REGULAR BOUNDARY VALUE PROBLEMS FOR ELLIPTIC DIFFERENTIAL-OPERATOR EQUATIONS OF THE FOURTH ORDER IN *UMD* BANACH SPACES

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Abstract. We study Fredholmness of nonlocal boundary value problems for fourth order elliptic differential-operator equations in *UMD* Banach spaces. The main condition is given in terms of R-boundedness of some families of bounded operators generated by inverse of the characteristic operator pencil of the equation. Then we prove an isomorphism of the problem on the semi-axis, for some special boundary conditions, in appropriate L_p spaces. This implies maximal L_p-regularity for the problem. We also present some relevant application of obtained abstract results to boundary value problems for fourth order elliptic and quasi-elliptic partial differential equations.

1 Introduction. In our previous studies, [4], [6], we have considered regular and irregular boundary value problems for second order elliptic differential-operator equations with the spectral parameter λ in the equation in *UMD* Banach spaces. By regularity we mean the following classical notion which originates from scalar ordinary differential problems. When considering a boundary value problem for homogeneous equations, we expand the determinant of a system, for finding two unknown constants of a solution, with respect to λ. Then, regularity means that the main coefficient of the expansion with respect to λ is not equal to zero. Otherwise, the problem is irregular. We have realized maximal L_p-regularity for regular problems and we have not succeeded to get maximal L_p-regularity for irregular problems, one should claim more smoothness from the right-hand known function f of the equation than just to be in $L_p((0,1);E)$. Moreover, we have showed a counterexample which proves that there is no maximal L_p-regularity for irregular problems.

In this paper, we continue our investigation for regular boundary value problems for fourth order elliptic differential-operator equations. We get, in particular, maximal L_p-regularity (it follows from Theorem 2). The main condition of theorems is given in terms of R-boundedness of some families of bounded operators generated by inverse of the polynomial characteristic operator pencil of the equation (see condition (4) of Theorem 1). In fact, this condition is the most difficult for checking from application point of view. There are many studies (see, e. g., books by R. Denk, M. Hieber, and J. Prüss [3] and P. C. Kunstmann and L. Weis [10] and various papers) of R-boundedness for very general elliptic partial differential operators but not for elliptic partial differential operator pencils in Banach spaces polynomially depended on the spectral parameter λ. On the other hand, condition (4) of Theorem 1 is very natural. When studying the problem in a Hilbert space then R-boundedness is replaced by norm-boundedness of the same sets in condition (4) of Theorem 1. And for norm-boundedness there are many results in application for such pencils. In early

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60s, S. Agmon and L. Nirenberg [1] and M. S. Agranovich and M. I. Vishik [2] intensively studied norm-bounded estimates for various sets generated by inverse of the polynomial characteristic operator pencil of differential equations. The study has been continued by various mathematicians (see [12] for a relevant bibliography on the subject). So, the study of R-boundedness of various sets generated by inverse of the polynomial characteristic operator pencil of differential equation is actual today from application point of view.

In section 4, we give the first result of such a kind. In section 5, with the help of section 4, we present some application of the obtained abstract result to elliptic PDEs. In sections 6 and 7, some application is shown for some other classes of elliptic and quasi-elliptic PDEs.

We have decided do not include in the paper, at this time, some introductory material, i.e., some comments about importance of the subject, some comparison of our results with others known, about stimulating reasons of the investigation. We have presented all these, in a detail, in our previous papers [5], [4], [6], but we each time remind the reader the same definitions and notations which are necessary in order to understand all calculations in the paper.

If E and F are Banach spaces, $B(E,F)$ denotes the Banach space of all bounded, linear operators from E into F with the norm equal to the operator norm; moreover, $B(E) := B(E,E)$. The spectrum of a linear operator A in E is denoted by $\sigma(A)$, its resolvent set by $\rho(A)$. The domain and range of an operator A is denoted by $D(A)$ and $R(A)$, respectively. The resolvent of an operator A is denoted by $R(\lambda, A) := (\lambda I - A)^{-1}$.

We use the notation Ff or $\widehat{f}$ for the Fourier transform of a function f belonging to a vector-valued L_p-space, i.e., $L_p(\mathbb{R};E)$

$$Ff := (Ff)(\sigma) := \widehat{f}(\sigma) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\sigma x} f(x)dx,$$

and the inverse Fourier transform

$$F^{-1}f := (F^{-1}f)(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\sigma x} f(\sigma)d\sigma.$$

A Banach space E is said to be of class HT, if the Hilbert transform is bounded on $L_p(\mathbb{R};E)$ for some (and then all) $p > 1$. Here the Hilbert transform H of a function $f \in S(\mathbb{R};E)$, the Schwartz space of rapidly decreasing E-valued functions, is defined by

$$Hf := \frac{1}{\pi} PV(\frac{1}{t}) * f,$$

i.e., $(Hf)(t) := \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|\tau|>\varepsilon} \frac{f(t-\tau)}{\tau} d\tau$. These spaces are often also called *UMD* Banach spaces, where the *UMD* stands for the property of *unconditional martingale differences*. We prefer the notion *UMD* in the framework of this paper.

