MORE ON $G^{\#}\alpha$-OPEN SETS IN DIGITAL PLANES

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Abstract. In this paper, we continue the study of the concept of $g^{\#}\alpha$-closed sets, $g^{\#}\alpha$-open sets and digital planes $(\mathbb{Z}^2, \kappa^2)$ (cf. [19]). In 1970, E.D. Khalimsky [10] introduced the concept of the digital line or so called *Khalimsky line* $(\mathbb{Z}, \kappa)$. The digital plane $(\mathbb{Z}^2, \kappa^2)$ (eg. [13]) is the topological product of two copies of $(\mathbb{Z}, \kappa)$. A subset A of a topological space (X, τ) is said to be $g^{\#}\alpha$-closed [19], if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is g-open set of (X, τ). The complement of a $g^{\#}\alpha$-closed set is said to be a $g^{\#}\alpha$-open set of (X, τ). The $g^{\#}\alpha$-openness in $(\mathbb{Z}^2, \kappa^2)$ is characterized (cf. Theorem 2.3 (ii)): for a subset B with some closed singletons of $(\mathbb{Z}^2, \kappa^2)$, B is $g^{\#}\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$ if and only if $(U(x))_{\kappa^2} \subset B$ holds for each closed singleton $\{x\} \subset B$, where $U(x)$ is the smallest open set containing x. The family of all $g^{\#}\alpha$-open sets of $(\mathbb{Z}^2, \kappa^2)$, say $G^{\#}\alpha O$, forms an alternative topology of $\mathbb{Z}^2$ (cf. Theorem A, Corollary B (i)). Let $(\mathbb{Z}^2, G^{\#}\alpha O)$ be a topological space obtained by changing the topology κ^2 of the digital plane $(\mathbb{Z}^2, \kappa^2)$ by $G^{\#}\alpha O$. We prove that this plane $(\mathbb{Z}^2, G^{\#}\alpha O)$ is a $T_{1/2}$-space (cf. Corollary B (ii) (ii-1), Remark 3.5); moreover it is shown that the plane $(\mathbb{Z}^2, G^{\#}\alpha O)$ is $T_{3/4}$ (cf. Corollary B (ii) (ii-2)). It is well known that the digital plane $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$ even if $(\mathbb{Z}, \kappa)$ is $T_{1/2}$.

1 Introduction and main results Throughout this paper, (X, τ) represents a nonempty topological space on which no separation axioms are assumed unless otherwise mentioned. In 1970, N. Levine [15] introduced the concept of the generalized closed sets in topological spaces. A subset A of a topological space (X, τ) is *generalized closed* (shortly, *g-closed*), if $Cl(A) \subset U$ whenever $A \subset U$ and U is any open set of (X, τ) ([15, Definition 2.1]). It is obvious that every closed set is g-closed. The complement of a g-closed set of (X, τ) is called *g-open* in (X, τ) ([15, Definition 4.1]). A subset B is g-open in (X, τ) if and only if $F \subset Int(B)$ whenever $F \subset B$ and F is any closed set of (X, τ) ([15, Theorem 4.2]). Moreover, using the concept of g-closed sets, he introduced the notion of the class of $T_{1/2}$-topological spaces which is properly placed between the class of T_1-spaces and T_0-spaces ([15, Definition 5.1]). A space is called a $T_{1/2}$-space if every g-closed set is closed. In 1977, W. Dunham [6, Theorem 2.5] proved that a topological space is $T_{1/2}$ if and only if every singleton is open or closed (cf. [11, p.7, line -6]). A typical example of the class of $T_{1/2}$-spaces is the *digital line* or so called *Khalimsky line*, say $(\mathbb{Z}, \kappa)$. It is not T_1. The definition of the digital line was published in Russia by E. Khalimsky in 1970 [10]. In 1990, E. Khalimsky, K. Kopperman and R. Meyer [11] developed a finite analog of the Jordan curve theorem motibated by a problem in computer graphics (cf. [12], [11]). In the present paper, the *digital plane* $(\mathbb{Z}^2, \kappa^2)$ is the topological product of two copies of the digital line $(\mathbb{Z}, \kappa)$, where $\mathbb{Z}^2 := \mathbb{Z} \times \mathbb{Z}$ and $\kappa^2 := \kappa \times \kappa$ (eg., [13, Definition 4], [21, p.335, p.336], [8], [7], [19], [2, Section 6]). It is well known that $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$ (cf. Section 2 below; eg., [2, line -3 in p.50], [8, p.32]). The digital plane is a mathematical model of the computer screen. The digital plane includes all 2-dimensional discrete objects in mathematical world.

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A subset B of (X, τ) is called *α-open* [18] in (X, τ) if $B \subset Int(Cl(Int(B)))$ holds in (X, τ); τ^α denotes the family of all α-open sets of (X, τ). It is well known that τ^α forms a topology of X ([18], eg., [20]). The complement of an α-open set is called an *α-closed* set of (X, τ). A subset E is α-closed if and only if $Cl(Int(Cl(E))) \subset E$ if and only if $\alpha Cl(E) = E$, where $\alpha Cl(D) := \bigcap\{F |\ D \subset F, F$ is α-closed in $(X, \tau)\}$ for a subset D of X. The *α-interior* of a subset E is defined as follows: $\alpha Int(E) := \bigcup\{U |\ U \subset E, U$ is α-open in $(X, \tau)\}$.

Definition 1.1 A subset A of (X, τ) is *$g^\#\alpha$-closed* ([19, Definition 2.1]), if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is any g-open set of (X, τ). The complement of a $g^\#\alpha$-closed set is called a *$g^\#\alpha$-open* set of (X, τ).

It is shown that: a subset B is $g^\#\alpha$-open in (X, τ) if and only if $F \subset \alpha Int(B)$ whenever $F \subset B$ and F is any g-closed set of (X, τ).

By [19, Remark 2.3], we obtain the folowing diagram of implications and none of these implications is reversible.

$$
\begin{array}{ccccccc}
 & \alpha\text{-open} & & & & & \\
\nearrow & & \searrow & & & & \\
\text{open} & & \text{g}^\#\alpha\text{-open} & \rightarrow & \alpha\text{g-open} & \rightarrow & \text{gs-open} \\
\searrow & & \nearrow & & & & \\
 & \text{g}^*\text{-open} & & & & &
\end{array}
$$

Some basic properties of $g^\#\alpha$-open sets and some properties of subsets on $(\mathbb{Z}^2, \kappa^2)$ are studied by [19] (cf. Theorem 2.1(ii), Theorem 3.1(i) below); in general, it is shown that:([19, Theorem 2.5 (i)]) for a topological space (X, τ), the intersection of two $g^\#\alpha$-open sets of (X, τ) is $g^\#\alpha$-open in (X, τ) (cf. Theorem 3.1 (i)). By [15, p.92 and Example 2.5], the union of two g-open sets is generally not g-open in a topological space. For the concept of $g^\#\alpha$-open sets of a topological space, we have the corresponding problem. Is the union of two $g^\#\alpha$-open sets of a topological space in generally not $g^\#\alpha$-open ?

