HYPERALGEBRAS AND HYPER-COALGEBRAS

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Abstract. In this paper we consider hyperalgebras and hyper-coalgebras of type τ, as an analogue of algebras of type τ and coalgebras of type τ (see [10]). Hyperalgebras and hyper-coalgebras are special cases of (F_1, F_2)-systems. Therefore many of the results in this paper will be instances of (F_1, F_2)-system results (see [8]). Nevertheless it is interesting and instructive to consider directly the theory of hyperalgebras and hyper-coalgebras of type τ. The aim of this paper is to create a unified theory of hyperalgebras and hyper-coalgebras which is based on (F_1, F_2)-systems.

1 Introduction Let $\tau = (n_i)_{i\in I}$ be a type, indexed by some set I. A *hyperalgebra* of type τ is a pair $(A; (f_i^A)_{i\in I})$ consisting of a non-empty set A and a set of finitary *hyperoperations* $f_i^A : A^{n_i} \to \mathcal{P}(A)$ where n_i is the *arity* of the hyperoperation f_i^A, A^{n_i} is the n_i-cartesian power of A and $\mathcal{P}(A)$ is the power set of A. We note that sometimes hyperalgebras are defined as pairs $(A; (f_i^A)_{i\in I})$ where f_i^A maps A^{n_i} into the set $\mathcal{P}(A)\backslash\{\emptyset\}$ and hyperalgebras in the sense of our definition are called power algebras. There exists an extended literature on power algebras and hyperalgebras (see e.g. [3], [1], [5]). In these papers one can find different definitions of the algebraic basic concepts, especially of the concept of a homomorphism. We want to give a unified approach which includes also hyper-coalgebras and is based on the concept of a (F_1, F_2)-system.

A *hyper-coalgebra* of type τ is a system $(A; (g_i^A)_{i\in I})$ consisting of a non-empty set A and a set of finitary *hyper-co-operations* $g_i^A : \mathcal{P}(A) \to A^{\sqcup n_i}$, where n_i is the *arity* of the hyper-co-operation g_i^A and $A^{\sqcup n_i}$ is the n_i-th copower of A. We recall that such copowers are defined by $A^{\sqcup n_i} := \underline{n_i} \times A$ with $\underline{n_i} = \{1, \ldots, n_i\}$. This means that each n_i-ary hyper-co-operation g_i^A is uniquely determined by a pair $((g_i^A)_1, (g_i^A)_2)$ of mappings, $(g_i^A)_1$ from $\mathcal{P}(A)$ to $\underline{n_i}$ and $(g_i^A)_2$ from $\mathcal{P}(A)$ to A. Therefore each n_i-ary hyper-co-operation satisfies

$$g_i^A(X) := ((g_i^A)_1(X), (g_i^A)_2(X))$$

for all sets $X \subseteq A$. The concept of a hyper-coalgebra of type τ generalizes that of a coalgebra of type τ (see e.g. [6], [7]).

Let $F_1,\ F_2 : \mathcal{S}et \to \mathcal{S}et$ be functors from the category $\mathcal{S}et$ of sets as objects and mappings between sets as morphisms into itself. An (F_1, F_2)-system is a pair $(A; \alpha_A)$ consisting of a non-empty set A and a function $\alpha_A : F_1(A) \to F_2(A)$ (see [8], [10]). If F_1 is the identity functor, then $(A; \alpha_A)$ is called F-coalgebra, and if F_2 is the identity functor, then $(A; \alpha_A)$ is called F-algebra. For the theory of F-coalgebras we refer to [13], [14] or [10].

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Any hyperalgebra of type τ can be regarded as an (F_1, F_2)-system for suitable functors $F_1,\ F_2 : \mathcal{S}et \to \mathcal{S}et$. The functor F_1 maps sets X to the coproduct, which is in the category $\mathcal{S}et$ the disjoint sum $\sum_{i\in I} X^{n_i}$. The functor F_1 takes mappings $f : X \to Y$ to mappings $F_1(f) : \sum_{i\in I} X^{n_i} \to \sum_{i\in I} Y^{\ n_i}$ defined by $(i, (a_1, \dots, a_{n_i})) \mapsto (i, (f(a_1), \dots, f(a_{n_i})))$, for all $i \in I$ and $a_1, \dots, a_{n_i} \in A$. The functor F_2 is the power set functor, that is, F_2 maps sets X to $\mathcal{P}(X)$, and $F_2(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ is defined by $C \mapsto \{f(c) \mid c \in C\}$ for all $C \subseteq X$. Then the type τ hyperalgebra $(A; (f_i^A)_{i\in I})$ is uniquely determined by $(A; \alpha_A)$ where $\alpha_A : F_1(A) \to F_2(A)$ is given by $(i, (a_1, \dots, a_{n_i})) \mapsto f_i^A(a_1, \dots, a_{n_i})$, and vice versa.

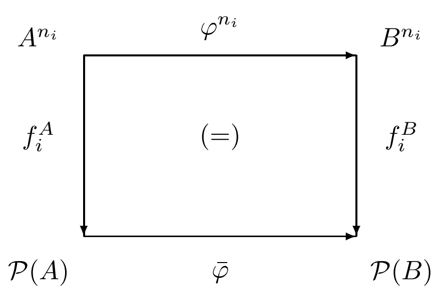
Similarly, any hyper-coalgebra of type τ can be regarded as an (F_1, F_2)-system that is, the functor F_1 is the power set functor, and the functor F_2 maps sets X to the direct product $\prod_{i\in I} X^{\sqcup n_i}$, and takes mappings $f : X \to Y$ to mappings $F_2(f) : \prod_{i\in I} X^{\sqcup n_i} \to \prod_{i\in I} Y^{\sqcup n_i}$ defined by $(k_i, a)_{i\in I} \mapsto (k_i, f(a))_{i\in I}$, where $k_i \in \underline{n_i}$. Then the type τ hyper-coalgebra $(A; (g_i^A)_{i\in I})$ is uniquely determined by $(A; \alpha_A)$ where $\alpha_A : F_1(A) \to F_2(A)$ is given by $X \mapsto (g_i^A(X))_{i\in I}$ for all $X \subseteq A$, and vice versa.

For basic concepts from Category Theory we refer to [12] and [2]. Basic concepts from Universal Algebra can be found in [4] , [9] or [11].

2 Hyperalgebras We begin by considering the definition of homomorphic images, subalgebras and congruences of hyperalgebras of type τ. In each case our definition is based on the type τ hyperalgebra structure, and we show that our definition is in fact equivalent to the (F_1, F_2)-system version for the functors F_1 and F_2 defined before. This guarantees to have the "right" definition. We recall that a mapping $\varphi : A \to B$ is a *homomorphism* from $(A; \alpha_A)$ to $(B; \alpha_B)$ if $\alpha_B \circ F_1(\varphi) = F_2(\varphi) \circ \alpha_A$. Therefore we define homomorphisms of hyperalgebras as follows.

Definition 2.1 Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ and $\mathcal{B} = (B; (f_i^B)_{i\in I})$ be hyperalgebras of type τ. A mapping $\varphi : A \to B$ is called a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ if $\bar{\varphi} \circ f_i^A = f_i^B \circ \varphi^{n_i}$ for all $i \in I$, where $\bar{\varphi}$ and φ^{n_i} are mappings which are defined by $\bar{\varphi}(X) = \{\varphi(x) \mid x \in X\}$ for all $X \subseteq A$ and $\varphi^{n_i}(a_1, \dots, a_{n_i}) = (\varphi(a_1), \dots, \varphi(a_{n_i}))$ for all $a_1, \dots, a_{n_i} \in A$.

Our definition of homomorphism means that the diagram below commutes.



Example 2.2 Consider the set $A = \{a, b\}$ and its cartesian square $A^2 = \{(a, a), (a, b), (b, a), (b, b)\}$. We define a binary hyperoperation $f^A : A^2 \to \mathcal{P}(A)$ by $(a, a) \mapsto \emptyset$, $(a, b) \mapsto A$, $(b, a) \mapsto \{b\}$ and $(b, b) \mapsto \{a\}$. Now let the set $B = \{u, v\}$ and let $f^B : B^2 \to \mathcal{P}(B)$ be given by $(u, u) \mapsto \{v\}$, $(u, v) \mapsto \{u\}$, $(v, u) \mapsto B$ and $(v, v) \mapsto \emptyset$. Consider the mapping $\varphi : A \to B$ given by $a \mapsto v$, $b \mapsto u$. Then it is easy to see that φ is a homomorphism.

As a result, our definition of homomorphism for type τ hyperalgebras is equivalent to the definition of an (F_1, F_2)-system homomorphism. This tells us that all the results for (F_1, F_2)-systems (see [8]), are valid in the case of type τ hyperalgebras as well. Then we obtain the following results and it is not necessary to reprove them in our new context.

Proposition 2.3 *Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$, $\mathcal{B} = (B; (f_i^B)_{i\in I})$ and $\mathcal{C} = (C; (f_i^C)_{i\in I})$ be hyperalgebras. Then*

(i) *If $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ are homomorphisms, then $\psi \circ \varphi : \mathcal{A} \to \mathcal{C}$ is a homomorphism.*

(ii) *If $id_A : A \to A$ is the identity mapping, then id_A is a homomorphism.*

Proposition 2.3 shows that the identity mapping id_A is a homomorphism on any type τ hyperalgebra $\mathcal{A}$ and that the composition of two homomorphisms is a homomorphism. The class of all hyperalgebras of type τ therefore forms a concrete category, which we shall call $\mathcal{H}yalg(\tau)$. By the same method, using results for (F_1, F_2)-systems, we see that every bijective homomorphism is an isomorphism in the category-theoretical sense.