Definition. *Let E be a complex Banach space, and A is a closed linear operator in E. The operator A is called* **sectorial** *if the following conditions are satisfied:*

1. $\overline{D(A)} = E$, $\overline{R(A)} = E$, $(-\infty, 0) \subset \rho(A)$;
2. $\|\lambda(\lambda + A)^{-1}\| \le M$ *for all* $\lambda > 0$, *and some* $M < \infty$.

Definition. *Let E and F be Banach spaces. A family of operators $\mathcal{T} \subset B(E,F)$ is called* R**-bounded**, *if there is a constant $C > 0$ and $p \ge 1$ such that for each natural number n,*

$T_j \in \mathcal{T}$, $u_j \in E$ and for all independent, symmetric, $\{-1,1\}$-valued random variables ε_j on $[0,1]$ (e.g., the Rademacher functions $\varepsilon_j(t) = \operatorname{sign}\sin(2^j\pi t)$) the inequality

$$\Big\|\sum_{j=1}^{n}\varepsilon_j T_j u_j\Big\|_{L_p((0,1);F)} \le C\Big\|\sum_{j=1}^{n}\varepsilon_j u_j\Big\|_{L_p((0,1);E)}$$

*is valid. The smallest such C is called R-**bound** of $\mathcal{T}$ and is denoted by $R\{\mathcal{T}\}_{E\to F}$. If $E = F$, the R-bound will be denoted by $R\{\mathcal{T}\}_E$.*

Remark 1. From the definition of R-boundedness it follows that every R-bounded family of operators is (uniformly) bounded (it is enough to take $n = 1$). On the other hand, in a Hilbert space H every bounded set is R-bounded (see, e.g., [10, p.75]). Therefore, in a Hilbert space, the notion of R-boundedness is equivalent to boundedness of a family of operators (see also [3, p.26]).

Definition. *A sectorial operator A is called R-**sectorial** if*

$$R_A(0) := R\{\lambda(\lambda + A)^{-1} \ : \ \lambda > 0\} < \infty.$$

The number

$$\phi_A^R := \inf\{\theta \in (0,\pi) \ : \ R_A(\pi - \theta) < \infty\},$$

*where $R_A(\theta) := R\{\lambda(\lambda + A)^{-1} \ : \ |\arg\lambda| \le \theta\}$, is called an R-**angle** of the operator A.*

For convenience, sometimes we write $\lambda + A$ instead of $\lambda I + A$.

For the operator A closed in E, the domain of definition $D(A^n)$ of the operator A^n is turned into a Banach space $E(A^n)$ with respect to the norm

$$\|u\|_{E(A^n)} := \Big(\sum_{k=0}^{n}\|A^k u\|^2\Big)^{\frac{1}{2}}.$$

The operator A^n from $E(A^n)$ into E is bounded.

Introduce the space $W_p^1(\mathbb{R}; E(A), E)$, $1 < p < \infty$, of functions with the norm

$$\|u\|_{W_p^1(\mathbb{R};E(A),E)} := \Big(\int_{-\infty}^{\infty}\|u(x)\|_{E(A)}^p dx + \int_{-\infty}^{\infty}\|u'(x)\|_E^p dx\Big)^{\frac{1}{p}}.$$

In a similar way, one can define the space $W_p^1((0,T); E(A), E)$ and, generally, $W_p^n((0,T); E_n, E_{n-1}, \ldots, E_0)$ for natural numbers n and Banach spaces E_j, $j = 0, \ldots, n$. If E and F are Banach spaces with continuous embedding $F \subset E$, we also consider the space $W_p^n((0,T); F, E)$ of functions with the norm

$$\|u\|_{W_p^n((0,T);F,E)} := \|u\|_{L_p((0,T);F)} + \|u^{(n)}\|_{L_p((0,T);E)}.$$

Similarly, one can define $W_p^n(\mathbb{R}; F, E)$.

By saying that an operator A has the **Fredholm property**, we mean that the range of A is closed and the $\dim \operatorname{Ker} A = \dim \operatorname{Coker} A < \infty$.

Let E_0 and E_1 be two Banach spaces continuously embedded into the Banach space E : $E_0 \subset E$, $E_1 \subset E$. Two such spaces are called an **interpolation couple** $\{E_0, E_1\}$. Consider the Banach space

$$E_0 + E_1 := \{u \ : \ u \in E, \ \exists u_j \in E_j, \ j = 0,1, \text{ where } u = u_0 + u_1, \quad \|u\|_{E_0+E_1} := \inf_{\substack{u=u_0+u_1\\ u_j\in E_j}}(\|u_0\|_{E_0} + \|u_1\|_{E_1})\}.$$

