:\text{nonnegative integer},\ i=1,2,\cdots,m,\ j=1,2,\cdots,n
\end{aligned}$$

where $C_{ij}^R(f_{ij})$ is piecewise linear convex function and defined as follows:

$$C_{ij}^R(f_{ij}) = (Rm_{ij}q + K_\alpha\sigma_{ij}^2 q^2) + \{Rm_{ij} + K_\alpha\sigma_{ij}^2(2q+1)\}(f_{ij}-q)$$

$$(q \leq f_{ij} \leq q+1),\ q = 0, 1, \cdots, q_{ij} (= \min\{a_i, b_j\})$$

for $(i,j) \in A_k$ and $C^R_{ij}(f_{ij}) = M$ (large number) for $(i,j) \notin A_k$.

In order to solve this problem, nonnegative condition is relaxed and the following problem $\overline{\mathrm{P}}^{\sim R}_k$ is considered.

$$\begin{aligned} \overline{\mathrm{P}}^{\sim R}_k : \quad & \text{Minimize} && \sum_{i=1}^{m} \sum_{j=1}^{n} C^R_{ij}(f_{ij}) \\ & \text{subject to} && \sum_{j=1}^{n} f_{ij} \leq a_i,\ i = 1, 2, \cdots, m \\ & && \sum_{i=1}^{m} f_{ij} \geq b_j,\ j = 1, 2, \cdots, n \\ & && f_{ij} \geq 0, \quad i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n \end{aligned}$$

This problem can be solved by using the algorithm [1]. If the optimal value is less than M, let an optimal transportation pattern of problem $\overline{\mathrm{P}}^{\sim R}_k$ be $(\overline{f}^{\sim R}_{ij}(k))$. Then we have the following theorem.

Theorem 2 $(\overline{f}^{\sim R}_{ij}(k))$ *is an integer transportation pattern, that is, an optimal solution of* P^R_k.

Proof of Theorem 2

$$\sum_{(i,j) \in A_k} C^R_{ij}(\overline{f}^{\sim R}_{ij}(k)) = \sum_{(i,j) \in A_k} C^R_{ij}(g_{ij} + r^0_{ij}) = \sum_{(i,j) \in A_k} \{Rm_{ij}(g_{ij} + r^0_{ij}) + K_\alpha \sigma^2_{ij}(g^2_{ij} + (2g_{ij}+1) r^0_{ij})\}$$

where $\overline{f}^{\sim R}_{ij}(k) = g_{ij} + r^0_{ij}$, g_{ij} are nonnegative integers and $0 \leq r^0_{ij} \leq 1$ for $(i,j) \in A_k$.

Further

$$\sum_{\{j | (i,j) \in A_k\}} (g_{ij} + r^0_{ij}) \leq a_i,\ i = 1, 2, \cdots, m, \qquad \sum_{\{i | (i,j) \in A_k\}} (g_{ij} + r^0_{ij}) \geq b_j,\ j = 1, 2, \cdots, n.$$

Now we consider the following problem $\mathrm{P}(r)$:

$$\begin{aligned} \mathrm{P}(r) : \quad & \text{Minimize} && \sum_{(i,j) \in A_k} \{Rm_{ij} + K_\alpha \sigma^2_{ij}(2g_{ij}+1)\} r_{ij} \\ & \text{subject to} && \sum_{\{j | (i,j) \in A_k\}} r_{ij} \leq a_i - \sum_{\{j | (i,j) \in A_k\}} g_{ij},\ i = 1, 2, \cdots, m \\ & && \sum_{\{i | (i,j) \in A_k\}} r_{ij} \geq b_j - \sum_{\{i | (i,j) \in A_k\}} g_{ij},\ j = 1, 2, \cdots, n \\ & && 0 \leq r_{ij} \leq 1 \text{ for } (i,j) \in A_k \end{aligned}$$

Note that (r_{ij}) are decision variable and $\mathrm{P}(r)$ is an usual transportation problem with each supply quantity $a_i - \sum_{\{j|(i,j)\in A_k\}} g_{ij}$ and each demand quantity $b_j - \sum_{\{i|(i,j)\in A_k\}} g_{ij}$ though it contains capacity constraint of the flow through the route (i,j). Further note that (r^0_{ij}) is a feasible solution of $\mathrm{P}(r)$. So since these quantities are integers, an optimal transportation pattern is integer one if using a flow type algorithm [3]. Thus $(\tilde{\overline{f}}_{ij}^{R}(k))$ is an integer transportation pattern. Since $\tilde{\overline{\mathrm{P}}}_k^R$ is a relaxation of $\overline{\mathrm{P}}_k^R$ and P_k^R is equivalent to $\overline{\mathrm{P}}_k^R$, so $(\tilde{\overline{f}}_{ij}^{R}(k))$ is an optimal solution of P_k^R.