Further we have:

Proposition 2.4 *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be hyperalgebras and let $f : A \to B$, $g : B \to C$ be mappings such that $\varphi := g \circ f : \mathcal{A} \to \mathcal{C}$ is a homomorphism. Then*

(i) *If f is a surjective homomorphism, then g is also a homomorphism.*

(ii) *If g is an injective homomorphism, then f is also a homomorphism.*

The so-called Diagram Lemma, well-known for the category $\mathcal{S}et$, extends to the category $\mathcal{H}yalg(\tau)$.

Proposition 2.5 *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be hyperalgebras and let $\varphi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{A} \to \mathcal{C}$ be homomorphisms. Let φ be surjective. Then there is a homomorphism $\chi : \mathcal{B} \to \mathcal{C}$ with $\chi \circ \varphi = \psi$ iff $Ker\ \varphi \subseteq Ker\ \psi$.*

Subsystems of (F_1, F_2)-systems are defined using homomorphisms, i.e., $(S; \alpha_S)$ is a subsystem of $(A; \alpha_A)$, if the embedding (injection) $\subseteq_S^A$: $S \hookrightarrow A$ is a homomorphism (see [8]). To define subalgebras of hyperalgebras of type τ, we use the restriction $f_i^A|B := \{f_i^A(b_1, \ldots, b_{n_i}) \mid b_1, \ldots, b_{n_i} \in B\}$ of a hyperoperation on a set A to a subset B of A.

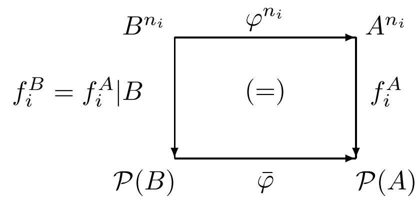
Definition 2.6 Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ and $\mathcal{B} = (B; (f_i^B)_{i\in I})$ be hyperalgebras of type τ, with $B \subseteq A$. Then $\mathcal{B}$ is called a sub-hyperalgebra of $\mathcal{A}$ if $f_i^B := f_i^A|B$ for all $i \in I$. We use the notation $\mathcal{B} \leq \mathcal{A}$ to indicate that $\mathcal{B}$ is a sub-hyperalgebra of $\mathcal{A}$.

From the definition of f_i^B and Definition 2.6, we get that $f_i^B(b_1, \ldots, b_{n_i}) = \emptyset$ iff $f_i^A(b_1, \ldots, b_{n_i}) = \emptyset$ for $b_1, \ldots, b_{n_i} \in B$.

To show that Definition 2.6 is equivalent to the definition of a subsystem for (F_1, F_2)-systems, we must verify that the embedding $\varphi : B \to A$ is a homomorphism. But for any $(b_1, \ldots, b_{n_i}) \in B^{n_i}$ and any $i \in I$, we have

$$\begin{aligned}
&\bar{\varphi}(f_i^B(b_1, \ldots, b_{n_i})) \\
&\quad = \bar{\varphi}(f_i^A(b_1, \ldots, b_{n_i})) \\
&\quad = f_i^A(b_1, \ldots, b_{n_i}) \\
&\quad = f_i^A(\varphi(b_1), \ldots, \varphi(b_{n_i})) \\
&\quad = f_i^A(\varphi^{n_i}(b_1, \ldots, b_{n_i})),
\end{aligned}$$

since $\varphi(b_j) = b_j$ for all $j \in \{1, \ldots, n_i\}$.



Example 2.7 Consider the set $A = \{a, b, c\}$ and its cartesian square

$$A^2 = \{(a, a), (a, b), (a, c), (b, b), (b, a), (b, c), (c, c), (c, a), (c, b)\}.$$

We define a binary hyperoperation $f^A : A^2 \to \mathcal{P}(A)$ by $(a, a) \mapsto \{a, c\}$, $(a, b) \mapsto \{a, b\}$, $(a, c) \mapsto \{a, c\}$, $(b, b) \mapsto \{a, c\}$, $(b, a) \mapsto \{c\}$, $(b, c) \mapsto \{a\}$, $(c, c) \mapsto \{c\}$, $(c, a) \mapsto \emptyset$, $(c, b) \mapsto \{b\}$. Then the following systems are sub-hyperalgebras of $\mathcal{A}$: $(\emptyset; \emptyset)$, $(\{a, c\}; f^A|\{a, c\})$, $(\{c\}; f^A|\{c\})$.

There is a "subalgebra criterion" for sub-hyperalgebras of type τ, similar to the one for algebras of type τ.

Lemma 2.8 *Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a hyperalgebra of type τ and let $B \subseteq A$ be a subset of A. Then the hyperalgebra $(B; (f_i^B)_{i\in I})$ of type τ is a sub-hyperalgebra of $(A; (f_i^A)_{i\in I})$ if and only if B is closed under all the hyperoperations f_i^A for $i \in I$; that is if and only if $f_i^A(b_1 \ldots, b_{n_i}) \in \mathcal{P}(B)$ for all $b_1, \ldots, b_{n_i} \in B$ and all $i \in I$.*

Proof: When $(B; (f_i^B)_{i\in I})$ is a sub-hyperalgebra of $(A; (f_i^A)_{i\in I})$, the mapping $f_i^B = f_i^A|B$ is an n_i-ary hyperoperation on B for all $i \in I$. Therefore $f_i^A(b_1 \ldots, b_{n_i}) = (f_i^A|B)(b_1 \ldots, b_{n_i}) \in \mathcal{P}(B)$ for all $b_1 \ldots, b_{n_i} \in B$ and all $i \in I$.
Conversely, suppose that B is closed with respect to f_i^A for all $i \in I$.
Then $(f_i^A|B)(b_1 \ldots, b_{n_i}) \in \mathcal{P}(B)$ for all $b_1 \ldots, b_{n_i} \in B$, so $f_i^A|B$ is an n_i-ary hyper operation on B and $(B; (f_i^B)_{i\in I})$ with $f_i^B = f_i^A|B$ is a sub-hyperalgebra of $(A; (f_i^A)_{i\in I})$. ■

The following usual properties of subalgebras of type τ algebras, also hold for sub-hyperalgebras of hyperalgebras, and we leave them for the reader to verify.

Corollary 2.9 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be hyperalgebras of type τ. Then*

(i) *If $\mathcal{A} \le \mathcal{B}$ and $\mathcal{B} \le \mathcal{C}$, then $\mathcal{A} \le \mathcal{C}$.*

(ii) *If $A \subseteq B \subseteq C$ and $\mathcal{A} \le \mathcal{C}$ and $\mathcal{B} \le \mathcal{C}$, then $\mathcal{A} \le \mathcal{B}$.*

Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Two standard properties of homomorphisms are that the image $\varphi(\mathcal{C})$ of a sub-hyperalgebra $\mathcal{C}$ of $\mathcal{A}$ should be a sub-hyperalgebra of $\mathcal{B}$, and the preimage $\varphi^{-1}(\mathcal{D})$ of a sub-hyperalgebra of $\mathcal{B}$ should be a sub-hyperalgebra of $\mathcal{A}$. For (F_1, F_2)-systems, the proof of the latter fact requires that the functor F_2 preserves pullbacks (see [8]). It can be proved that the functor F_2 we are using for hyperalgebras of type τ, defined by $X \mapsto \mathcal{P}(X)$ for every set X, preserves pullbacks. Thus this pre-image fact will also hold for hyperalgebras of type τ. However we can also prove this fact directly, in a simpler fashion.

Theorem 2.10 *Let $\mathcal{A}$ and $\mathcal{B}$ be hyperalgebras of type τ and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism.*

(i) *If $\mathcal{C} \leq \mathcal{A}$, then $\varphi(\mathcal{C}) \leq \mathcal{B}$.*

(ii) *If $\mathcal{D} \leq \mathcal{B}$, then $\varphi^{-1}(\mathcal{D}) \leq \mathcal{A}$.*

Proof. (i) We know that $\varphi(C) \subseteq B$ and want to show that the hyperalgebra $(\varphi(C); (f_i^B|\varphi(C))_{i\in I})$ is a sub-hyperalgebra of $(B; (f_i^B)_{i\in I})$. Assume that $b_1, \ldots, b_{n_i} \in \varphi(C)$. Then there are elements $c_1, \ldots, c_{n_i} \in C$ such that $b_j = \varphi(c_j)$ for all $j = 1, \ldots, n_i$. So

$$\begin{aligned} f_i^B(b_1, \ldots, b_{n_i}) &= f_i^B(\varphi(a_1), \ldots, \varphi(a_{n_i})) \\ &= (f_i^B \circ \varphi^{n_i})(a_1, \ldots, a_{n_i}) \\ &= (\bar{\varphi} \circ f_i^A)(a_1, \ldots, a_{n_i}) \\ &= \bar{\varphi}(f_i^A(a_1, \ldots, a_{n_i})) \\ &= \bar{\varphi}(f_i^C(a_1, \ldots, a_{n_i})), \end{aligned}$$

that is, $f_i^B(b_1, \ldots, b_{n_i}) \subseteq \varphi(C)$, which shows that $\varphi(C)$ is closed under taking of f_i^B for all $i \in I$. By the sub-hyperalgebra criterion (Lemma 2.8) we have $\varphi(\mathcal{C}) \leq \mathcal{B}$.