The purpose of the present paper is to solve the above problem for the digital plane $(\mathbb{Z}^2, \kappa^2)$; we have an answer to the above problem for $(\mathbb{Z}^2, \kappa^2)$ and related properties as follows (cf. Theorem A, Corollary B, Theorem 2.3 below).

Theorem A (i) *The union of any collection of $g^\#\alpha$-open sets of $(\mathbb{Z}^2, \kappa^2)$ is $g^\#\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$.*

(ii) *The intersection of any collection of $g^\#\alpha$-closed sets of $(\mathbb{Z}^2, \kappa^2)$ is $g^\#\alpha$-closed in $(\mathbb{Z}^2, \kappa^2)$.*

It is well known that the digital line $(\mathbb{Z}, \kappa)$ is $T_{1/2}$; but the digital plane $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$. As corollary of Theorem A, we have a new topology, say $G^\#\alpha O(\mathbb{Z}^2, \kappa^2)$ of $\mathbb{Z}^2$. We change the topology κ^2 of $(\mathbb{Z}^2, \kappa^2)$ by new topology $G^\#\alpha O(\mathbb{Z}^2, \kappa^2)$ (cf. [21, Section 3. Change the topologies]). Consequently we get a new $T_{1/2}$-topological space, say $(\mathbb{Z}^2, G^\#\alpha O(\mathbb{Z}^2, \kappa^2))$, associated to $(\mathbb{Z}^2, \kappa^2)$.

Corollary B *Let $G^\#\alpha O(\mathbb{Z}^2, \kappa^2)$ be the family of all $g^\#\alpha$-open sets in $(\mathbb{Z}^2, \kappa^2)$. Then, the following properties hold.*

(i) *The family $G^\#\alpha O(\mathbb{Z}^2, \kappa^2)$ is a topology of $\mathbb{Z}^2$.*

(ii) *Let $(\mathbb{Z}^2, G^\#\alpha O(\mathbb{Z}^2, \kappa^2))$ be a topological space obtained by changing the topology κ^2 of the digital plane $(\mathbb{Z}^2, \kappa^2)$ by $G^\#\alpha O(\mathbb{Z}^2, \kappa^2)$.*

(ii-1) *$(\mathbb{Z}^2, G^\#\alpha O(\mathbb{Z}^2, \kappa^2))$ is a $T_{1/2}$-topological space.*

(ii-2) *Moreover,* $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2,\kappa^2))$ *is* $T_{3/4}$ *and it is not* T_1.

We shall prove the above main results in Section 3 (cf. the end of Section 3 for the notion of $T_{3/4}$-spaces). For some undefined or related concepts, the reader refered to some papers in References of the present paper, [21], [17], [3] and [1].

2 Charactrizations of $g^{\#}\alpha$-open sets in the digital plane We first recall some notations and concepts for properties on $(\mathbb{Z}^2,\kappa^2)$ as follows. The *digital line* is the set of the integers, $\mathbb{Z}$, equipped with the topology κ having $\{\{2m-1,2m,2m+1\}|\ m\in\mathbb{Z}\}$ as a subbase (cf. Section 1, e.g., [8, Sections 1–3]). This topological space is denoted by $(\mathbb{Z},\kappa)$. A subset U of $\mathbb{Z}$ is open if and only if whether $x\in U$ is an even integer, then $x-1, x+1\in U$. Let $(\mathbb{Z}^2,\kappa^2)$ be the topological product of two copies of the digital line $(\mathbb{Z},\kappa)$, where $\mathbb{Z}^2=\mathbb{Z}\times\mathbb{Z}$ and $\kappa^2=\kappa\times\kappa$. This topological space $(\mathbb{Z}^2,\kappa^2)$ is called the *digital plane* in the present paper (eg., [13, Definition 4], [8, p.32 and Sections 4, 5], [7, p.164], [22], [19, Section 5], [2, Section 6]; cf. [11, Definition 4.1], [12, p.907, Section 4]). The concept of the *smallest open set* $U(x)$ containing a point x of $(\mathbb{Z}^2,\kappa^2)$ is usefull; let $x\in\mathbb{Z}^2$,

$U(x) := \{2s-1,2s,2s+1\}\times\{2u-1,2u,2u+1\}$ if $x=(2s,2u)$;
$U(x) := \{(2s+1,2u+1)\}$ if $x=(2s+1,2u+1)$;
$U(x) := \{2s-1,2s,2s+1\}\times\{2u+1\}$ if $x=(2s,2u+1)$;
$U(x) := \{2s+1\}\times\{2u-1,2u,2u+1\}$ if $x=(2s+1,2u)$, where $s,u\in\mathbb{Z}$.

By a property of κ^2, it is well known that, for an open set G containing a point x, $x\in U(x)\subset G$ hold. We call the set $U(x)$ by the *smallest open set containing* x ([7, line +8 in p.164 etc], [19, line −17 in p.21 etc], [2, line −15 in p.50 etc]). In [8, line −7 in p.38 and Lemma 4.2 etc], the set $U(x)$ is called as the *basic open neighbourhood of* x. It follows from the definition of κ^2 that every singleton $\{(2s,2u)\}$ is closed and every singleton $\{(2s+1,2u+1)\}$ is open in $(\mathbb{Z}^2,\kappa^2)$, where $s,u\in\mathbb{Z}$. Moreover, singletons $\{(2s+1,2u)\}$ and $\{(2s,2u+1)\}$ are not open in $(\mathbb{Z}^2,\kappa^2)$; they are not closed in $(\mathbb{Z}^2,\kappa^2)$; such points $(2s+1,2u)$ and $(2s,2u+1)$ are called *mixed* ([11, p.11], [12, p.907]); the singletons $\{(2s+1,2u)\}$ and $\{(2s,2u+1)\}$ are nowhere dense, i.e., $Int(Cl(\{(2s+1,2u)\}))=Int(Cl(\{(2s,2u+1)\}))=\emptyset$. The above properties show that $(\mathbb{Z}^2,\kappa^2)$ is not $T_{1/2}$. We use the following notation ([7], [19], [8], [2]):

$(\mathbb{Z}^2)_{\kappa^2} := \{x\in\mathbb{Z}^2|\ \{x\}$ is open in $(\mathbb{Z}^2,\kappa^2)\}$;
$(\mathbb{Z}^2)_{\mathcal{F}^2} := \{x\in\mathbb{Z}^2|\ \{x\}$ is closed in $(\mathbb{Z}^2,\kappa^2)\}$;
$(\mathbb{Z}^2)_{mix} := \mathbb{Z}^2\setminus((\mathbb{Z}^2)_{\kappa^2}\cup(\mathbb{Z}^2)_{\mathcal{F}^2})$;

for a subset E of $(\mathbb{Z}^2,\kappa^2)$, $E_{\kappa^2} := E\cap((\mathbb{Z}^2)_{\kappa^2})$; $E_{\mathcal{F}^2} := E\cap((\mathbb{Z}^2)_{\mathcal{F}^2})$;
$E_{mix} := E\cap((\mathbb{Z}^2)_{mix})$ and $U(E) := \bigcup\{U(x)|x\in E\}$,

where $U(x)$ is the smallest open set containing x.