Due to H. Triebel [11, section 1.3.1], the functional

$$K(t,u) := \inf_{\substack{u=u_0+u_1\\ u_j\in E_j}} \Big(\|u_0\|_{E_0} + t\|u_1\|_{E_1}\Big), \quad u \in E_0+E_1,$$

is continuous on $(0,\infty)$ in t, and the following estimate holds:

$$\min\{1,t\}\|u\|_{E_0+E_1} \le K(t,u) \le \max\{1,t\}\|u\|_{E_0+E_1}.$$

An **interpolation space** for $\{E_0,E_1\}$ by the K-method is defined as follows:

$$\begin{aligned}(E_0,E_1)_{\theta,p} &:= \Big\{u \ :\ u\in E_0+E_1,\quad \|u\|_{(E_0,E_1)_{\theta,p}}\\ &:= \Big(\int_0^\infty t^{-1-\theta p}K^p(t,u)\,dt\Big)^{\frac{1}{p}} < \infty\Big\}, \quad 0<\theta<1,\ 1\le p<\infty,\\ (E_0,E_1)_{\theta,\infty} &:= \Big\{u \ :\ u\in E_0+E_1,\quad \|u\|_{(E_0,E_1)_{\theta,\infty}}\\ &:= \sup_{t\in(0,\infty)} t^{-\theta}K(t,u) < \infty\Big\}, \quad 0<\theta<1.\end{aligned}$$

2 Coerciveness on the space variable and Fredholmness. Consider, in a *UMD* Banach space E, a boundary value problem in [0,1] for the fourth order elliptic equation

$$L(D)u := u''''(x) + A_2u''(x) + A_4u(x) + \sum_{k=0}^{3} B_k(x)u^{(k)}(x) = f(x), \tag{2.1}$$

$$L_ku := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) + \sum_{j=0}^{m_k-1}\sum_{s=1}^{N_{kj}} T_{kjs}u^{(j)}(x_{kjs}) = \varphi_k, \quad k=1,\dots,4, \tag{2.2}$$

where $0\le m_1,m_2\le 1,\ m_3=m_1+2,\ m_4=m_2+2$, and α_k and β_k are complex numbers, $x_{kjs}\in[0,1],\ D:=\frac{d}{dx};\ A_2,\ A_4,\ B_k(x)$ for $x\in[0,1]$ and T_{kjs} are, generally speaking, unbounded operators in E.

Theorem 1. *Let the following conditions be satisfied*:

1. *an operator A_4 is closed, densely defined and invertible in a UMD Banach space E and $R\{\lambda R(\lambda,A_4)\ :\ \arg\lambda=\pi\}_E<\infty$;*[1]
2. *the embedding $E(A_4)\subset E$ is compact*;
3. *an operator A_2 from E_2 into E is bounded, where $E_2 := E(A_4^{\frac{1}{2}})$*;
4. *for $\lambda=i\sigma,\ \sigma\in\mathbb{R}$, and $|\sigma|\ge\sigma_0$ (for some $\sigma_0\ge 0$), the characteristic operator pencil $L(\lambda):=\lambda^4I+\lambda^2A_2+A_4$ is invertible in E and*

$$\begin{aligned}&R\{\sigma^4L(i\sigma)^{-1}\ :\ |\sigma|\ge\sigma_0\}_E<\infty;\quad R\{A_4L(i\sigma)^{-1}\ :\ |\sigma|\ge\sigma_0\}_E<\infty;\\ &R\{\sigma^4L(i\sigma)^{-1}\ :\ |\sigma|\ge\sigma_0\}_{E_2}<\infty;\quad R\{A_4L(i\sigma)^{-1}\ :\ |\sigma|\ge\sigma_0\}_{E_2}<\infty;\end{aligned}$$

[1] In fact, this is equivalent to that A_4 is an invertible R-sectorial operator in E with the R-angle $\phi^R_{A_4}<\pi$ and, therefore, in particular, there exist fractional powers of A_4 (see, e.g., [3, Theorem 2.3]).

5.

$$\begin{vmatrix} \alpha_1(-1)^{m_1} & 0 & \beta_1 & 0 \\ 0 & \alpha_3(-1)^{m_1} & 0 & \beta_3 \\ \alpha_2(-1)^{m_2} & 0 & \beta_2 & 0 \\ 0 & \alpha_4(-1)^{m_2} & 0 & \beta_4 \end{vmatrix} \neq 0;$$

for $m_1 \neq m_2, \ \ \alpha_k = \alpha_{k+2}, \ \ \beta_k = \beta_{k+2}, \ \ k = 1, 2;$

6. for any $\varepsilon > 0$ *and for almost all* $x \in [0,1]$, $k = 0, \dots, 3$,

$$\|B_k(x)u\|_E \leq \varepsilon \|u\|_{E(A_4^{1-\frac{k}{4}})} + C(\varepsilon)\|u\|_E, \quad u \in E(A_4^{1-\frac{k}{4}});$$

for $u \in E(A_4^{1-\frac{k}{4}})$ *the function* $B_k(x)u$ *is measurable on* $[0,1]$ *in* E;

7. operators T_{kjs} *from* $(E(A_4), E)_{\frac{j}{4}+\frac{1}{4p},p}$ *into* $(E(A_4), E)_{\frac{m_k}{4}+\frac{1}{4p},p}$ *are compact, where* $p \in (1, \infty)$.