(ii) Again it is clear that $\varphi^{-1}(D) \subseteq A$, and we need to show that the hyperalgebra $(\varphi^{-1}(D); (f_i^A|\varphi^{-1}(D))_{i\in I})$ is a sub-hyperalgebra of $(A; (f_i^A)_{i\in I})$. Let $a_1, \ldots, a_{n_i} \in \varphi^{-1}(D)$, so that $\varphi(a_j) \in D$ for all $j = 1, \ldots, n_i$. Since $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism and $\mathcal{D} \leq \mathcal{B}$, we have

$$\begin{aligned} \bar{\varphi}(f_i^A(a_1, \ldots, a_{n_i})) &= (\bar{\varphi} \circ f_i^A)(a_1, \ldots, a_{n_i}) \\ &= (f_i^B \circ \varphi^{n_i})(a_1, \ldots, a_{n_i}) \\ &= f_i^B(\varphi(a_1), \ldots, \varphi(a_{n_i})) \\ &= f_i^D(\varphi(a_1), \ldots, \varphi(a_{n_i})) \\ &\subseteq D. \end{aligned}$$

This gives $f_i^A(a_1, \ldots, a_{n_i}) \subseteq \varphi^{-1}(D)$. By Lemma 2.8 the hyperalgebra $(\varphi^{-1}(D); (f_i^A|\varphi^{-1}(D))_{i\in I})$ is a sub-hyperalgebra of $(A; (f_i^A)_{i\in I})$. ■

One of the main results for (F_1, F_2)-systems is that the intersection of subsystems of a given (F_1, F_2)-system $\mathcal{A}$ is a subsystem under the condition that the functor F_2 preserves pullbacks (see [8]). Since any hyperalgebra of type τ can be regarded as (F_1, F_2)-system and since we have seen that the suitable functor F_2 is the power set functor, it is not difficult to prove that the power set functor preserves pullbacks, so the intersection of sub-hyperalgebras of a hyperalgebra $\mathcal{A}$ is a sub-hyperalgebra. We can also use the sub-hyperalgebra criterion to prove this fact directly.

Theorem 2.11 *If $(\mathcal{B}_j)_{j\in J}$ is a family of sub-hyperalgebras of a hyperalgebra $\mathcal{A} = (A; (f_i^A)_{i\in I})$ of type τ, then $\bigcap_{j\in J} \mathcal{B}_j$ is a sub-hyperalgebra of $\mathcal{A}$.*

Proof: Since each B_j is a subset of A, we clearly have $\bigcap_{j\in J} B_j \subseteq A$. Let $(b_1, \ldots, b_{n_i}) \in \bigcap_{j\in J} B_j$, this makes $(b_1, \ldots, b_{n_i}) \in B_j$ for all $j \in J$. Since $f_i^{B_j}(b_1, \ldots, b_{n_i}) = (f_i^A|B_j)(b_1, \ldots, b_{n_i})$ for all $j \in J$, then $f_i^{B_j}(b_1, \ldots, b_{n_i}) \subseteq \bigcap_{j\in J} B_j$ for all $j \in J$.

Therefore $(\bigcap_{j\in J} B_j; (f_i^A|(\bigcap_{j\in J} B_j))_{i\in I})$ is a sub-hyperalgebra of $(A; (f_i^A)_{i\in I})$. ■

This result allows us to define the sub-hyperalgebra $\langle S\rangle$ generated by a subset S of a hyperalgebra $\mathcal{A}$.

Definition 2.12 Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a hyperalgebra of type τ and let $S \subseteq A$ be a subset. Then the intersection of all the sub-hyperalgebras of $\mathcal{A}$ which contains S forms the least sub-hyperalgebra of $\mathcal{A}$ containing S. This sub-hyperalgebra is called the hyperalgebra generated by S, and is denoted by $\langle S\rangle$.

Example 2.13 Let us consider set A and the hyperoperation f^A from Example 2.7. The following sub-hyperalgebras of $(A; f^A)$ are generated by the given subsets of A:

$$\begin{array}{lllllll} \langle\{b\}\rangle & = & \langle\{a,b\}\rangle & = & \langle\{b,c\}\rangle & = & (A; f^A), \\ \langle\{a\}\rangle & = & \langle\{a,c\}\rangle & = & (\{a,c\}; f^A|\{a,c\}) \text{ and} & & \\ \langle\{c\}\rangle & = & (\{c\}; f^A|\{c\}). & & & & \end{array}$$

An important problem is the following one: given a hyperalgebra $\mathcal{A}$ of type τ, and a subset S of A, determine all elements of the carrier set of the sub-hyperalgebra of $\mathcal{A}$ which is generated by S. To do this, we set

$$E(S) := S \cup f_i^A(s_1, \dots, s_{n_i})$$

for all $s_1, \dots, s_{n_i} \in S$ and all $i \in I$. Then we inductively define $E^0(S) := S$, and $E^{k+1}(S) := E(E^k(S))$, for all $k \in \mathbb{N}$. With this notation we obtain a similar result as in the case of an algebra of type τ.

Corollary 2.14 *For any hyperalgebra $\mathcal{A}$ of type τ and for any non-empty subset $S \subseteq A$, we have $\langle S\rangle = \bigcup\limits_{k=0}^{\infty} E^k(S)$.*

The proof corresponds to that one for algebras of type τ.

Congruences of (F_1, F_2)-systems are defined as kernels of homomorphisms. For hyperalgebras of type τ, we define a congruence to be an equivalence relation with a certain additional property. We shall see shortly that this is equivalent to a congruence being a kernel of a homomorphism in this case too.

Definition 2.15 Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a hyperalgebra of type τ. A congruence relation θ on $\mathcal{A}$ is an equivalence relation on A which satisfies the condition that,if $(a_1, b_1) \dots, (a_{n_i}, b_{n_i}) \in \theta$, then

$$[(f_i^A(a_1, \dots, a_{n_i})]_\theta = [f_i^A(b_1, \dots, b_{n_i})]_\theta$$

where

$$[(f_i^A(a_1, \dots, a_{n_i})]_\theta = \{[a]_\theta \mid a \in f_i^A(a_1, \dots, a_{n_i})\}$$

and

$$[(f_i^A(b_1, \dots, b_{n_i})]_\theta = \{[b]_\theta \mid b \in f_i^A(b_1, \dots, b_{n_i})\}.$$

Example 2.16 Let us consider $A = \{a, b\}$ and let the binary hyperoperation f^A be given as in Example 2.2. The equivalences on A are given by the following sets

$$\begin{array}{lll} \theta_1 & := & \{(a,a),(b,b)\}, \\ \theta_2 & := & \{(a,a),(a,b),(b,b)\}, \\ \theta_3 & := & \{(a,a),(b,a),(b,b)\}, \text{ and} \\ \theta_4 & := & A \times A. \end{array}$$

Since $(a,a),(b,a) \in \theta_2, \theta_3$ and $f^A(a,b) = A$, $f^A(a,a) = \emptyset$, the relations θ_2, θ_3 are not congruence relations on A. Similarly we have that, since $(a, a), (a, b) \in \theta_2$, the relation θ_2 is not a congruence relation. Since $(a, a), (b, b) \in \theta_1$ we have that $f^A(a, a) = \emptyset$, $f^A(b, b) = \{a\}$, $f^A(a, b) = A$ and $f^A(b, a) = \{b\}$. This gives

$$\begin{array}{rcl} [f^A(a,b)]_{\theta_1} & = & [f^A(a,b)]_{\theta_1}, \\ [f^A(b,a)]_{\theta_1} & = & [f^A(b,a)]_{\theta_1}, \\ [f^A(b,b)]_{\theta_1} & = & [f^A(b,b)]_{\theta_1}. \end{array}$$

Therefore θ_1 is a congruence relation on A.

It is easy to see that Δ_A is the least congruence on A and Example 2.16 shows that the greatest congruence on A need not to be $A \times A$.

Lemma 2.17 *Let $(\theta_j)_{j\in J}$ be a family of congruence relations on A. Then $\bigcap_{j\in J} \theta_j$ is a congruence on A.*

Proof. Let $(a_1, b_1), \ldots , (a_{n_i}, b_{n_i}) \in \bigcap_{j\in J} \theta_j$. This gives

$$(a_1, b_1), \ldots , (a_{n_i}, b_{n_i}) \in \theta_j$$

for all $j \in J$. By the definition of a congruence relation we get that

$$[f_i^A(a_1, \ldots , a_{n_i})]_{\theta_j} = [f_i^A(b_1, \ldots , b_{n_i})]_{\theta_j}$$

for all $j \in J$, making $[f^A(a_1, \ldots , a_{n_i})]_{\bigcap_{j\in J} \theta_i} = [f^A(b_1, \ldots , b_{n_i})]_{\bigcap_{i\in I} \theta_j}$. Therefore $\bigcap_{i\in I} \theta_i$ is a congruence relation on A. ■

As in the algebra case, congruences can be used to produce quotient algebras.

Definition 2.18 Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a hyperalgebra of type τ and let θ be a congruence relation on $\mathcal{A}$. We define hyperoperations on the quotient set A/θ by

$$f_i^{A/\theta}([a_1]_\theta, \ldots , [a_{n_i}]_\theta) \ := \ \{[a]_\theta \mid a \in f_i^A(a_1, \ldots , a_{n_i})\},$$

for all $a_1, \ldots , a_{n_i} \in A$. Then the hyperalgebra $\mathcal{A}/\theta = (A/\theta; (f_i^{A/\theta})_{i\in I})$ is called the *quotient hyperalgebra* of $\mathcal{A}$ by θ.