It is well known that: for a subset E of $(\mathbb{Z}^2,\kappa^2)$,

$E_{mix} = E\setminus(E_{\kappa^2}\cup E_{\mathcal{F}^2})$;
$E_{\kappa^2} = \{(2s+1,2u+1)\in E|s,u\in\mathbb{Z}\}$;
$E_{\mathcal{F}^2} = \{(2s,2u)\in E|\ s,u\in\mathbb{Z}\}$;
$E_{mix} = \{(2s+1,2u)\in E|\ s,u\in\mathbb{Z}\}\cup\{(2s,2u+1)\in E|\ s,u\in Z\}$;

$\mathbb{Z}^2=(\mathbb{Z}^2)_{\mathcal{F}^2}\cup(\mathbb{Z}^2)_{\kappa^2}\cup(\mathbb{Z}^2)_{mix}$ (disjoint union) and $E=E_{\mathcal{F}^2}\cup E_{\kappa^2}\cup E_{mix}$ (disjoint union) hold for any subset E of $(\mathbb{Z}^2,\kappa^2)$.

In [2, p.52 etc], [19, p.21 etc] and [8, p.38 etc], the notation $(\mathbb{Z}^2)_{\mathcal{F}^2}$ was written by $(\mathbb{Z}^2)_{\mathcal{F}}$. Sometimes, E_{κ^2}, $E_{\mathcal{F}^2}$ and E_{mix} are written by $(E)_{\kappa^2}$, $(E)_{\mathcal{F}^2}$ and $(E)_{mix}$, respectively. For example, a notation $U((X\setminus E)_{mix})$ means the following subset $\bigcup\{U(x)|\ x\in(X\setminus E)_{mix}\} = \bigcup\{U(x)|\ x\in X\setminus E, x\in(\mathbb{Z}^2)_{mix}\}$.

A subset V is *preopen* [16] in a topological space (X,τ), if $V\subset Int(Cl(V))$ holds in (X,τ). A subset W is *semi-open* [14] in a topological space (X,τ), if $W\subset Cl(Int(W))$

holds in (X,τ). Let $PO(X,\tau)$ (resp. $SO(X,\tau)$) be the family of all preopen (resp. semi-open) sets of (X,τ).

We recall some properties needed later on $g^{\#}\alpha$-open sets, $g^{\#}\alpha$-closed sets, preopen sets and semi-open sets in $(\mathbb{Z}^2,\kappa^2)$.

Theorem 2.1 (i) ([2, Theorem 6.1]; eg., [7, Theorem 2.1(ii)]) *$PO(\mathbb{Z}^2,\kappa^2) \subset SO(\mathbb{Z}^2,\kappa^2)$ and $(\kappa^2)^\alpha = PO(\mathbb{Z}^2,\kappa^2)$ hold.*

(ii) ([19, Corollary 5.3 (ii)]) *If a subset F is g-closed and $F \subset A_{mix} \cup A_{\kappa^2}$ holds for some subset A of $(\mathbb{Z}^2,\kappa^2)$, then $F = \emptyset$.*

We prove a lemma needed later.

Lemma 2.2 *Let A and B be subsets of $(\mathbb{Z}^2,\kappa^2)$ and x,y points of $(\mathbb{Z}^2,\kappa^2)$.*

(i) *If A is a g-closed set of $(\mathbb{Z}^2,\kappa^2)$ and $y \in A_{mix}$, then $Cl(\{y\})\backslash\{y\} \subset A$ and hence $Cl(\{y\}) \subset A$ in $(\mathbb{Z}^2,\kappa^2)$.*

(ii) *For a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, a subset $\{x\} \cup (U(x))_{\kappa^2}$ is preopen in $(\mathbb{Z}^2,\kappa^2)$ and hence it is α-open in $(\mathbb{Z}^2,\kappa^2)$* (cf. Theorem 2.1(i)).

Proof. **(i)** Since $y \in A_{mix}$, we can set $y = (2s, 2u+1)$ or $y = (2s+1, 2u)$, where $s,u \in \mathbb{Z}$. Then, $Cl(\{y\}) = \{2s\} \times \{2u, 2u+1, 2u+2\} = \{y, y^+, y^-\}$ if $y = (2s, 2u+1)$, where $y^+ := (2s, 2u+2)$ and $y^- := (2s, 2u)$; $Cl(\{y\}) = \{2s, 2s+1, 2s+2\} \times \{2u\} = \{y^-, y, y^+\}$ if $y = (2s+1, 2u)$, where $y^+ := (2s+2, 2u)$ and $y^- := (2s, 2u)$. Thus, we have that $Cl(\{y\}) \setminus \{y\} = \{y^-, y, y^+\} \setminus \{y\} = \{y^-, y^+\}$. It is noted that $\{y^+\}$ and $\{y^-\}$ are closed singletons of $(\mathbb{Z}^2,\kappa^2)$. We suppose that $y^+ \notin A$ or $y^- \notin A$. If $y^+ \notin A$, then $y^+ \in Cl(\{y\}) \subset Cl(A)$ and so $y^+ \in Cl(A) \setminus A$. Then, $Cl(A) \setminus A$ contains a closed set $\{y^+\}$; this contradicts to [15, Theorem 2.2], i.e., for a topological space (X,τ), a subset A of X is g-closed if and only if $Cl(A) \setminus A$ does not contain nonempty closed subset of (X,τ). If $y^- \notin A$, then $y^- \in Cl(\{y\}) \subset Cl(A)$ and so $y^- \in Cl(A) \setminus A$; the subset $Cl(A) \setminus A$ contains a closed set $\{y^-\}$. Thus, for the case where $y^- \notin A$, we have also a contradiction. Therefore, we prove that $y^+ \in A$ and $y^- \in A$ and hence $Cl(\{y\}) \setminus \{y\} \subset A$ and we have that $Cl(\{y\}) \subset A$, because $y \in A_{mix} \subset A$.