Then, the operator $\mathbb{L} : u \to \mathbb{L}u := \big(L(D)u, L_1u, L_2u, L_3u, L_4u\big)$ *from* $W_p^4((0,1); E(A_4), E)$ *into* $L_p((0,1); E) \dot{+}_{k=1}^{4} (E(A_4), E)_{\frac{m_k}{4}+\frac{1}{4p},p}$ *is bounded and Fredholm*[2].

Proof. Consider the principal part of problem (2.1)–(2.2), i.e.,

$$L_0(D)u := u''''(x) + A_2u''(x) + A_4u(x) = f(x), \quad x \in (0,1), \tag{2.3}$$

$$L_{k0}u := \alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1) = \varphi_k, \quad k = 1, \dots, 4. \tag{2.4}$$

By the substitution

$$v(x) := \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} := \begin{pmatrix} u(x) \\ u''(x) \end{pmatrix},$$

problem (2.3)–(2.4) is reduced to the equivalent problem

$$\begin{aligned} &v''(x) = \mathbb{A}v(x) + F(x), \quad x \in (0,1), \\ &a_k v^{(m_k)}(0) + b_k v^{(m_k)}(1) = \Phi_k, \quad k = 1, 2, \end{aligned} \tag{2.5}$$

where,

$$\mathbb{A} := \begin{pmatrix} 0 & I \\ -A_4 & -A_2 \end{pmatrix}, \quad a_k := \begin{pmatrix} \alpha_k I & 0 \\ 0 & \alpha_{k+2} I \end{pmatrix}, \quad b_k := \begin{pmatrix} \beta_k I & 0 \\ 0 & \beta_{k+2} I \end{pmatrix},$$

$$F(x) := \begin{pmatrix} 0 \\ f(x) \end{pmatrix}, \quad \Phi_k := \begin{pmatrix} \varphi_k \\ \varphi_{k+2} \end{pmatrix}.$$

[2]By virtue of [5, Theorem 7 and Corollary 8], the embedding $W_p^4((0,1); E(A_4), E) \subset W_p^2((0,1); E(A_4^{\frac{1}{2}}))$ is continuous. Then, by virtue of condition (3), $A_2u'' \in L_p((0,1); E)$.

We consider the operator $\mathbb{A}$ in the space $\mathcal{E} := E_2 \dot{+} E$. Let $D(\mathbb{A}) := E(A_4) \dot{+} E_2$ and $F := (f_1, f_2) \in \mathcal{E} = E_2 \dot{+} E$. From the first equation of the system

$$(\lambda^2 I - \mathbb{A})v = F \tag{2.6}$$

we find

$$v_2 = \lambda^2 v_1 - f_1.$$

Substituting this expression into the second equation of system (2.6) we have

$$\lambda^2(\lambda^2 v_1 - f_1) = -A_4 v_1 - A_2(\lambda^2 v_1 - f_1) + f_2.$$

Hence,

$$L(\lambda)v_1 = \lambda^2 f_1 + A_2 f_1 + f_2,$$

i.e., by condition (4), for $\lambda = i\sigma,\ \sigma \in \mathbb{R}$, and $|\sigma| \geq \sigma_0$,

$$v_1 = \lambda^2 L(\lambda)^{-1} f_1 + L(\lambda)^{-1} A_2 f_1 + L(\lambda)^{-1} f_2. \tag{2.7}$$

Consequently,

$$v_2 = \lambda^4 L(\lambda)^{-1} f_1 + \lambda^2 L(\lambda)^{-1} A_2 f_1 - f_1 + \lambda^2 L(\lambda)^{-1} f_2. \tag{2.8}$$

Since (2.7) and (2.8) define $(\lambda^2 I - \mathbb{A})^{-1}$ then one can get that

$$\mathbb{A}(\lambda^2 I - \mathbb{A})^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{2.9}$$

where

$$\begin{aligned}
A_{11} &= \lambda^4 L(\lambda)^{-1} + \lambda^2 L(\lambda)^{-1} A_2 - I, \\
A_{12} &= \lambda^2 L(\lambda)^{-1}, \\
A_{21} &= -\lambda^2 A_4 L(\lambda)^{-1} - A_4 L(\lambda)^{-1} A_2 - \lambda^4 A_2 L(\lambda)^{-1} - \lambda^2 A_2 L(\lambda)^{-1} A_2 + A_2, \\
A_{22} &= -A_4 L(\lambda)^{-1} - \lambda^2 A_2 L(\lambda)^{-1}.
\end{aligned}$$

From Venni's proposition (see [5, p.500, under $X = Y = E$, $\alpha = 0$, $\beta = \frac{1}{2}$, $\gamma = 1$, $f(\sigma) = \sigma^2$, $B(\sigma) = L(i\sigma)^{-1}$]) and two first inequalities in condition (4), we get that

$$R\{\sigma^2 A_4^{\frac{1}{2}} L(i\sigma)^{-1} \ :\ |\sigma| \geq \sigma_0\}_E < \infty. \tag{2.10}$$