For this definition to be valid we have to verify that the hyperoperations $f_i^{A/\theta}$ defined on A/θ are well-defined. To check this, let

$$([a_1]_\theta, \ldots , [a_{n_i}]_\theta) = ([b_1]_\theta, \ldots , [b_{n_i}]_\theta).$$

This gives that $(a_1, b_1), \ldots , (a_{n_i}, b_{n_i}) \in \theta$, and that

$$[f^A(a_1, \ldots , a_{n_i})]_\theta = [f^A(b_1, \ldots , b_{n_i})]_\theta.$$

Therefore

$$\begin{array}{rcl} f_i^{A/\theta}([a_1]_\theta, \ldots , [a_{n_i}]_\theta) & = & [f^A(a_1, \ldots , a_{n_i})]_\theta \\ & = & [f^A(b_1, \ldots , b_{n_i})]_\theta \\ & = & f_i^{A/\theta}([b_1]_\theta, \ldots , [b_{n_i}]_\theta). \end{array}$$

Example 2.19 Let $\mathcal{A} = (A; f^A)$ be a hyperalgebra as defined in Example 2.2 and $\theta = \theta_1$ as defined in Example 2.17. Then $A/\theta = \{[a]_\theta, [b]_\theta\}$ and the hyperoperation $f^{A/\theta} : (A/\theta)^2 \to \mathcal{P}(A/\theta)$ is defined as follows:

$$\begin{array}{lcl} f^{A/\theta}([a]_\theta, [a]_\theta) & = & \emptyset \\ f^{A/\theta}([a]_\theta, [b]_\theta) & = & \{[a]_\theta, [b]_\theta\} \\ f^{A/\theta}([b]_\theta, [a]_\theta) & = & \{[b]_\theta\} \\ f^{A/\theta}([b]_\theta, [b]_\theta) & = & \{[a]_\theta\}. \end{array}$$

Proposition 2.20 *Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a hyperalgebra of type τ and let θ be a congruence on $\mathcal{A}$. Then the natural mapping $\gamma : A \to A/\theta$ defined by $a \mapsto [a]_\theta$ is a surjective homomorphism from $\mathcal{A}$ onto $\mathcal{A}/\theta$.*

Proof. For any $(a_1, \ldots, a_{n_i}) \in A^{n_i}$, we have

$$\begin{array}{lcl} (\bar{\varphi} \circ f_i^A)(a_1 \ldots, a_{n_i}) & = & \bar{\varphi}(f_i^A(a_1, \ldots, a_{n_i})) \\ & = & \{[x]_\theta \mid x \in f_i^A(a_1, \ldots, a_{n_i})\} \\ & = & f_i^{A/\theta}([a_1]_\theta, \ldots, [a_{n_i}]_\theta) \\ & = & f_i^{A/\theta}(\varphi(a_1), \ldots, \varphi(a_{n_i})) \\ & = & (f_i^{A/\theta} \circ \varphi^{n_i})(a_1, \ldots, a_{n_i}). \end{array}$$

This shows that γ is a homomorphism.

■

Theorem 2.21 *Let $\mathcal{A}$ be a hyperalgebra of type τ. Then an equivalence relation θ on A is a congruence on $\mathcal{A}$ if and only if θ is the kernel of some homomorphism from $\mathcal{A}$ to some hyperalgebra $\mathcal{B}$.*

Proof. When θ is a congruence, it is clear that θ is the kernel of the natural mapping $\gamma : \mathcal{A} \to \mathcal{A}/\theta$ since

$$(a, b) \in \theta \Leftrightarrow [a]_\theta = [b]_\theta \Leftrightarrow \gamma(a) = \gamma(b) \Leftrightarrow (a, b) \in Ker\ \gamma.$$

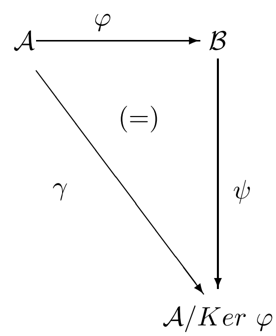
Conversely, let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism with $Ker\ \varphi$ as its kernel. Then $Ker\ \varphi$ is an equivalence relation on A. For any $(a_1, b_1), \ldots, (a_{n_i}, b_{n_i}) \in Ker\ \varphi$, we have $\varphi(a_j) = \varphi(b_j)$ for all $j = 1, \ldots, n_i$, so that

$$\begin{array}{lcl} \bar{\varphi}(f_i^A(a_1, \ldots, a_{n_i})) & = & (\bar{\varphi} \circ f_i^A)(a_1, \ldots, a_{n_i}) \\ & = & f_i^B(\varphi(a_1), \ldots, \varphi(a_{n_i})) \\ & = & f_i^B(\varphi(b_1), \ldots, \varphi(b_{n_i})) \\ & = & (f_i^B \circ \varphi^{n_i})(b_1, \ldots, b_{n_i}) \\ & = & (\bar{\varphi} \circ f_i^A)(b_1, \ldots, b_{n_i}) \\ & = & \bar{\varphi}(f_i^A(b_1, \ldots, b_{n_i}), \end{array}$$

that is $[f_i^A(a_1, \ldots, a_{n_i})]_{Ker\ \varphi} = [f_i^A(b_1, \ldots, b_{n_i})]_{Ker\ \varphi}$. Therefore $Ker\ \varphi$ is a congruence relation on $\mathcal{A}$. ■

If θ is a congruence on the hyperalgebra $\mathcal{A}$ of type τ, then θ is the kernel of some homomorphism corresponding to a homomorphism of the corresponding (F_1, F_2)-system and θ is the kernel of this (F_1, F_2)-system homomorphism and thus a congruence on the (F_1, F_2)-system and conversely. This shows that congruences on hyperalgebras of type τ correspond to congruences of the corresponding (F_1, F_2)-systems. The analogue of the Homomorphic Image Theorem for algebras also holds for hyperalgebras of type τ. This fact follows directly from the Diagram Lemma of (F_1, F_2)-systems, but we will also give a direct proof using the definition of hyperalgebras of type τ.

Theorem 2.22 *Let $\mathcal{A}$ and $\mathcal{B}$ be hyperalgebras of type τ, with φ a surjective homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then $\mathcal{B}$ is isomorphic to the quotient hyperalgebra $\mathcal{A}/Ker\ \varphi$ and the diagram below commutes*



Proof. We use the natural mapping γ considered in Proposition 2.20 to define a mapping $\psi : \mathcal{B} \to \mathcal{A}/Ker\ \varphi$ by $\psi(b) = \gamma(a)$ for $b = \varphi(a)$ and any $a \in A$. This mapping is well-defined, since $\varphi(c) = b = \varphi(a) \in B$ implies that $(c, a) \in Ker\ \varphi$ and so $[c]_{Ker\ \varphi} = [a]_{Ker\ \varphi}$. It is clear that ψ is onto, and we show that ψ is one-to-one. If $\psi(b_1) = \psi(b_2)$ for some $b_1, b_2 \in B$, then there are elements $a_1, a_2 \in A$ with $b_1 = \varphi(a_1)$ and $b_2 = \varphi(a_2)$. Since $[a_1]_{Ker\ \varphi} = [a_2]_{Ker\ \varphi}$, that is $(a_1, a_2) \in Ker\ \varphi$, we get $b_1 = b_2$. Also, since we have $\psi(\varphi(a)) = \psi(b) = \gamma(a)$ for all $a \in A$, the diagram commutes. Using the fact that γ is a homomorphism, we can show that ψ is also a homomorphism. For any $b_1, \ldots, b_{n_i} \in B$, there are $a_1, \ldots, a_{n_i} \in A$ such that $b_j = \varphi(a_j)$ for all $j = 1, \ldots, n_i$, then

$$
\begin{aligned}
(\bar{\psi} \circ f_i^B)(b_1, \ldots, b_{n_i}) &= \bar{\psi}(f_i^B(b_1, \ldots, b_{n_i})) \\
&= \bar{\psi}(f_i^B(\varphi(a_1), \ldots, \varphi(a_{n_i}))) \\
&= \bar{\psi}((f_i^B \circ \varphi^{n_i})(a_1, \ldots, a_{n_i})) \\
&= \bar{\psi}((\bar{\varphi} \circ f_i^A)(a_1, \ldots, a_{n_i})) \\
&= \bar{\psi} \circ (\bar{\varphi} \circ f_i^A))(a_1, \ldots, a_{n_i}) \\
&= ((\bar{\psi} \circ \bar{\varphi}) \circ f_i^A)(a_1, \ldots, a_{n_i}) \\
&= ((\psi \mathbin{\bar{\circ}} \varphi) \circ f_i^A)(a_1, \ldots, a_{n_i}) \\
&= (\bar{\gamma} \circ f_i^A)(a_1, \ldots, a_{n_i}) \\
&= (f_i^{A/Ker\ \varphi} \circ \gamma^{n_i})(a_1, \ldots, a_{n_i}) \\
&= f_i^{A/Ker\ \varphi}(\gamma(a_1), \ldots, \gamma(a_{n_i})) \\
&= f_i^{A/Ker\ \varphi}(\psi(b_1), \ldots, \psi(b_{n_i})) \\
&= (f_i^{A/Ker\ \varphi} \circ \psi^{n_i})(b_1, \ldots, b_{n_i}).
\end{aligned}
$$

Then ψ is a homomorphism. ■

3 Hyper-coalgebras

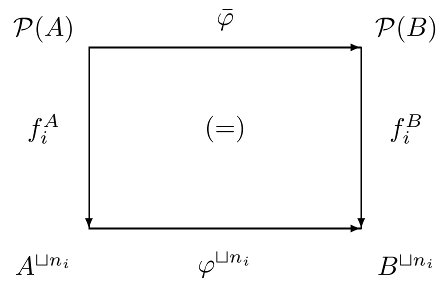
In this section we want consider the definitions of homomorphic images, subcoalgebras and congruences of hyper-coalgebras of type τ, and show that our definitions are in fact equivalent to the corresponding definitions for (F_1, F_2)-systems for suitable functors F_1, F_2. Homomorphisms of hyper-coalgebras of type τ are defined as follows:

Definition 3.1 Let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ and $\mathcal{B} = (B; (f_i^B)_{i \in I})$ be hyper-coalgebras of type τ. A mapping $\varphi : A \to B$ is called a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ if the following equations are satisfied for all $i \in I$ and all sets $X \subseteq A$:

(i) $(f_i^A)_1(X) = (f_i^B)_1(\bar{\varphi}(X))$, and

(ii) $\varphi((f_i^A)_2(X)) = (f_i^B)_2(\bar{\varphi}(X))$.