(ii) We set $x := (2s, 2u)$, where $s,u \in \mathbb{Z}$ are integers, because $\{x\}$ is closed. Let $p_1 := (2s-1, 2u-1), p_2 := (2s-1, 2u+1), p_3 := (2s+1, 2u-1)$ and $p_4 := (2s+1, 2u+1)$. Then, $U(x) = \{2s-1, 2s, 2s+1\} \times \{2u-1, 2u, 2u+1\}$ and $\{x, p_1, p_2, p_3, p_4\} \subset U(x)$. We have that $Cl((U(x))_{\kappa^2}) = Cl(\bigcup\{\{p_i\} | i \in \{1,2,3,4\}\}) = \bigcup\{Cl(\{p_i\}) | i \in \{1,2,3,4\}\} = (\{2s-2, 2s-1, 2s\} \times \{2u-2, 2u-1, 2u\}) \cup (\{2s-2, 2s-1, 2s\} \times \{2u, 2u+1, 2u+2\}) \cup (\{2s, 2s+1, 2s+2\} \times \{2u, 2u+1, 2u+2\}) \cup (\{2s, 2s+1, 2s+2\} \times \{2u-2, 2u-1, 2u\})$ and so $Cl((U(x))_{\kappa^2}) = \{a \in \mathbb{Z} | 2s-2 \leq a \leq 2s+2\} \times \{b \in \mathbb{Z} | 2u-2 \leq b \leq 2u+2\}$; $Cl(\{x\} \cup (U(x))_{\kappa^2}) = (\{2s\} \times \{2u\}) \cup (\{a \in \mathbb{Z} | 2s-2 \leq a \leq 2s+2\} \times \{b \in \mathbb{Z} | 2u-2 \leq b \leq 2u+2\}) = Cl((U(x))_{\kappa^2})$. Then, it is obtained that $Int(Cl(\{x\} \cup (U(x))_{\kappa^2})) = Int(\{a \in \mathbb{Z} | 2s-2 \leq a \leq 2s+2\}) \times Int(\{b \in \mathbb{Z} | 2u-2 \leq b \leq 2u+2\}) = \{2s-1, 2s, 2s+1\} \times \{2u-1, 2u, 2u+1\} = U(x)$. Thus, we have that $\{x\} \cup (U(x))_{\kappa^2} \subset \{x\} \cup U(x) = U(x) = Int(Cl(\{x\} \cup (U(x))_{\kappa^2}))$, i.e., $\{x\} \cup (U(x))_{\kappa^2}$ is preopen in $(\mathbb{Z}^2,\kappa^2)$. It is well known that, for a subset W of $(\mathbb{Z}^2,\kappa^2)$, W is preopen in $(\mathbb{Z}^2,\kappa^2)$ if and only W is α-open in $(\mathbb{Z}^2,\kappa^2)$(cf. Theorem 2.1 (i)). Concequently, we have that the set $\{x\} \cup (U(x))_{\kappa^2}$ is α-open in $(\mathbb{Z}^2,\kappa^2)$. □.

Theorem 2.3 *Let B be a nonempty subset of $(\mathbb{Z}^2,\kappa^2)$.*

(i) *If $B_{\mathcal{F}^2} = \emptyset$, then B is a $g^{\#}\alpha$-open set of $(\mathbb{Z}^2,\kappa^2)$.*

(ii) *For a subset B such that $B_{\mathcal{F}^2} \neq \emptyset$, the following properties are equivalent in $(\mathbb{Z}^2,\kappa^2)$:*

(1) *The subset B is a $g^{\#}\alpha$-open set of $(\mathbb{Z}^2,\kappa^2)$;*

(2) *$(U(x))_{\kappa^2} \subset B$ holds for each point $x \in B_{\mathcal{F}^2}$.*

Proof. **(i)** Let F be a g-closed set such that $F \subset B$. Since $B_{\mathcal{F}^2} = \emptyset$, we have obviously that $B = B_{mix} \cup B_{\kappa^2}$ and so $F \subset B_{mix} \cup B_{\kappa^2}$. Then, by Theorem 2.1 (ii) it is obtained that $F = \emptyset$, because F is g-closed in $(\mathbb{Z}^2, \kappa^2)$. Thus, we conclude that whenever F is g-closed and $F \subset B, F = \emptyset \subset \alpha Int(B)$. Namely B is a $g^{\#}\alpha$-open set of $(\mathbb{Z}^2, \kappa^2)$.

(ii) (1)⇒(2) Let $x \in B_{\mathcal{F}^2}$. Since $\{x\}$ is closed, $\{x\}$ is a g-closed set and $\{x\} \subset B$. By (1), $\{x\} \subset \alpha Int(B) = B \cap Int(Cl(Int(B)))$ and so $x \in Int(Cl(Int(B)))$. Namely, x is an interior point of the set $Cl(Int(B))$. Thus, we have that, for the smallest open set $U(x)$ containing $x, U(x) \subset Cl(Int(B))$. We can set $x := (2s, 2u)$ for some integers s and u, because $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$. Since $U((2s, 2u)) = \{2s-1, 2s, 2s+1\} \times \{2u-1, 2u, 2u+1\}$, it is shown that $(U(x))_{\kappa^2} = \{(x_1, x_2) \in U(x) | x_1 \text{ and } x_2 \text{ are odd}\} = \{p_1, p_2, p_3, p_4\}$, where $p_1 := (2s-1, 2u-1), p_2 := (2s-1, 2u+1), p_3 := (2s+1, 2u-1)$ and $p_4 := (2s+1, 2u+1)$. For each point $p_i (1 \leq i \leq 4)$, $p_i \in Cl(Int(B))$ and so $\{p_i\} \cap Int(B) \neq \emptyset$. Therefore, $p_i \in B$ for each i with $1 \leq i \leq 4$ and hence $((U(x))_{\kappa^2} \subset B$.

(2)⇒(1) It follows from assumption that, for each point $x \in B_{\mathcal{F}^2}$, $\{x\} \cup (U(x))_{\kappa^2} \subset B$ and so $\bigcup\{\{x\} \cup (U(x))_{\kappa^2} |\ x \in B_{\mathcal{F}^2}\} \subset B$. Put $V_B := \bigcup\{\{x\} \cup (U(x))_{\kappa^2} | x \in B_{\mathcal{F}^2}\}$ and so $V_B \neq \emptyset, V_B \subset B$. By Lemma 2.2(ii), V_B is preopen and it is α-open (cf. Theorem 2.1 (i)). We have that $B = V_B \cup (B \setminus V_B) = V_B \cup \{(B \setminus V_B)_{\mathcal{F}^2} \cup (B \setminus V_B)_{\kappa^2} \cup (B \setminus V_B)_{mix})\} = V_B \cup (B \setminus V_B)_{\kappa^2} \cup (B \setminus V_B)_{mix}$. We note that, for a point $y \in (B \setminus V_B)_{mix}, U(y) \subset B$ or $U(y) \not\subset B$. We put:

$(B \setminus V_B)^1_{mix} := \{y \in (B \setminus V_B)_{mix} |\ U(y) \subset B\}$,
$U((B \setminus V_B)^1_{mix}) := \bigcup\{U(y) |\ y \in (B \setminus V_B)^1_{mix}\}$ and
$(B \setminus V_B)^2_{mix} := \{y \in (B \setminus V_B)_{mix} |\ U(y) \not\subset B\}$.

Then, $(B \setminus V_B)_{mix}$ is decomposed as $(B \setminus V_B)_{mix} = (B \setminus V_B)^1_{mix} \cup (B \setminus V_B)^2_{mix}$. Thus, we have that:

$(*^1)$ $B = V_B \cup (B \setminus V_B)_{\kappa^2} \cup (B \setminus V_B)^1_{mix} \cup (B \setminus V_B)^2_{mix}$.

By using Lemma 2.2(ii), V_B is α-open in $(\mathbb{Z}^2, \kappa^2)$; the set $(B \setminus V_B)_{\kappa^2}$ is open in $(\mathbb{Z}^2, \kappa^2)$ and so it is α-open in $(\mathbb{Z}^2, \kappa^2)$; $U((B \setminus V_B)^1_{mix})$ is open and so α-open in $(\mathbb{Z}^2, \kappa^2)$. Thus, we have that:

$(*^2)$ the subset $V_B \cup (B \setminus V_B)_{\kappa^2} \cup U((B \setminus V_B)^1_{mix})$ is α-open in $(\mathbb{Z}^2, \kappa^2)$.