Similarly, from two last inequalities in condition (4), we get

$$R\{\sigma^2 A_4^{\frac{1}{2}} L(i\sigma)^{-1} \ :\ |\sigma| \geq \sigma_0\}_{E_2} < \infty. \tag{2.11}$$

Using now the definition of R-boundedness, conditions (3) and (4), and formulas (2.10) and (2.11) we obtain, from (2.9), that

$$R\{\mathbb{A}(\lambda^2 I - \mathbb{A})^{-1} \ : \lambda = i\sigma,\ \sigma \in \mathbb{R},\ |\sigma| \geq \sigma_0\}_{\mathcal{E}} < \infty.$$

From this and from the identity

$$\lambda^2(\lambda^2 I - \mathbb{A})^{-1} = \mathbb{A}(\lambda^2 I - \mathbb{A})^{-1} + I,$$

we have, using, e.g., [3, Proposition 3.4],

$$R\{\lambda^2(\lambda^2 I - \mathbb{A})^{-1} \ :\ \lambda = i\sigma,\ \sigma \in \mathbb{R},\ |\sigma| \geq \sigma_0\}_{\mathcal{E}} < \infty,$$

i.e., for some $M \geq 0$,

$$R\{\lambda(\lambda I - \mathbb{A})^{-1} \ :\ \lambda \leq -M\}_{\mathcal{E}} < \infty. \tag{2.12}$$

From condition (5), for $m_1 \neq m_2$, it follows that $(-1)^{m_1}\alpha_1\beta_2 - (-1)^{m_2}\alpha_2\beta_1 \neq 0$ and $a_k v^{(m_k)}(0) + b_k v^{(m_k)}(1) = \alpha_k v^{(m_k)}(0) + \beta_k v^{(m_k)}(1)$ in (2.5). Then, by virtue of [4, Theorem 5 and Remark 4 (only for $m_1 = m_2$)], the operator

$$\mathbb{P}_0 : v \to \mathbb{P}_0 v := ((D^2 - \mathbb{A})v(x), a_1 v^{(m_1)}(0) + b_1 v^{(m_1)}(1), a_2 v^{(m_2)}(0) + b_2 v^{(m_2)}(1))$$

from $W_p^2((0,1);\mathcal{E}(\mathbb{A}),\mathcal{E})$ into $L_p((0,1);\mathcal{E})\dot{+}(\mathcal{E}(\mathbb{A}),\mathcal{E})_{\frac{m_1}{2}+\frac{1}{2p},p}\dot{+}(\mathcal{E}(\mathbb{A}),\mathcal{E})_{\frac{m_2}{2}+\frac{1}{2p},p}$ is bounded and Fredholm. From (2.12), it follows that the operator $\mathbb{A}$ is closed. Consequently, $\mathcal{E}(\mathbb{A}) = E(A_4)\dot{+}E_2$.

We have $(\mathcal{E}(\mathbb{A}),\mathcal{E})_{\theta,p} = (E(A_4)\dot{+}E_2, E_2\dot{+}E)_{\theta,p} = (E(A_4),E_2)_{\theta,p}\dot{+}(E_2,E)_{\theta,p}$. Since $E_2 := E(A_4^{\frac{1}{2}})$ then, by virtue of [11, Theorem 1.3.3 and formula 1.15.4/(2)],

$$\begin{aligned}(E(A_4),E(A_4^{\frac{1}{2}}))_{\frac{m_k}{2}+\frac{1}{2p},p} &= (E(A_4^{\frac{1}{2}}),E(A_4))_{1-\frac{m_k}{2}-\frac{1}{2p},p} = (E,E(A_4))_{1-\frac{m_k}{4}-\frac{1}{4p},p}\\ &= (E(A_4),E)_{\frac{m_k}{4}+\frac{1}{4p},p}, \qquad k=1,2.\end{aligned}$$

Since $m_{k+2} = m_k + 2,\ k = 1,2$, then, by calculations similar to the previous ones, using also, e.g., [11, Theorem 1.15.2], one can get

$$\begin{aligned}(E(A_4^{\frac{1}{2}}),E)_{\frac{m_k}{2}+\frac{1}{2p},p} &= (E,E(A_4^{\frac{1}{2}}))_{1-\frac{m_k}{2}-\frac{1}{2p},p} = (E,E(A_4))_{\frac{1}{2}-\frac{m_k}{4}-\frac{1}{4p},p}\\ &= (E(A_4),E)_{\frac{1}{2}+\frac{m_k}{4}+\frac{1}{4p},p} = (E(A_4),E)_{\frac{m_{k+2}}{4}+\frac{1}{4p},p}, \quad k=1,2.\end{aligned}$$