Let us set $\varphi^{\sqcup n_i}(f_i^A(X)) = ((f_i^A)_1(X), \varphi((f_i^A)_2(X)))$. Then we see our definition of homomorphism means that the diagram below commutes, since $f_i^B(\bar{\varphi}(X)) = ((f_i^B)_1(\bar{\varphi}(X)), (f_i^B)_2(\bar{\varphi}(X))) = ((f_i^A)_1(X), \varphi((f_i^A)_2(X))) = \varphi^{\sqcup n_i}(f_i^A(X))$.



Example 3.2 Consider the set $A = \{a, b\}$ and its copower

$$A^{\sqcup 2} = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c), (1, d), (2, d)\}.$$

We define a binary hyper-co-operation $f^A : \mathcal{P}(A) \to A^{\sqcup 2}$ by $\emptyset \mapsto (2, b)$, $A \mapsto (1, a)$, $\{a\} \mapsto (2, a)$ and $\{b\} \mapsto (1, b)$. Now let the set $B = \{x, y\}$ and let $f^B : \mathcal{P}(B) \to B^{\sqcup 2}$ be given by $\emptyset \mapsto (2, x)$, $B \mapsto (1, y)$, $\{x\} \mapsto (1, x)$ and $\{y\} \mapsto (2, y)$. Then it is easy to see that the mapping $\varphi : A \to B$ given by $a \mapsto y$ and $b \mapsto x$ is a homomorphism.

The definition of a homomorphism for hyper-coalgebras of type τ is equivalent to the definition of an (F_1, F_2)-system homomorphism. Then the following homomorphism properties are valid as in the case of (F_1, F_2)-systems.

Proposition 3.3 *Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$, $\mathcal{B} = (B; (f_i^B)_{i\in I})$ and $\mathcal{C} = (C; (f_i^C)_{i\in I})$ be hyper-coalgebras. Then*

(i) *If $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ are homomorphisms, then $\psi \circ \varphi : \mathcal{A} \to \mathcal{C}$ is a homomorphism.*

(ii) *If $id_A : A \to A$ is the identity mapping, then id_A is a homomorphism.*

Proposition 3.3 shows that the class of all hyper-coalgebras of type τ together with homomorphisms between them forms a concrete category, which we shall call $\mathcal{H}ycoalg(\tau)$. The next proposition characterizes isomorphisms of hyper-coalgebra of type τ.

Proposition 3.4 *Assume that $\mathcal{A} = (A; (f_i^A)_{i\in I})$ and $\mathcal{B} = (B; (f_i^B)_{i\in I})$ be hyper-coalgebras of type τ and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a bijective homomorphism. Then φ is an isomorphism.*

Further we have:

Proposition 3.5 *Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ and $\mathcal{B} = (B; (f_i^B)_{i\in I})$, $\mathcal{C} = (C; (f_i^C)_{i\in I})$ be hyper-coalgebras of type τ and let $f : A \to B$, $g : B \to C$ be mappings such that $\varphi := g \circ f : \mathcal{A} \to \mathcal{C}$ is a homomorphism. Then*

(i) *If f is a surjective homomorphism, then g is also a homomorphism.*

(ii) *If g is an injective homomorphism, then f is also a homomorphism.*

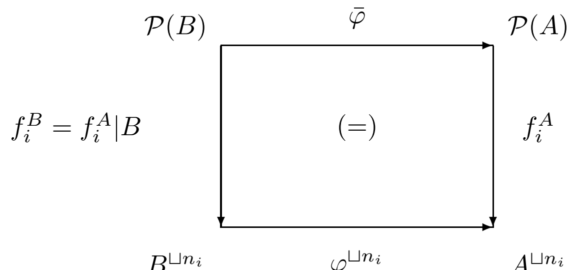
It is easy too see that the Diagram Lemma extends from the category $\mathcal{S}et$ to the category $\mathcal{H}ycoalg(\tau)$.

Proposition 3.6 *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be hyper-coalgebras and let $\varphi : \mathcal{A} \to \mathcal{B}$, $\psi : \mathcal{A} \to \mathcal{C}$ be homomorphisms. Let φ be surjective. Then there is a homomorphism $\chi : \mathcal{B} \to \mathcal{C}$ with $\chi \circ \varphi = \psi$ iff $Ker\ \varphi \subseteq Ker\ \psi$.*

To define subcoalgebras of hyper-coalgebras of type τ, we use the restriction $f_i^A|\mathcal{P}(B) := \{((f_i^A)_1(X), (f_i^A)_2(X)) \mid X \subseteq B\}$ of a hyper-co-operation on a set A to a subset $\mathcal{P}(B)$ of $\mathcal{P}(A)$.

Definition 3.7 Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ and $\mathcal{B} = (B; (f_i^B)_{i\in I})$ be hyper-coalgebras of type τ, with $B \subseteq A$. Then $\mathcal{B}$ is called a sub-hyper-coalgebra of $\mathcal{A}$ if $f_i^B := f_i^A|\mathcal{P}(B)$ for all $i \in I$. We use the notation $\mathcal{B} \preceq \mathcal{A}$ to indicate that $\mathcal{B}$ is a sub-hyper-coalgebra of $\mathcal{A}$.

To show that this definition is equivalent to the definition of a subsystem for (F_1, F_2)-systems, we must verify that the embedding $\varphi : B \to A$ is a homomorphism. But for any $X \subseteq B$ and any $i \in I$, we have $\varphi^{\sqcup n_i}(f_i^B(X)) = \varphi^{\sqcup n_i}(f_i^A(X)) = ((f_i^A)_1(X), (f_i^A)_2(\bar{\varphi}(X)))$ $= f_i^A(\bar{\varphi}(X))$, since $(f_i^A)_1(\bar{\varphi}(X)) = (f_i^A)_1(X)$ and the diagram below commutes.



Example 3.8 Let $A = \{a, b, c\}$, with copower

$$A^{\sqcup 2} = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}.$$

We define the binary hyper-co-operation $f^A : \mathcal{P}(A) \to A^{\sqcup 2}$ by $\emptyset \mapsto (1, a)$, $A \mapsto (2, a)$, $\{a\} \mapsto (1, a)$, $\{b\} \mapsto (1, a)$, $\{c\} \mapsto (1, c)$, $\{a, b\} \mapsto (2, a)$, $\{a, c\} \mapsto (2, c)$ and $\{b, c\} \mapsto (2, b)$. Then the subcoalgebras of $(A; f^A)$ are $(\emptyset; \emptyset)$, $(A; f^A)$, $(\{a\}; f^A|\mathcal{P}(\{a\}))$, $(\{a, b\}; f^A|\mathcal{P}(\{a, b\}))$ and $(\{a, c\}; f^A|\mathcal{P}(\{a, c\}))$.

There is a " subcoalgebra criterion " for sub-hyper-coalgebras of type τ, similar to the one for algebras of type τ.

Lemma 3.9 *Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a hyper-coalgebra of type τ and let $B \subseteq A$ be a subset of A. Then the hyper-coalgebra $(B; (f_i^B)_{i\in I})$ of type τ is a sub-hyper-coalgebra of $(A; (f_i^A)_{i\in I})$ if and only if B is closed under all the hyper-co-operations f_i^A for $i \in I$; that is, if and only if $f_i^A(X) \in B^{\sqcup n_i}$ for all $X \subseteq B$ and all $i \in I$.*

Proof: When $(B; (f_i^B)_{i\in I})$ is a sub-hyper-coalgebra of $(A; (f_i^A)_{i\in I})$, the mapping $f_i^B = f_i^A|\mathcal{P}(B)$ is an n_i-ary hyper-co-operation on B for all $i \in I$.
Therefore $f_i^B(X) = (f_i^A|\mathcal{P}(B))(X) \in B^{\sqcup n_i}$ for all $X \subseteq B$ and all $i \in I$.
Conversely, suppose that B is closed with respect to f_i^A for all $i \in I$. Then $(f_i^A|\mathcal{P}(B))(Y) \in B^{\sqcup n_i}$ for all $Y \subseteq B$, so $f_i^A|\mathcal{P}(B)$ is an n_i-ary hyper-co-operation on B and $(B; (f_i^B)_{i\in I})$ with $f_i^B = f_i^A|\mathcal{P}(B)$ is a sub-hyper-coalgebra of $(A; (f_i^A)_{i\in I})$. ■

The following properties which are valid for sub-hyperalgebras also hold for sub-hyper-coalgebras, and we leave them for the reader to verify.