(**Solution Method for** P_k)

Solving P_k^R by changing R suitably (for example, using a binary method) and checking whether $R = 2\sqrt{\sum_{(i,j)\in A_k} \sigma_{ij}^2 (f_{ij}^R(k))^2}$ or not, we can find an optimal solution of P_k. Another method is making an efficient frontier of the space

$$\{(\sum_{(i,j)\in A_k} m_{ij} f_{ij}, \sum_{(i,j)\in A_k} \sigma_{ij}^2 f_{ij}^2) | \sum_{\{j|(i,j)\in A_k\}} f_{ij} \le a_i,\ i = 1, 2, \cdots, m,$$

$$\sum_{\{i|(i,j)\in A_k\}} f_{ij} \ge b_j,\ j = 1, 2, \cdots, n,\ f_{ij} : \text{nonnegative integer}\}$$

and checking transportation patterns corresponding to vertices of the efficient frontier whether $R = 2\sqrt{\sum_{(i,j)\in A_k} \sigma_{ij}^2 (f_{ij}^R(k))^2}$ or not by using the very similar manner to [4].

Now we are ready to propose the main algorithm for the original problem P.

(**Main Algorithm**)

Step 1: Set $k = 1$ and $NDT=\phi$. Go to Step 2.

Step 2: Solve P_k and obtain its optimal transportation pattern $\mathbf{f}(k) = (f_{ij}(k))$. If $\mathbf{f}(k)$ is dominated by some transportation patterns in NDT, then go to Step 3 directly. Otherwise set $NDT=NDT \cup \mathbf{f}(k)$ and go to Step 3.

Step 3: If $k = g$, then terminate and NDT is a set of some non-dominated transportation patterns. Otherwise set $k \leftarrow \max\{l, k+1\}$ (where $\mu^l = \min_{i,j} \{\mu_{ij} | f_{ij}(k) > 0\}$) and return to Step 2.

4. Conclusion. In this paper, we considered a bi-criteria transportation problem on a fuzzy network and developed an algorithm to find non-dominated solutions. But unfortunately we cannot show the complexity of our method rigorously. Therefore we should endeavor to construct an efficient procedure by utilizing structure of the problem. Further we should consider the flexibility of supply quantities and demand quantities, that is, treat the case $\sum_{i=1}^m a_i < \sum_{j=1}^n b_j$. This case makes the problem three criteria one and we are now attacking this case. Anyway, there remain many other network problems with both fuzzy factors and random factors to be investigated.

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References

1. R.K. Ahuja, J.L. Batra, S.K. Gupta, *A parametric algorithm for convex cost network flow and related problems*. European Journal of Operational Research, 16 (1984), 222-235.
2. M.H. Chen, H. Ishii, C.X. Wu, *Transportation problems on a fuzzy network*. International Journal of Innovative Computing, Information and Control, 4 (2008), 1105-1109.
3. L.R. Ford,Jr., D.R. Fulkerson, Flows in Networks. Princeton University Press, New Jersey, 1962.

4. S. Geetha, K.P.K. Nair, *On stochastic spanning tree problem.* Networks, 23 (1993), 675-679.
5. F.L. Hitchcock, *The distribution of a product from several sources to numerous locations.* Journal of Mathematical Physics, 20 (1941), 224-230.
6. H. Ishii, *Competitive transportation problem.* Central European Journal of operational research, 12 (2004), 71-78.
7. H. Ishii, *Transportation problems on a fuzzy network.* Proceedings of the 3th Taiwan-Japan Symposium on Fuzzy System and Innovational Computation, Japan, 284-287, 2006.
8. H. Ishii, S. Shiode, T. Nishida, *Chance constrained spanning tree problem*, Journal of the Operations Research Society of Japan, 24 (1981), 147-157.
9. H. Ishii, M. Tada, T. Nishida, *Fuzzy Transportation Problem.* Journal of Japan Society for Fuzzy Theory and System, 2 (1990), 79-84 (in Japanese).
10. J. Munkres, *Algorithms for the Assignment and Transportation Problems.* Journal of Society for Industrial and Applied Mathematics, 5 (1957), 32-38.
11. M. Tada, H. Ishii, *An integer fuzzy transportation Problem.* Computers and Mathematics with Applications, 31 (1996), 71-87.
12. M. Tada, H. Ishii, T. Nishida, *Fuzzy Transportation Problem with Integral Flow.* Mathematica Japonica, 35 (1990), 335-341.

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