Hence, the operator that corresponds to problem (2.3)–(2.4),

$$\mathbb{L}_0 : u \to \mathbb{L}_0 u := \Big(L_0(D)u, L_{10}u, L_{20}u, L_{30}u, L_{40}u\Big),$$

from $W_p^4((0,1);E(A_4),E)$ into $L_p((0,1);E)\dot{+}_{k=1}^{4}(E(A_4),E)_{\frac{m_k}{4}+\frac{1}{4p},p}$, is bounded and Fredholm. It is enough now to note that the operator $\mathbb{L}$ has the form

$$\mathbb{L} = \mathbb{L}_0 + \mathbb{T}, \tag{2.13}$$

where

$$\mathbb{T} : u \to \mathbb{T}u := \Big(\sum_{k=0}^{3} B_k(x)u^{(k)}(x), T_1u, T_2u, T_3u, T_4u\Big)$$

and $T_k u := \sum_{j=0}^{m_k-1}\sum_{s=1}^{N_{kj}} T_{kjs}u^{(j)}(x_{kjs})$. By condition (6), [12, Lemma 5.2.1/2], and the boundedness of the embeddings $W_p^{4-k}((0,1);E(A_4^{1-\frac{k}{4}}),E) \subset W_p^1((0,1);E(A_4^{1-\frac{k}{4}}),E)$, $k = 0,...,3$, the operator $B_k\ :\ u(x) \to B_k u|_x := B_k(x)u(x)$ from $W_p^{4-k}((0,1);E(A_4^{1-\frac{k}{4}}),\ E)$ into $L_p((0,1);E)$ is compact. Using [5, Theorem 7 and Corollary 8] about intermediate derivatives, the operator $u(x) \to u^{(k)}(x)$ from $W_p^4((0,1);E(A_4),E)$ into $W_p^{4-k}((0,1);E(A_4^{1-\frac{k}{4}}),E)$

is bounded. Hence, the operator $B: \ u(x) \to B_k u|_x := \sum_{k=0}^{3} B_k(x)u^{(k)}(x)$ from $W_p^4((0,1); E(A_4), E)$ into $L_p((0,1); E)$ is compact.

In view of [11, Theorem 1.8.2] (see also [12, Theorem 1.7.7/1]), the operator $u(x) \to u^{(j)}(x_0)$ from $W_p^4((0,1);\ E(A_4), E)$ into $(E(A_4), E)_{\frac{j}{4}+\frac{1}{4p},p}$ is bounded. Thus, condition (7) implies that operators T_k from $W_p^4((0,1); E(A_4), E)$ into $(E(A_4), E)_{\frac{m_k}{4}+\frac{1}{4p},p}$ are compact. Consequently, the operator $\mathbb{T}$ from $W_p^4((0,1); E(A_4), E)$ into $L_p((0,1); E) \dot{+}_{k=1}^{4} (E(A_4), E)_{\frac{m_k}{4}+\frac{1}{4p},p}$ is compact. Now, it is enough to apply the perturbation theorem of Fredholm operators (see, e.g., [12, Theorem 1.2.8]) to operator (2.13). □

3 Isomorphism of problems on the semi-axis. In a *UMD* Banach space E, consider a boundary value problem in $[0, \infty)$ for the fourth order elliptic equation

$$L(D)u := u''''(x) + A_2 u''(x) + A_4 u(x) = f(x), \quad x > 0, \tag{3.1}$$

$$\begin{aligned} L_1 u &:= \alpha u(0) + \beta u'(0) = \varphi_1, \\ L_2 u &:= \alpha u''(0) + \beta u'''(0) = \varphi_2, \end{aligned} \tag{3.2}$$

where α and β are complex numbers.

Theorem 2. *Let the following conditions be satisfied*:

1. *an operator A_4 is closed, densely defined and invertible in a UMD Banach space E and $R\{\lambda R(\lambda, A_4) \ :\ \arg\lambda = \pi\}_E < \infty$;*[3]
2. *an operator A_2 from E_2 into E is bounded, where $E_2 := E(A_4^{\frac{1}{2}})$;*
3. *at least one of two numbers α and β is not equal to zero; $\Re\alpha\beta^{-1} \le 0$ if $\beta \ne 0$;*
4. *for $\lambda = i\sigma$, $\sigma \in \mathbb{R}$, the characteristic operator pencil $L(\lambda) := \lambda^4 I + \lambda^2 A_2 + A_4$ is invertible in E and*

$$\begin{aligned} &R\{\sigma^4 L(i\sigma)^{-1} \ :\ \sigma \in \mathbb{R}\}_E < \infty; \quad R\{A_4 L(i\sigma)^{-1} \ :\ \sigma \in \mathbb{R}\}_E < \infty; \\ &R\{\sigma^4 L(i\sigma)^{-1} \ :\ \sigma \in \mathbb{R}\}_{E_2} < \infty; \quad R\{A_4 L(i\sigma)^{-1} \ :\ \sigma \in \mathbb{R}\}_{E_2} < \infty; \end{aligned}$$

Then, the operator $\mathbb{L} : u \to \mathbb{L}u := (L(D)u, L_1 u, L_2 u)$ from $W_p^4((0,\infty); E(A_4), E)$ into $L_p((0,\infty); E) \dot{+} (E(A_4), E)_{\frac{m}{4}+\frac{1}{4p},p} \dot{+} (E(A_4), E)_{\frac{m+2}{4}+\frac{1}{4p},p}$, where $m = 0$ if $\beta = 0$ and $m = 1$ if $\beta \ne 0$, $p \in (1, \infty)$, is an isomorphism.