Moreover, we conclude that:

$(*^3)$ $B = V_B \cup (B \setminus V_B)_{\kappa^2} \cup U((B \setminus V_B)^1_{mix}) \cup (B \setminus V_B)^2_{mix}$ holds.

Proof of $(*^3)$: since $(B\setminus V_B)^1_{mix} \subset U((B\setminus V_B)^1_{mix})$, it is shown that $B \subset V_B \cup (B\setminus V_B)_{\kappa^2} \cup U((B \setminus V_B)^1_{mix}) \cup (B \setminus V_B)^2_{mix}$ (cf. $(*^1)$). Conversely, we have that $V_B \cup (B \setminus V_B)_{\kappa^2} \cup U((B \setminus V_B)^1_{mix}) \cup (B \setminus V_B)^2_{mix} \subset B$ holds, because $U((B \setminus V_B)^1_{mix}) \subset B, V_B \subset B, (B \setminus V_B)_{\kappa^2} \subset B$ and $(B \setminus V_B)^2_{mix} \subset B$ hold. Thus we have the required equality $(*^3)$.

Let F be a nonempty g-closed set of $(\mathbb{Z}^2, \kappa^2)$ such that $F \subset B$. We claim that:

$(*^4)$ $F \cap ((B \setminus V_B)^2_{mix}) = \emptyset$ holds.

Proof of $(*^4)$: suppose that there exists a point $y \in F \cap (B \setminus V_B)^2_{mix}$. Then we have that:

$(**)$ $y \in B_{mix}, y \in F_{mix}$ and $U(y) \not\subset B$.

By Lemma 2.2 (i) for the g-closed set F and the point y, it is obtained that $Cl(\{y\})\setminus\{y\} \subset F$. Since $y \in (\mathbb{Z}^2)_{mix}$, we may put $y := (2s, 2u+1)$ (resp. $y := (2s+1, 2u)$); $y^+ := (2s, 2u+2)$ (resp. $y^+ := (2s+2, 2u)$); $y^- := (2s, 2u)$ (resp. $y^- := (2s, 2u)$), where $s, u \in \mathbb{Z}$. Then, $Cl(\{y\}) = \{y^+, y, y^-\}$ and $y^+, y^- \in (\mathbb{Z}^2)_{\mathcal{F}^2}$ (cf. proof of Lemma 2.2 (i)). Thus, we have that $Cl(\{y\}) \setminus \{y\} = \{y^+, y^-\} \subset F$ and so $y^+ \in F_{\mathcal{F}^2}$ and $y^- \in F_{\mathcal{F}^2}$. Since $F \subset B$, we have that $y^+ \in B_{\mathcal{F}^2}$ and $y^- \in B_{\mathcal{F}^2}$. For the point y^+, it follows from the assumption (2) that $\{y^+\} \cup (U(y^+))_{\kappa^2} \subset B$ and so $U(y) \subset B$. Indeed, $(U(y))_{\kappa^2} \subset (U(y^+))_{\kappa^2} \subset B, y \in F_{mix} \subset B_{mix} \subset B$ and $U(y) = \{y\} \cup ((U(y))_{\kappa^2})$ hold. The obtained property $U(y) \subset B$ contradicts to $(**)$ above. Thus, we claimed that $F \cap ((B \setminus V_B)^2_{mix}) = \emptyset$.

By using $(*^3)$ and $(*^4)$, it is shown that, for the g-closed set F such that $F \subset B$, $F = B \cap F = [V_B \cup (B \setminus V_B)_{\kappa^2} \cup U((B \setminus V_B)^1_{mix}) \cup (B \setminus V_B)^2_{mix}] \cap F \subset V_B \cup (B \setminus V_B)_{\kappa^2} \cup U((B \setminus V_B)^1_{mix})$. We put $E := V_B \cup (B \setminus V_B)_{\kappa^2} \cup U((B \setminus V_B)^1_{mix})$ and so $F \subset E \subset B$ and E is α-open. Using $(*^2)$ and $(*^3)$, we have that $F \subset E = \alpha Int(E) \subset \alpha Int(B)$ and hence $F \subset \alpha Int(B)$ holds. Namely, B is $g^{\#}\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$. □

Example 2.4 By Theorem 2.3, for examples, the following subsets B_1, B_2 and B_3 of $(\mathbb{Z}^2, \kappa^2)$ are $g^{\#}\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$: $B_1 := \bigcup\{\{x\} \cup (U(x))_{\kappa^2} |\ x \in E_{\mathcal{F}^2}\}$, where E is a subset of $(\mathbb{Z}^2, \kappa^2)$; $B_2 := E_{\mathcal{F}^2} \cup (U(E_{\mathcal{F}^2}))_{\kappa^2} \cup F_{\kappa^2} \cup F_{mix}$, where E and F are nonempty subsets of $(\mathbb{Z}^2, \kappa^2)$; $B_3 := U(E_{\mathcal{F}^2}) \cup F_{\kappa^2} \cup F_{mix}$, where E and F are nonempty subsets of $(\mathbb{Z}^2, \kappa^2)$. Moreover, we note that B_1 is α-open in $(\mathbb{Z}^2, \kappa^2)$. In the definitions of B_2 and B_3, we take subsets E and F as follows: $E := \{(0,0),(2,0)\}, F = \{(3,1),(4,1)\}$. Then resulting subsets B_2 and B_3 are not α-open in $(\mathbb{Z}^2, \kappa^2)$.

3 Proofs of Theorem A and Corollary B In the present section, first we prove Theorem A.

Proof of Theorem A. **(i)** We prove that the union of any collection of $g^{\#}\alpha$-open sets of $(\mathbb{Z}^2, \kappa^2)$ is $g^{\#}\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$. Let $\{B_i | i \in J\}$ be a collection of $g^{\#}\alpha$-open sets of $(\mathbb{Z}^2, \kappa^2)$, where J is an index set and put $V := \bigcup\{B_i|\ i \in J\}$. First we assume that $V_{\mathcal{F}^2} \neq \emptyset$; there exists a point $x \in (B_j)_{\mathcal{F}^2}$ for some $j \in J$. By Theorem 2.3 (ii), it is obtained that $(U(x))_{\kappa^2} \subset B_j$ and hence $(U(x))_{\kappa^2} \subset V$. Again using Theorem 2.3 (ii), we conclude that V is $g^{\#}\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$. Finally, we assume that $V_{\mathcal{F}^2} = \emptyset$, i.e., $V = V_{\kappa^2} \cup V_{mix}$. By Theorem 2.3 (i), for this case, V is also $g^{\#}\alpha$-open set in $(\mathbb{Z}^2, \kappa^2)$. **(ii)** We recall that a subset E is $g^{\#}\alpha$-closed if and only if the complement of E is $g^{\#}\alpha$-open. It follows from (i) and definitions that the intersection of any collection of $g^{\#}\alpha$-closed sets of $(\mathbb{Z}^2, \kappa^2)$ is $g^{\#}\alpha$-closed in $(\mathbb{Z}^2, \kappa^2)$. □

We recall the following properties:

Theorem 3.1 *Let (X, τ) be a topological space.*

(i) ([19, Theorem 2.5](i)) *The union of two $g^{\#}\alpha$-closed sets of (X, τ) is $g^{\#}\alpha$-closed in (X, τ) ; hence the intersection of two $g^{\#}\alpha$-open sets of (X, τ) is $g^{\#}\alpha$-open in (X, τ).*

(ii) ([6, Theorem 2.5]; cf.[11, p.7, line −6]) *A topological space (X, τ) is $T_{1/2}$ if and only if every singleton $\{x\}$ is open or closed in (X, τ), where $x \in X$.*

We need the following proposition.