Corollary 3.10 *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be hyper-coalgebras of type τ. Then*

(i) *If* $\mathcal{A} \preceq \mathcal{B}$ *and* $\mathcal{B} \preceq \mathcal{C}$, *then* $\mathcal{A} \preceq \mathcal{C}$.

(ii) *If* $A \subseteq B \subseteq C$ *and* $\mathcal{A} \preceq \mathcal{C}$ *and* $\mathcal{B} \preceq \mathcal{C}$, *then* $\mathcal{A} \preceq \mathcal{B}$.

Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism. The functor F_2 which defines hyper-coalgebras of type τ as (F_1, F_2)-systems is given by $X \mapsto \prod_{i \in I} X^{\sqcup n_i}$ for all sets X. It is not difficult to prove that F_2 preserves pullbacks. This can be used to prove that the preimages of sub-hyper-coalgebras of $\mathcal{B}$ are sub-hyper-coalgebras of $\mathcal{A}$. However we can also prove this fact directly.

Theorem 3.11 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be hyper-coalgebras of type* τ *and let* $\varphi : \mathcal{A} \to \mathcal{B}$ *be a homomorphism.*

(i) *If* $\mathcal{C} \preceq \mathcal{A}$, *then* $\varphi(\mathcal{C}) \preceq \mathcal{B}$.

(ii) *If* $\mathcal{D} \preceq \mathcal{B}$, *then* $\varphi^{-1}(\mathcal{D}) \preceq \mathcal{A}$.

Proof. (i) We know that $\varphi(C) \subseteq B$ and want to show that the hyper-coalgebra $(\varphi(C); (f_i^B|\mathcal{P}(\varphi(C)))_{i\in I})$ is a sub-hyper-coalgebra of $(B; (f_i^B)_{i\in I})$. Assume that $X \subseteq \varphi(C)$. Then there is a subset Y of C such that $X = \bar{\varphi}(Y)$. So

$$\begin{aligned} f_i^B(X) &= f_i^B(\bar{\varphi}(Y)) \\ &= (f_i^B \circ \bar{\varphi})(Y) \\ &= (\varphi^{\sqcup n_i} \circ f_i^A)(Y) \\ &= \varphi^{\sqcup n_i}(f_i^A(Y)) \\ &= (\varphi^{\sqcup n_i}((f_i^A)_1(Y), (f_i^A)_2(Y)) \\ &= ((f_i^A)_1(Y), \varphi((f_i^A)_2(Y))). \end{aligned}$$

There follows $f_i^B(X) \in \varphi(C)^{\sqcup n_i}$ for all $i \in I$. By the sub-hyper-coalgebra criterion (Lemma 3.9) we have $\varphi(\mathcal{C}) \preceq \mathcal{B}$.
(ii) Again it is clear that $\varphi^{-1}(D) \subseteq A$, and we need to show that the hyper-coalgebra $(\varphi^{-1}(D); (f_i^A|\mathcal{P}(\varphi^{-1}(D)))_{i\in I})$ is a sub-hyper-coalgebra of $(A; (f_i^A)_{i\in I})$. Let $X \subseteq \varphi^{-1}(D)$, so that $\varphi(X) \subseteq D$. Since $\varphi : \mathcal{A} \to \mathcal{B}$ is a homomorphism and $\mathcal{D} \preceq \mathcal{B}$, we have

$$\begin{aligned} \varphi^{\sqcup n_i}(f_i^A(X)) &= (\varphi^{\sqcup n_i} \circ f_i^A)(X) \\ &= (f_i^B \circ \bar{\varphi})(X) \\ &= f_i^B(\bar{\varphi}(X)) \\ &= f_i^D(\bar{\varphi}(X)) \in D^{\sqcup n_i}. \end{aligned}$$

This gives $\varphi^{\sqcup n_i}(f_i^A(X)) \in D^{\sqcup n_i}$, and thus $f_i^A(X) \in \varphi^{-1}(D)^{\sqcup n_i}$. By Lemma 3.9 the hyper-coalgebra $(\varphi^{-1}(D); (f_i^A|\mathcal{P}(\varphi^{-1}(D)))_{i\in I})$ is a sub-hyper-coalgebra of $(A; (f_i^A)_{i\in I})$. ■

One of the main results for (F_1, F_2)-systems is that the union of subsystems of a given (F_1, F_2)-system $\mathcal{A}$ is a subsystem under the condition that the functor F_1 preserves sums (see [8]). Any hyper-coalgebra of type τ can be regarded as (F_1, F_2)-system. We have seen that F_1 is the power set functor. It is not difficult to prove that the power set functor does not preserve sums, so the union of sub-hyper-coalgebras of hyper-coalgebra $\mathcal{A}$ is not a sub-hyper-coalgebra. Let us consider a counterexample

Example 3.12 Let $A = \{a, b, c, d, e, \}$ with co-operation $f^A : A \to A^{\sqcup 2}$ be a hyper-coalgebra and let $B = \{a, b\}$, and $C = \{a, d, c\}$ be subsets of A. Assume that $\mathcal{B} = (B; f^A|\mathcal{P}(B))$, and $\mathcal{C} = (C; f^A|\mathcal{P}(C))$ are sub-hyper-coalgebras of $\mathcal{A}$. If $f^A(\{b, c\}) = (1, e)$, then $f^A|\mathcal{P}(B \cup C)$ is not a hyper-co-operation on $B \cup C$, since $(1, e) \notin (B \cup C)^{\sqcup 2}$. Therefore $B \cup C$ is not a sub-hyper-coalgebra of $(A; f^A)$.

Another result for (F_1, F_2)-systems is that arbitrary sums of (F_1, F_2)-systems exist if the functor F_1 preserves sums. But the power set functor does not preserve sums. Do sum exist in $\mathcal{H}ycoalg(\tau)$? We consider the following erxample.

Example 3.13 Let $A = \{a\}$ and $B = \{b\}$. Let $\mathcal{A} = (A; f^A)$ and $\mathcal{B} = (B; f^B)$ be hyper-coalgebras such that f^A and f^B are defined by

$$f^A : \emptyset \mapsto (1, a),\ A \mapsto (2, a) \text{ and } f^B : \emptyset \mapsto (2, b),\ B \mapsto (2, b).$$

Suppose that $(A \oplus B; f^{A\oplus B})$ together with homomorphisms $e_A : (A; f^A) \to (A \oplus B; f^{A\oplus B})$ and $e_B : (B; f^B) \to (A \oplus B; f^{A\oplus B})$ is the sum of $\mathcal{A}$ and $\mathcal{B}$. Let us consider the mappings $e_A : A \to A \oplus B$ and $e_B : B \to A \oplus B$. Since

$$\begin{array}{lcl} (e_A^{\sqcup 2} \circ f^A)(\emptyset) & = & e_A^{\sqcup 2}(f^A(\emptyset)) \\ & = & e_A^{\sqcup 2}(1, a) \\ & = & (1, e_A(a)) \qquad \text{and} \\ (f^{A\oplus B} \circ \bar{e_A})(\emptyset) & = & f^{A\oplus B}(\bar{e_A}(\emptyset)) \\ & = & f^{A\oplus B}(\emptyset), \end{array}$$

we obtain $f^{A\oplus B}(\emptyset) = (1, e_A(a))$.

Since

$$\begin{array}{lcl} (e_B^{\sqcup 2} \circ f^B)(\emptyset) & = & e_B^{\sqcup 2}(f^B(\emptyset)) \\ & = & e_B^{\sqcup 2}(2, b) \\ & = & (2, e_B(b)) \qquad \text{and} \\ (f^{A\oplus B} \circ \bar{e_B})(\emptyset) & = & f^{A\oplus B}(\bar{e_B}(\emptyset)) \\ & = & f^{A\oplus B}(\emptyset), \end{array}$$

we get $f^{A\oplus B}(\emptyset) = (2, e_B(b))$. This contradiction shows that the sum $(A \oplus B; f^{A\oplus B})$ does not exist.

Congruences of hyper-coalgebras $\mathcal{A}$ are defined as equivalence relations preserving the mappings $(f_i^A)_1$ and $(f_i^A)_2$. Let θ be an equivalence relation on A. For any subset X of θ we define $\pi_1(X) = \{x_1 \in A \mid \exists y \in A\ ((x_1, y) \in X)\} =: X_1$ and $\pi_2(X) = \{x_2 \in A \mid \exists y \in A\ ((y, x_2) \in X)\} =: X_2$ such that $\pi_1 : X \to A$ and $\pi_2 : X \to A$ are the first and the second projections, respectively.

Definition 3.14 Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a hyper-coalgebra of type τ. A congruence relation θ on $\mathcal{A}$ is an equivalence relation on A which satisfies the condition that for any $X \subseteq \theta$, $((f_i^A)_2(X_1), (f_i^A)_2(X_2)) \in \theta$ and $(f_i^A)_1(X_1) = (f_i^A)_1(X_2)$ for all $i \in I$.

Let $(A; (f_i^A)_{i\in I})$ be a hyper-coalgebra of type τ. For any $X \subseteq A \times A$ and if $X_1 = X_2$, we have $(f_i^A)_1(X_1) = (f_i^A)_1(X_2)$ and $(f_i^A)_2(X_1) = (f_i^A)_2(X_2)$. There follows that Δ_A is a congruence relation on $\mathcal{A}$. But $A \times A$ in general is not a congruence relation on $\mathcal{A}$.