Proof. By [11, Theorem 1.8.2] (see also [12, Theorem 1.7.7/1]) and condition (3), the operator $\mathbb{L}$ acts continuously from $W_p^4((0,\infty); E(A_4), E)$ into $L_p((0,\infty); E) \dot{+} (E(A_4), E)_{\frac{m}{4}+\frac{1}{4p},p} \dot{+} (E(A_4), E)_{\frac{m+2}{4}+\frac{1}{4p},p}$. Prove that for any $f \in L_p((0,\infty); E)$, $\varphi_1 \in (E(A_4), E)_{\frac{m}{4}+\frac{1}{4p},p}$, and $\varphi_2 \in (E(A_4), E)_{\frac{m+2}{4}+\frac{1}{4p},p}$, problem (3.1)–(3.2) has a unique solution that belongs to $W_p^4((0,\infty);\ E(A_4), E)$.

Let us show that a solution of problem (3.1)–(3,2) is represented in the form $u(x) = u_1(x) + u_2(x)$, where $u_1(x)$ is the restriction on $[0, \infty)$ of a solution $\tilde{u}_1(x)$ of the equation

$$\tilde{u}_1''''(x) + A_2 \tilde{u}_1''(x) + A_4 \tilde{u}_1(x) = \tilde{f}(x), \quad x \in \mathbb{R}, \tag{3.3}$$

[3]See the corresponding footnote of Theorem 1.

where $\tilde{f}(x) := f(x)$ if $x \in [0, \infty)$ and $\tilde{f}(x) := 0$ if $x \in (-\infty, 0)$, and $u_2(x)$ is a solution of the problem

$$u_2''''(x) + A_2 u_2''(x) + A_4 u_2(x) = 0, \quad x > 0, \tag{3.4}$$

$$\begin{aligned} \alpha u_2(0) + \beta u_2'(0) &= -L_1 u_1 + \varphi_1, \\ \alpha u_2''(0) + \beta u_2'''(0) &= -L_2 u_1 + \varphi_2. \end{aligned} \tag{3.5}$$

Apply [5, Theorem 1] to equation (3.3). Conditions (1) and (2) of [FY1, Theorem 1] are obvious. Condition (3) of [5, Theorem 1] follows from two first inequalities in condition (4). Hence, by virtue of [5, Theorem 1], equation (3.3) has a unique solution $\tilde{u}_1 \in W_p^4(\mathbb{R}; E(A_4), E(A_4^{\frac{3}{4}}), E(A_4^{\frac{1}{2}}), E(A_4^{\frac{1}{4}}), E)$. Then, $u_1 \in W_p^4((0,\infty); E(A_4),\, E)$.

Let us now prove that for any $\varphi_1 \in (E(A_4), E)_{\frac{m}{4}+\frac{1}{4p},p}$, $\varphi_2 \in (E(A_4), E)_{\frac{m+2}{4}+\frac{1}{4p},p}$ problem (3.4)–(3.5) has a unique solution $u_2(x)$ that belongs to $W_p^4((0,\infty);\, E(A_4), E)$. By the substitution

$$v(x) := \begin{pmatrix} v_1(x) \\ \\ v_2(x) \end{pmatrix} := \begin{pmatrix} u_2(x) \\ \\ u_2''(x) \end{pmatrix}$$

problem (3.4)–(3.5) is reduced to the equivalent problem

$$v''(x) = \mathbb{A} v(x), \quad x > 0, \tag{3.6}$$

$$\alpha v(0) + \beta v'(0) = \Phi_0, \tag{3.7}$$

where

$$\mathbb{A} := \begin{pmatrix} 0 & I \\ \\ -A_4 & -A_2 \end{pmatrix}, \qquad \Phi_0 := \begin{pmatrix} -L_1 u_1 + \varphi_1 \\ \\ -L_2 u_1 + \varphi_2 \end{pmatrix}.$$

We consider the operator $\mathbb{A}$ in the space $\mathcal{E} := E_2 \dot{+} E$. Let $D(\mathbb{A}) := E(A_4) \dot{+} E_2$. Like to (2.12) in the proof of Theorem 1, we get here the estimate

$$R\{\lambda(\lambda I - \mathbb{A})^{-1} :\ \arg\lambda = \pi\}_{\mathcal{E}} < \infty.$$