Proposition 3.2 *Let x be a point of $(\mathbb{Z}^2, \kappa^2)$. The following properties on the singleton $\{x\}$ hold.*

(i) *If $x \in (\mathbb{Z}^2)_{\kappa^2}$, then $\{x\}$ is $g^{\#}\alpha$-open; it is not $g^{\#}\alpha$-closed in $(\mathbb{Z}^2, \kappa^2)$.*

(ii) *If $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, then $\{x\}$ is $g^{\#}\alpha$-closed; it is not $g^{\#}\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$.*

(iii) *If $x \in (\mathbb{Z}^2)_{mix}$, then $\{x\}$ is both $g^{\#}\alpha$-closed and $g^{\#}\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$.*

Proof. **(i)** It follows from assumption that $\{x\}$ is open in $(\mathbb{Z}^2, \kappa^2)$ and so it is $g^{\#}\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$(cf. the diagram in Section 1), i.e., $\{x\}$ is "open" in $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$. We prove that $\{x\}$ is not $g^{\#}\alpha$-closed. Indeed, let $x = (2s+1, 2u+1) \in (\mathbb{Z}^2)_{\kappa^2}$, where $s, u \in \mathbb{Z}$. We take a point $y := (2s, 2u) \in \mathbb{Z}^2 \setminus \{x\}$; then $y \in (\mathbb{Z}^2 \setminus \{x\})_{\mathcal{F}^2} \neq \emptyset$. Thus we have that $x = (2s+1, 2u+1) \in U(y) := \{2s-1, 2s, 2s+1\} \times \{2u-1, 2u, 2u+1\}$ and $x \in (U(y))_{\kappa^2}$ hold and so $(U(y))_{\kappa^2} \not\subset \mathbb{Z}^2 \setminus \{x\}$. By Theorem 2.3 (ii), $\mathbb{Z}^2 \setminus \{x\}$ is not $g^{\#}\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$. Namely, $\{x\}$ is not $g^{\#}\alpha$-closed in $(\mathbb{Z}^2, \kappa^2)$. [*An alternative proof of* (i): since $\{x\}$ is open and $x = (2s+1, 2u+1)$ for some $s, u \in \mathbb{Z}$, $\{x\}$ is g-open (cf. the diagram in Section 1). Then, there exists a g-open set $U := \{x\}$ such that $\alpha Cl(\{x\}) \not\subset \{x\}$, because

$\alpha Cl(\{x\}) = \{x\} \cup Cl(Int(Cl(\{x\}))) = Cl(Int(\{2s, 2s+1, 2s+2\} \times \{2u, 2u+1, 2u+2\})) = Cl(\{(2s+1, 2u+1)\}) = \{2s, 2s+1, 2s+2\} \times \{2u, 2u+1, 2u+2\}$. Namely, by Definition 1.1, $\{x\}$ is not $g^{\#}\alpha$-closed in $(\mathbb{Z}^2, \kappa^2)$]. **(ii)** For the case where $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, $\{x\}$ is closed in $(\mathbb{Z}^2, \kappa^2)$ and so it is $g^{\#}\alpha$-closed in $(\mathbb{Z}^2, \kappa^2)$, i.e., $\{x\}$ is "closed" in $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$. On the other hand, by Theorem 2.3(ii), $\{x\}$ is not $g^{\#}\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$, because $x \in \{x\}_{\mathcal{F}^2} \neq \emptyset$ and $(U(x))_{\kappa^2} \not\subset \{x\}$. **(iii)** For this case, we put $V := \mathbb{Z}^2 \setminus \{x\}$. It is shown that $V_{\mathcal{F}^2} \neq \emptyset$. Indeed, let $x = (2s, 2u+1)$ (resp. $x = (2s+1, 2u)$), where $s, u \in \mathbb{Z}$; then, we can take a point $y := (2s, 2u) \in V_{\mathcal{F}^2}$ (resp. $y := (2s, 2u) \in V_{\mathcal{F}^2}$). Moreover, for each point $a \in V_{\mathcal{F}^2}$, we have that $V_{\kappa^2} = (\mathbb{Z}^2)_{\kappa^2}$, $(U(a))_{\kappa^2} \subset (\mathbb{Z}^2)_{\kappa^2} = V_{\kappa^2} \subset V$. Thus, for each point $a \in V_{\mathcal{F}^2}$, $(U(a))_{\kappa^2} \subset V$. By using Theorem 2.3(ii), $V := \mathbb{Z}^2 \setminus \{x\}$ is a $g^{\#}\alpha$-open set in $(\mathbb{Z}^2, \kappa^2)$. Namely, $\{x\}$ is $g^{\#}\alpha$-closed in $(\mathbb{Z}^2, \kappa^2)$. Moreover, by using Theorem 2.3 (ii), it is easily shown that the singleton $\{x\}$ is $g^{\#}\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$, because $\{x\}_{\mathcal{F}^2} = \emptyset$. □

Finally, using Theorem A, Theorem 3.1 and Proposition 3.2, we prove Corollary B as follows.

Proof of Corollary B (i) and (ii) (ii-1). **(i)** It is obvious from Theorem A (i), Theorem 3.1(i) and definitions that the family $G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2)$ is a topology of $\mathbb{Z}^2$. **(ii) (ii-1)** Let $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$ be a topological space with a new topology $G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2)$. Then, it is claimed that the topological space $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$ is $T_{1/2}$ (in the sense of Levine ([15], cf. Theorem 3.1(ii))). By Proposition 3.2 (i) (resp. (ii), (iii)), a singleton $\{x\}$ is "open" (resp. "closed", "closed" and "open") in $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$, where $x \in (\mathbb{Z}^2)_{\kappa^2}$ (resp. $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}, x \in (\mathbb{Z}^2)_{mix}$). Therefore, every singleton $\{x\}$ of $\mathbb{Z}^2$ is "open" or "closed" in $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$ and so, by Theorem 3.1(ii) due to W. Dunham, the space $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$ is $T_{1/2}$. □

Sometimes, we abbreviate the topology $G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2)$ by $G^{\#}\alpha O$.

For a subset A of $\mathbb{Z}^2$, we denote the closure of A, interior of A and the kernel of A with respect to $G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2)$ by $G^{\#}\alpha O$-$Cl(A)$, $G^{\#}\alpha O$-$Int(A)$ and $G^{\#}\alpha O$-$Ker(A)$, respectively. The kernel is defined by $G^{\#}\alpha O\text{-}Ker(A) := \bigcap\{V |\ V \in G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2),\ A \subset V\}$. We need a property as follows:

$(*)$ if B is a $g^{\#}\alpha$-open set containing a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, then $\{x\} \cup (U(x))_{\kappa^2} \subset B$.
Indeed, by Theorem 2.3(ii) (1)⇒(2), $(U(x))_{\kappa^2} \subset B$ and $x \in B_{\mathcal{F}^2} \subset B$.