Example 3.15 Let $A = \{a, b\}$ and let f^A be a binary hyper-co-operation on A which is defined by $\emptyset \mapsto (2, a)$, $A \mapsto (1, a)$, $\{a\} \mapsto (1, b)$ and $\{b\} \mapsto (2, b)$. Consider the set $X = \{(a, b), (b, b)\}$. We have $X_1 = \{a, b\}$ and $X_2 = \{b\}$. But $(f^A)_1(X_1) = 1$ and $(f^A)_1(X_2) = 2$. This means that $A \times A$ is not a congruence on $\mathcal{A}$.

Lemma 3.16 *Let $(\theta_j)_{j\in J}$ be a family of congruence relations on $\mathcal{A}$. Then $\bigcap\limits_{j\in J} \theta_j$ is a congruence relation on $\mathcal{A}$.*

Proof. The intersection of equivalence relations on A is again an equivalence relation on A. Now it is left to prove that any subset X of $\bigcap_{j\in J}\theta_j$ satisfies Definition 3.14. Let $X \subseteq \bigcap_{j\in J}\theta_j$ be a subset of $\bigcap_{j\in J}\theta_j$. Then $X \subseteq \theta_j$ for all $j \in J$. This gives $((f_i^A)_2(X_1),(f_i^A)_2(X_2)) \in \theta_j$ and $(f_i^A)_1(X_1) = (f_i^A)_1(X_2)$ for all $i \in I$ and all $j \in J$. Therefore $((f_i^A)_2(X_1),(f_i^A)_2(X_2)) \in \bigcap_{j\in J}\theta_j$ and $(f_i^A)_1(X_1) = (f_i^A)_1(X_2)$ for all $i \in I$. ∎

As in the hyperalgebra case, congruences can be used to produce quotient hyper-coalgebras.

Definition 3.17 Let $\mathcal{A} = (A;(f_i^A)_{i\in I})$ be a hyper-coalgebra of type τ and let θ be a congruence relation on $\mathcal{A}$. We define hyper-co-operations on the quotient set A/θ by

$$f_i^{A/\theta}(X_\theta) \;:=\; ((f_i^A)_1(X),[(f_i^A)_2(X)]_\theta),$$

where $X \subseteq A$ and $X_\theta := \{[x]_\theta \mid x \in X\}$. Then the hyper-coalgebra $\mathcal{A}/\theta = (A/\theta;(f_i^{A/\theta})_{i\in I})$ is called the *quotient hyper-coalgebra* of $\mathcal{A}$ by θ.

The definition means that

$$\begin{aligned}(f_i^{A/\theta})_1([X_\theta]) &= (f_i^A)_1(X)\\ \text{and}\\ (f_i^{A/\theta})_2([X_\theta]) &= [(f_i^A)_2(X)]_\theta.\end{aligned}$$

For this definition to be valid we have to verify that the co-operations $f_i^{A/\theta}$ defined on A/θ are well-defined. To check this, let $[X]_\theta = [Y]_\theta$. This gives $\{[x]_\theta \mid x \in X\} = \{[y]_\theta \mid y \in Y\}$, this means that for all $x \in X$, there is $y' \in Y$ such that $(x,y') \in \theta$, and for all $y \in Y$, there is $x' \in X$ such that $(x',y) \in \theta$. We define

$$W := \{(x,y') \in X \times Y \mid \forall x \in X\ \exists y' \in Y\ ((x,y') \in \theta)\}$$

$$\cup\{(x',y) \in X \times Y \mid \forall y \in Y\ \exists x' \in X\ ((x',y) \in \theta)\}.$$

Then we have $W \subseteq \theta$. Let $W_1 := \pi_1(W)$ and $W_2 := \pi_2(W)$. By the definition of a congruence relation on a hyper-coalgebra we get that $(f_i^A)_1(W_1) = (f_i^A)_1(W_2)$ and $((f_i^A)_2(W_1),(f_i^A)_1(W_2)) \in \theta$. Since $W_1 = \pi_1(W) = X$ and $W_2 = \pi_2(W) = Y$, we have $(f_i^A)_1(X) = (f_i^A)_1(Y)$ and $((f_i^A)_2(X),(f_i^A)_1(Y)) \in \theta$. Therefore

$$\begin{aligned}f_i^{A/\theta}([X]_\theta) &= ((f_i^A)_1(X),[(f_i^A)_2(X)]_\theta)\\ &= ((f_i^A)_1(Y),[(f_i^A)_2(Y)]_\theta)\\ &= f_i^{A/\theta}([Y]_\theta).\end{aligned}$$

Example 3.18 Let $A = \{a,b,c\}$ and let the equivalence relation θ on A be given by the partition $\{\{a,b\},\{c\}\}$. Then $A/\theta = \{[a]_\theta,[c]_\theta\}$. Let the binary hyper-co-operation f^A on A be defined by $f^A(\emptyset) = (2,b)$, $f^A(A) = (1,c)$, $f^A(\{a\}) = (1,a)$, $f^A(\{b\}) = (1,a)$, $f^A(\{c\}) = (1,c)$, $f^A(\{a,b\}) = (1,a)$, $f^A(\{a,c\}) = (1,c)$ and $f^A(\{b,c\}) = (1,c)$. Then $f^{A/\theta} : \mathcal{P}(A/\theta) \to (A/\theta)^{\sqcup 2}$ is defined by $\emptyset \mapsto (2,[a]_\theta)$, $\{[a]_\theta,[c]_\theta\} \mapsto (1,[c]_\theta)$, $\{[a]_\theta\} \mapsto (1,[a]_\theta)$ and $\{[c]_\theta\} \mapsto (1,[c]_\theta)$.

Proposition 3.19 *Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be a hyper-coalgebra of type τ and let θ be a congruence on $\mathcal{A}$. Then the natural mapping $\gamma : A \to A/\theta$ defined by $a \mapsto [a]_\theta$ is a surjective homomorphism from $\mathcal{A}$ onto $\mathcal{A}/\theta$.*

Proof. For any $X \subseteq A$, we have

$$\begin{aligned}(\gamma^{\sqcup n_i} \circ f_i^A)(X) &= \gamma^{\sqcup n_i}(f_i^A(X)) \\ &= \gamma^{\sqcup n_i}((f_i^A)_1(X), (f_i^A)_2(X)) \\ &= ((f_i^A)_1(X), \gamma((f_i^A)_2(X))) \\ &= ((f_i^A)_1(X), [(f_i^A)_2(X)]_\theta) \\ &= f_i^{A/\theta}([X]_\theta) \\ &= f_i^{A/\theta}(\bar{\gamma}(X)) \\ &= (f_i^{A/\theta} \circ \bar{\gamma})(X).\end{aligned}$$

This shows that γ is a homomorphism. ■

Congruences on (F_1, F_2)-systems are defined as kernels of homomorphisms. Now we prove that any congruence of a hyper-coalgebra of type τ corresponds to a congruence of the corresponding (F_1, F_2)-system.

Theorem 3.20 *Let $\mathcal{A}$ be a hyper-coalgebra of type τ. Then an equivalence relation θ on A is a congruence on $\mathcal{A}$ if and only if θ is the kernel of some homomorphism from $\mathcal{A}$ to some hyper-coalgebra $\mathcal{B}$.*

Proof. When θ is a congruence, it is clear that θ is the kernel of the natural mapping $\gamma : \mathcal{A} \to \mathcal{A}/\theta$ since

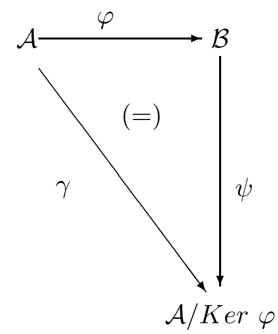
$$(a, b) \in \theta \Leftrightarrow [a]_\theta = [b]_\theta \Leftrightarrow \gamma(a) = \gamma(b) \Leftrightarrow (a, b) \in Ker\ \gamma.$$

Conversely, let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism with $Ker\ \varphi$ as its kernel. Then $Ker\ \varphi$ is an equivalence relation on A. For any $X \subseteq Ker\ \varphi$, we have $\bar{\varphi}(X_1) = \bar{\varphi}(X_2)$, so that $f_i^B(\bar{\varphi}(X_1)) = f_i^B(\bar{\varphi}(X_2))$ and thus $(f_i^B \circ \bar{\varphi})(X_1) = (f_i^B \circ \bar{\varphi})(X_2)$ since φ is a homomorphism. This implies that $(\varphi^{\sqcup n_i} \circ f_i^A)(X_1) = (\varphi^{\sqcup n_i} \circ f_i^A)(X_2)$, so that $\varphi^{\sqcup n_i}((f_i^A)_1(X_1), (f_i^A)_2(X_1)) = \varphi^{\sqcup n_i}((f_i^A)_1(X_2), (f_i^A)_2(X_2))$
and thus $((f_i^A)_1(X_1), \varphi((f_i^A)_2(X_1))) = ((f_i^A)_1(X_2), \varphi((f_i^A)_2(X_2)))$, which makes $(f_i^A)_1(X_1) = (f_i^A)_1(X_2)$ and $\varphi((f_i^A)_2(X_1)) = \varphi((f_i^A)_2(X_2))$. This means that $((f_i^A)_2(X_1), (f_i^A)_2(X_2)) \in Ker\ \varphi$. Therefore $Ker\ \varphi$ is a congruence relation on $\mathcal{A}$. ■

Thus, if θ is a congruence on the hyper-coalgebra $\mathcal{A}$ of type τ, then θ is the kernel of some homomorphism of the hyper-coalgebra $\mathcal{A}$ of type τ. But this homomorphism corresponds to a homomorphism of the corresponding (F_1, F_2)-system and θ is the kernel of this (F_1, F_2)-system homomorphism and thus a congruence on the (F_1, F_2)-system and conversely. This shows that congruences on hyper-coalgebras of type τ correspond to congruences of the corresponding (F_1, F_2)-systems.