Proposition 3.3 *For the topological space $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$, we have the properties on the singletons as follows. Let x be a point of $\mathbb{Z}^2$ and $s, u \in \mathbb{Z}$.*

(i) (i-1) *If $x \in (\mathbb{Z}^2)_{\kappa^2}$, then $G^{\#}\alpha O\text{-}Ker(\{x\}) = \{x\}$ and $G^{\#}\alpha O\text{-}Ker(\{x\}) \in G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2)$.*

(i-2) *If $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, then $G^{\#}\alpha O\text{-}Ker(\{x\}) = \{x\} \cup (U(x))_{\kappa^2} = \{(2s, 2u)\} \cup \{(2s+1, 2u+1), (2s+1, 2u-1), (2s-1, 2u+1), (2s-1, 2u-1)\}$, where $x = (2s, 2u)$, and $G^{\#}\alpha O$-$Ker(\{x\}) \in G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2)$.*

(i-3) *If $x \in (\mathbb{Z}^2)_{mix}$, then $G^{\#}\alpha O\text{-}Ker(\{x\}) = \{x\}$ and $G^{\#}\alpha O\text{-}Ker(\{x\}) \in G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2)$.*

(ii) (ii-1) *If $x \in (\mathbb{Z}^2)_{\kappa^2}$, then $G^{\#}\alpha O\text{-}Cl(\{x\}) = \{(2s+1, 2u+1), (2s, 2u+2), (2s, 2u), (2s+2, 2u+2), (2s+2, 2u)\}$, where $x = (2s+1, 2u+1)$; and hence $\{x\}$ is not "closed" in $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$.*

(ii-2) *If $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, then $G^{\#}\alpha O\text{-}Cl(\{x\}) = \{x\}$.*

(ii-3) *If $x \in (\mathbb{Z}^2)_{mix}$, then $G^{\#}\alpha O\text{-}Cl(\{x\}) = \{x\}$.*

(iii) (iii-1) *If $x \in (\mathbb{Z}^2)_{\kappa^2}$, then $G^{\#}\alpha O\text{-}Int(\{x\}) = \{x\}$.*

(iii-2) *If $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, then $G^{\#}\alpha O\text{-}Int(\{x\}) = \emptyset$.*

(iii-3) *If $x \in (\mathbb{Z}^2)_{mix}$, then $G^{\#}\alpha O\text{-}Int(\{x\}) = \{x\}$.*

(iv) *If* $x \in (\mathbb{Z}^2)_{mix}$*, i.e.,* $x = (2s, 2u+1)$ *or* $(2s+1, 2u)$*, then* $\{x\}$ *is "regular open" and "regular closed" in* $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$.

(v) *If* $x \in (\mathbb{Z}^2)_{\kappa^2}$*, i.e.,* $x = (2s+1, 2u+1)$*, then* $\{x\}$ *is not "regular closed" in* $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$*; it is "semi-open" in* $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$*; moreover,* $\{x\}$ *is "regular open" in* $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$.

Proof. **(i)** **(i-1)** For a point $x \in (\mathbb{Z}^2)_{\kappa^2}$, by Proposition 3.2 (i), $\{x\}$ is $g^{\#}\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$. Thus, we have that $G^{\#}\alpha O\text{-}Ker(\{x\}) = \{x\}$ and $G^{\#}\alpha O\text{-}Ker(\{x\}) \in G^{\#}\alpha O$. **(i-2)** Let B be any $g^{\#}\alpha$-open set of $(\mathbb{Z}^2, \kappa^2)$ containing the point $x = (2s, 2u) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$. Then, by $(*)$ above, $\{x\} \cup (U(x))_{\kappa^2} \subset B$ holds and $\{x\} \cup (U(x))_{\kappa^2} \in G^{\#}\alpha O$ (cf. Lemma 2.2(ii)). Thus, we have that $G^{\#}\alpha O\text{-}Ker(\{x\}) = \bigcap\{V \mid \{x\} \subset V, V \in G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2)\} = \{x\} \cup (U(x))_{\kappa^2} = \{(2s, 2u), (2s+1, 2u+1), (2s+1, 2u-1), (2s-1, 2u+1), (2s-1, 2u-1)\}$. By Lemma 2.2(ii) and a fact that $(\kappa^2)^\alpha \subset G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2)$(cf. the diagram in Section 1), the kernel $G^{\#}\alpha O\text{-}Ker(\{x\})$ is $g^{\#}\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$, i.e., $G^{\#}\alpha O\text{-}Ker(\{x\}) \in G^{\#}\alpha O$. **(i-3)** Let $x \in (\mathbb{Z}^2)_{mix}$. The singleton $\{x\}$ is $g^{\#}\alpha$-open, because $\{x\}_{\mathcal{F}^2} = \emptyset$ (cf. Theorem 2.3 (i)). Thus, we have that $G^{\#}\alpha O\text{-}Ker(\{x\}) = \{x\}$ and $G^{\#}\alpha O\text{-}Ker(\{x\}) \in G^{\#}\alpha O$.

(ii) Let $x \in \mathbb{Z}^2$. By (i), it is shown that, for a point $y \in \mathbb{Z}^2$, $y \in G^{\#}\alpha O\text{-}Cl(\{x\})$ holds if and only if $x \in G^{\#}\alpha O\text{-}Ker(\{y\})$ holds.

(ii-1) For a point $x \in (\mathbb{Z}^2)_{\kappa^2}$, we can put $x = (2s+1, 2u+1)$, where $s, u \in \mathbb{Z}$. For a point $y \in (\mathbb{Z}^2)_{\kappa^2}, y \in G^{\#}\alpha O\text{-}Cl(\{x\})$ holds (i.e., $y \in (G^{\#}\alpha O\text{-}Cl(\{x\}))_{\kappa^2}$) if and only if $x \in G^{\#}\alpha O\text{-}Ker(\{y\})$ holds (i.e., $y = x$) (cf. (i-1)). Thus we have that $(G^{\#}\alpha O\text{-}Cl(\{x\}))_{\kappa^2} = \{x\}$. For a point $y \in (\mathbb{Z}^2)_{\mathcal{F}^2}, y \in G^{\#}\alpha O\text{-}Cl(\{x\})$ holds (i.e., $y \in (G^{\#}\alpha O\text{-}Cl(\{x\}))_{\mathcal{F}^2}$) if and only if $x \in G^{\#}\alpha O\text{-}Ker(\{y\})$ holds (i.e., $x \in \{y\} \cup (U(y))_{\kappa^2}$ and $x \neq y$ holds) (cf. (i-2)). Thus, we have that $(G^{\#}\alpha O\text{-}Cl(\{x\}))_{\mathcal{F}^2} = \{y \in (\mathbb{Z}^2)_{\mathcal{F}^2} \mid x \in \{y\} \cup (U(y))_{\kappa^2}\} = W_x$, where $W_x := \{(2s, 2u), (2s, 2u+2), (2s+2, 2u), (2s+2, 2u+2)\}$ and $x = (2s+1, 2u+1)$. For a point $y \in (\mathbb{Z}^2)_{mix}, y \in G^{\#}\alpha O\text{-}Cl(\{x\})$ holds (i.e., $y \in (G^{\#}\alpha O\text{-}Cl(\{x\}))_{mix}$) if and only if $x \in G^{\#}\alpha O\text{-}Ker(\{y\}) = \{y\}$ holds (cf. (i-3)). Since $y \neq x$, we have that $(G^{\#}\alpha O\text{-}Cl(\{x\}))_{mix} = \emptyset$.

Therefore, we obtain that $G^{\#}\alpha O\text{-}Cl(\{x\}) = \{x\} \cup W_x$, because $E = E_{\kappa^2} \cup E_{\mathcal{F}^2} \cup E_{mix}$ holds for any subset E. **(ii-2)** For a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, by Proposition 3.2 (ii), it is obtained that $G^{\#}\alpha O\text{-}Cl(\{x\}) = \{x\}$. **(ii-3)** For a point $x \in (\mathbb{Z}^2)_{mix}$, by Proposition 3.2 (iii), it is obtained that $G^{\#}\alpha O\text{-}Cl(\{x\}) = \{x\}$.

(iii) For a point $x \in (\mathbb{Z}^2)_{\kappa^2}$ (resp. $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}, x \in (\mathbb{Z}^2)_{mix}$), by Proposition 3.2 (i) (resp. (ii), (iii)), it is shown that $G^{\#}\alpha O\text{-}Int(\{x\}) = \{x\}$ (resp. $G^{\#}\alpha O\text{-}Int(\{x\}) = \emptyset$, $G^{\#}\alpha O\text{-}Int(\{x\}) = \{x\}$) holds.

(iv) For a point $x \in (\mathbb{Z}^2)_{mix}$, by (iii-3) and (ii-3), $G^{\#}\alpha O\text{-}Cl[G^{\#}\alpha O\text{-}Int(\{x\})] = \{x\}$, i.e., the singleton $\{x\}$ is "regular closed" in $(\mathbb{Z}^2, G^{\#}\alpha O)$. Similarly, we have that $G^{\#}\alpha O\text{-}Int[G^{\#}\alpha O\text{-}Cl(\{x\})] = \{x\}$, i.e., $\{x\}$ is "regular open" in $(\mathbb{Z}^2, G^{\#}\alpha O)$.

(v) Let $x = (2s+1, 2u+1) \in (\mathbb{Z}^2)_{\kappa^2}$, where $s, u \in \mathbb{Z}$. By (iii-1) and (ii-1), it is obtained that $G^{\#}\alpha O\text{-}Cl[G^{\#}\alpha O\text{-}Int(\{(2s+1, 2u+1)\})] = G^{\#}\alpha O\text{-}Cl(\{(2s+1, 2u+1)\}) \supset \{(2s+1, 2u+1)\}$ and hence the singleton $\{(2s+1, 2u+1)\}$ is not "regular closed" in $(\mathbb{Z}^2, G^{\#}\alpha O)$; explicitely the singleton $\{(2s+1, 2u+1)\}$ is "semi-open" in $(\mathbb{Z}^2, G^{\#}\alpha O)$. By (ii-1) and (iii), it is obtained that $G^{\#}\alpha O\text{-}Int[G^{\#}\alpha O\text{-}Cl(\{(2s+1, 2u+1)\})] = \{(2s+1, 2u+1)\}$ and hence a singleton $\{(2s+1, 2u+1)\}$ is "regular open" in $(\mathbb{Z}^2, G^{\#}\alpha O)$. □

A topological space (X, τ) is said to be $T_{3/4}$, if every δ-generalized closed subset is closed in (X, τ) ([4, Definition 4]). A space (X, τ) is $T_{3/4}$ if and only if every singleton $\{x\}$ of X is closed or regular open in (X, τ) ([4, Theorem 4.3]). The notion of the class of $T_{3/4}$-topological spaces which is properly placed between the class of T_1-spaces and $T_{1/2}$-spaces ([4]).

Proof of Corollary B (ii) (ii-2). Let $(\mathbb{Z}^2, \kappa^2)$ be the digital plane and $(\mathbb{Z}^2, G^{\#}\alpha O)$ a topological space obtained by changing the topology κ^2 by $G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2)$. Then, by Proposition 3.3 (v), a singleton $\{x\}$ is "regular open" in $(\mathbb{Z}^2, G^{\#}\alpha O)$, where $x \in (\mathbb{Z}^2)_{\kappa^2}$; by Proposition 3.3 (ii-2), a singleton $\{x\}$ is "closed" in $(\mathbb{Z}^2, G^{\#}\alpha O)$, where $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$; by Proposition 3.3 (iv), a singleton $\{x\}$ is "closed" in $(\mathbb{Z}^2, G^{\#}\alpha O)$, where $x \in (\mathbb{Z}^2)_{mix}$. Therefore, every singleton $\{x\}$ is "regular open" or "closed" in $(\mathbb{Z}^2, G^{\#}\alpha O)$. Namely, it is a $T_{3/4}$-topological space (cf. [4, Theorem 4.3 (3)]). Moreover, it is shown that $(\mathbb{Z}^2, G^{\#}\alpha O)$ is not T_1. Indeed, by Proposition 3.3 (ii-1), a singleton $\{(2s+1, 2u+1)\}$ is not "closed" in $(\mathbb{Z}^2, G^{\#}\alpha O)$, where $s, u \in \mathbb{Z}$. □

Remark 3.4 We note that the digital line $(\mathbb{Z}, \kappa)$ is a typical example of $T_{3/4}$-spaces (and so $T_{1/2}$) ([4, Example 4.6]); unfortunately, the digital plane $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$ (and so it is not $T_{3/4}$). Corollary B (ii) (resp. (iii)) shows that, by changing the topology, $\mathbb{Z}^2$ can be given a $T_{1/2}$-space structure (resp. $T_{3/4}$-space structure), i.e., $(\mathbb{Z}^2, G^{\#}\alpha O(\mathbb{Z}^2, \kappa^2))$ is $T_{1/2}$ (resp. $T_{3/4}$).

Present authors take the opportunity of noting the following typographical errors in [19] and [2].

Notice. We have not any change of the theorems in the papers [19] and [2].

Eratum in [19]

· page 18, line +1:
Replace "...$x \in \tau^{\alpha}$-$Cl(A)$..." by "...$x \in GO$-$Ker(A)$...".

· page 21, line +24:
Replace "...$\{2n+1, 2n, 2n+1\}$..." by "...$\{2n-1, 2n, 2n+1\}$...".

Eratum in [2]

· page 52, line +10:
Replace "$E_{mix} := \mathbb{Z}^2 \setminus (E_{\kappa^2} \cup E_{\mathcal{F}})$." by "$E_{mix} := E \setminus (E_{\kappa^2} \cup E_{\mathcal{F}})$.".

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