The next theorem shows the Homomorphic Image Theorem for hyper-coalgebras of type τ.

Theorem 3.21 *Let $\mathcal{A}$ and $\mathcal{B}$ be hyper-coalgebras of type τ, with φ a surjective homomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Then $\mathcal{B}$ is isomorphic to the quotient hyper-coalgebra $\mathcal{A}/Ker\ \varphi$ and the diagram below commutes:*



Proof. We use the natural mapping γ considered in Proposition 3.19 to define a mapping $\psi : \mathcal{B} \to \mathcal{A}/Ker\ \varphi$ by $\psi(b) = \gamma(a)$ for $b = \varphi(a)$ and any $a \in A$. This mapping is well-defined, since $\varphi(c) = b = \varphi(a) \in B$ implies that $(c, a) \in Ker\ \varphi$ and so $[c]_{Ker\ \varphi} = [a]_{Ker\ \varphi}$. It is clear that ψ is onto, and we show that ψ is one-to-one. If $\psi(b_1) = \psi(b_2)$ for some $b_1, b_2 \in B$, then there are elements $a_1, a_2 \in A$ with $b_1 = \varphi(a_1)$ and $b_2 = \varphi(a_2)$. Since $[a_1]_{Ker\ \varphi} = [a_2]_{Ker\ \varphi}$, that is $(a_1, a_2) \in Ker\ \varphi$, we get $b_1 = b_2$. Also, $\psi \circ \varphi = \gamma$ since $\psi(\varphi(a)) = \psi(b) = \gamma(a)$. It is left to prove that ψ is a homomorphism, i.e., $f_i^{A/Ker\ \varphi} \circ \bar{\psi} = \psi^{\sqcup n_i} \circ f_i^B$ for all $i \in I$. Let $X \subseteq B$. Since φ is surjective, there is $Y \subseteq A$ such that $\bar{\varphi}(Y) = X$. Since φ is a homomorphism, then $f_i^B(\bar{\varphi}(Y)) = \varphi^{\sqcup n_i}(f_i^A(Y))$. There follows $((f_i^B)_1(X), (f_i^B)_2(X)) = ((f_i^A)_1(Y), \varphi((f_i^A)_2(Y)))$. This implies that $(f_i^B)_1(X) = (f_i^A)_1(Y)$, $(f_i^B)_2(X) = \varphi((f_i^A)_2(Y))$, $(f_i^B)_1(X) = (f_i^A)_1(Y)$ and $\psi((f_i^B)_2(X)) = [(f_i^A)_2(Y)]_{Ker\ \varphi}$. Therefore

$$
\begin{aligned}
&(\psi^{\sqcup n_i} \circ f_i^B)(X)\\
&= \psi^{\sqcup n_i}(f_i^B(X))\\
&= \psi^{\sqcup n_i}((f_i^B)_1(X), (f_i^B)_2(X))\\
&= ((f_i^B)_1(X), \psi((f_i^B)_2(X)))\\
&= ((f_i^A)_1(Y), [(f_i^A)_2(Y)]_{Ker\ \varphi})\\
&= ((f_i^A)_1(Y), \gamma((f_i^A)_2(Y)))\\
&= \gamma^{\sqcup n_i}((f_i^A)_1(Y), (f_i^A)_2(Y))\\
&= \gamma^{\sqcup n_i}(f_i^A(Y))\\
&= (\gamma^{\sqcup n_i} \circ f_i^A)(Y)\\
&= (f_i^{A/Ker\ \varphi} \circ \bar{\gamma})(Y) \qquad (\ \gamma \text{ is a homomorphism})\\
&= f_i^{A/Ker\ \varphi}(\bar{\gamma}(Y))\\
&= f_i^{A/Ker\ \varphi}(\{\gamma(y) \mid y \in Y\})\\
&= f_i^{A/Ker\ \varphi}(\{\psi(\varphi(y)) \mid y \in Y\})\\
&= f_i^{A/Ker\ \varphi}(\{\psi(x) \mid x \in X\})\\
&= f_i^{A/Ker\ \varphi}(\bar{\psi}(X))\\
&= (f_i^{A/Ker\ \varphi} \circ \bar{\psi})(X).
\end{aligned}
$$

Hence ψ is a homomorphism. ■

At the end of this section we want to present bisimulations for hyper-coalgebras of type τ and show that our definition here coincides with the definition of a bisimulation for the corresponding (F_1, F_2)-system. Indeed, a relation $R \subseteq A \times B$ is called a bisimulation between (F_1, F_2)-systems A and B if there is a mapping $\alpha_R : F_1(R) \to F_2(R)$ such that the projections $\pi_A : R \to A$ and $\pi_B : R \to B$ are homomorphisms.

Definition 3.22 Let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ and $\mathcal{B} = (B; (f_i^B)_{i \in I})$ be hyper-coalgebras of type τ, and let $R \subseteq A \times B$ be a binary relation. Then R is said to be a bisimulation between $\mathcal{A}$ and $\mathcal{B}$ if for all $i \in I$ and for all $X \subseteq R$, we have $(f_i^A)_1(X_1) = (f_i^B)_1(X_2)$ and

$((f_i^A)_2(X_1), (f_i^B)_2(X_2)) \in R$ whenever $\bar{\pi}_A(X) = X_1$ and $\bar{\pi}_B(X) = X_2$.

Now we prove that for hyper-coalgebras of type τ this definition is equivalent to the definition of bisimulations for (F_1, F_2)-systems.

Theorem 3.23 *Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ and $\mathcal{B} = (B; (f_i^B)_{i\in I})$ be hyper-coalgebras of type τ and let $R \subseteq A \times B$ be a binary relation. Then R is a bisimulation between $\mathcal{A}$ and $\mathcal{B}$ if and only if R is a bisimulation between the corresponding (F_1, F_2)-systems.*

Proof. First let R be a bisimulation between the (F_1, F_2)-systems $\mathcal{A}$ and $\mathcal{B}$. Then for each $i \in I$ there are hyper-co-operations $f_i^R : \mathcal{P}(R) \to R^{\sqcup n_i}$ on R such that the projections $\pi_A : R \to A$ and $\pi_B : R \to B$ are homomorphisms. Thus for each subset $X \subseteq R$ we have

$$\begin{aligned}
&((f_i^R)_1(X), \pi_A((f_i^R)_2(X)))\\
&= \pi_A^{\sqcup n_i}(f_i^R(X))\\
&= f_i^A(\bar{\pi}_A(X))\\
&= f_i^A(X_1)\\
&= ((f_i^A)_1(X_1), (f_i^A)_2(X_1)),
\end{aligned}$$

and

$$\begin{aligned}
&((f_i^R)_1(X), \pi_B((f_i^R)_2(X)))\\
&= \pi_B^{\sqcup n_i}(f_i^R(X))\\
&= f_i^B(\bar{\pi}_B(X))\\
&= f_i^B(X_2)\\
&= ((f_i^B)_1(X_2), (f_i^B)_2(X_2)).
\end{aligned}$$

This means that

$$(f_i^A)_1(X_1) = (f_i^B)_1(X_2) \text{ and } (f_i^R)_2(X) = ((f_i^A)_2(X_1), (f_i^B)_2(X_2)) \in R.$$

This shows that R is a bisimulation between the hyper-coalgebras $(A; (f_i^A)_{i\in I})$ and $(B; (f_i^B)_{i\in I})$ of type τ.

Conversely, suppose that R is a bisimulation between the coalgebras $\mathcal{A} = (A; (f_i^A)_{i\in I})$ and $\mathcal{B} = (B; (f_i^B)_{i\in I})$. By definition we have

$$(f_i^A)_1(X_1) = (f_i^B)_1(X_2) \text{ and } ((f_i^A)_2(X_1), (f_i^B)_2(X_2)) \in R,$$

for each $i \in I$ and each $X \subseteq R$. For each $i \in I$ we define n_i-ary hyper-co-operations on R by $(f_i^R)_1(X) = (f_i^A)_1(X_1) = (f_i^B)_1(X_2)$ and $(f_i^R)_2(X) = ((f_i^A)_2(X_1), (f_i^B)_2(X_2))$ for all $X \subseteq R$. Then $\mathcal{R} = (R; (f_i^R)_{i\in I})$ is a hyper-coalgebra of type τ, and it suffices to show that the projections $\pi_A : R \to A$ and $\pi_B : R \to B$ are homomorphisms. For any $X \subseteq R$,

$$\begin{aligned}
(\pi_A^{\sqcup n_i} \circ f_i^R)(X) &= \pi_A^{\sqcup n_i}(f_i^R(X))\\
&= \pi_A^{\sqcup n_i}((f_i^R)_1(X), (f_i^R)_2(X))\\
&= ((f_i^R)_1(X), \pi_A((f_i^R)_2(X)))\\
&= ((f_i^A)_1(X_1), (f_i^A)_2(X_1))\\
&= f_i^A(X_1)\\
&= f_i^A(\bar{\pi}_A(X))\\
&= (f_i^A \circ \bar{\pi}_A)(X).
\end{aligned}$$

This shows that π_A is a homomorphism, and the proof for π_B is similar. ■

This equivalence of our definition of bisimulation for hyper-coalgebras of type τ and the definition for (F_1, F_2)-systems means that all the results for bisimulations of (F_1, F_2)-systems are also valid for bisimulations of hyper-coalgebras of type τ.

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