Hence, by virtue of, e.g., [12, Theorem 1.5.3] and Remark 1, there exists an operator $\mathrm{e}^{-x\mathbb{A}^{\frac{1}{2}}}$ and for some $\omega > 0$

$$\|e^{-x\mathbb{A}^{\frac{1}{2}}}\| \le Ce^{-\omega x}, \quad x \ge 0.$$

Repeating the beginning of the proof of [4, Theorem 2] (just take $\varphi = 0$), one can show that an arbitrary solution of (3.6) that belongs to $W_p^2((0,\infty); \mathcal{E}(\mathbb{A}),\, \mathcal{E})$ has the form

$$v(x) = e^{-x\mathbb{A}^{\frac{1}{2}}} g, \tag{3.8}$$

where $g \in (\mathcal{E}(\mathbb{A}), \mathcal{E})_{\frac{1}{2p},p}$. To this end, one should use S. G. Krein [8, Theorem 3.2.11]. Function (3.8) satisfies boundary condition (3.7) if

$$\alpha g - \beta \mathbb{A}^{\frac{1}{2}} g = \Phi_0. \tag{3.9}$$

Since $u_1 \in W_p^4((0,\infty); E(A_4), E)$, by virtue of [11, Theorem 1.8.2] (see also [12, Theorem 1.7.7/1]),

$$L_1 u_1 \in (E(A_4), E)_{\frac{m}{4}+\frac{1}{4p},p}, \qquad L_2 u_1 \in (E(A_4), E)_{\frac{m+2}{4}+\frac{1}{4p},p}.$$

Obviously, we have

$$(\mathcal{E}(\mathbb{A}),\mathcal{E})_{q,p} = (E(A_4)\dot{+}E_2, E_2\dot{+}E)_{q,p} = (E(A_4),E_2)_{q,p}\dot{+}(E_2,E)_{q,p}.$$

Since $E_2 := E(A_4^{\frac{1}{2}})$ then, in a similar way as in the proof of Theorem 1, we get

$$(E(A_4),E_2)_{\frac{m}{2}+\frac{1}{2p},p} = (E(A_4),E)_{\frac{m}{4}+\frac{1}{4p},p},$$
$$(E_2,E)_{\frac{m}{2}+\frac{1}{2p},p} = (E(A_4),E)_{\frac{m+2}{4}+\frac{1}{4p},p}.$$

Consequently, $\Phi_0 \in (\mathcal{E}(\mathbb{A}),\mathcal{E})_{\frac{m}{2}+\frac{1}{2p},p}$.

For $\beta = 0$, a unique solution of problem (3.6)–(3.7) has the form (using (3.8) and (3.9))

$$v(x) = \alpha^{-1}e^{-x\mathbb{A}^{\frac{1}{2}}}\Phi_0.$$

Since $\Phi_0 \in (\mathcal{E}(\mathbb{A}),\mathcal{E})_{\frac{1}{2p},p}$ (remind that $m = 0$ when $\beta = 0$) then $v \in W_p^2((0,\infty);\mathcal{E}(\mathbb{A}),\mathcal{E})$. Therefore, a unique solution u_2 of problem (3.4)-(3.5) belongs to $W_p^4((0,\infty);E(A_4),E)$.

For $\beta \neq 0$, by condition (5), a unique solution of problem (3.6)–(3.7) has the form (using (3.8) and (3.9))

$$v(x) = e^{-x\mathbb{A}^{\frac{1}{2}}}(\alpha I - \beta\mathbb{A}^{\frac{1}{2}})^{-1}\Phi_0.$$

By [11, Theorem 1.15.2], the operator $\mathbb{A}^{\frac{1}{2}}$ from $(\mathcal{E}(\mathbb{A}),\mathcal{E})_{\frac{1}{2p},p}$ onto $(\mathcal{E}(\mathbb{A}),\mathcal{E})_{\frac{p+1}{2p},p}$ is an isomorphism. Since $\Phi_0 \in (\mathcal{E}(\mathbb{A}),\mathcal{E})_{\frac{p+1}{2p},p}$ (remind that $m = 1$ when $\beta = 1$) then $(\alpha I - \beta\mathbb{A}^{\frac{1}{2}})^{-1}\Phi_0 \in (\mathcal{E}(\mathbb{A}),\mathcal{E})_{\frac{1}{2p},p}$, i.e., $v \in W_p^2((0,\infty);\mathcal{E}(\mathbb{A}),\mathcal{E})$. Therefore, a unique solution u_2 of problem (3.4)-(3.5) belongs to $W_p^4((0,\infty);E(A_4),E)$.

The uniqueness of a solution of problem (3.1)–(3.2) follows from the uniqueness of a solution of problem (3.4)–(3.5). Indeed, if problem (3.1)–(3.2) has two solutions, $u(x)$, $\tilde{u}(x)$, then functions $u_2(x) := u(x) - u_1(x)$ and $\tilde{u}_2(x) := \tilde{u}(x) - u_1(x)$, where $u_1(x)$ is the restriction on $[0,\infty)$ of the solution $\tilde{u}_1(x)$ of equation (3.3), are two different solutions of problem (3.4)–(3.5), which is a contradiction. □

4 R-boundedness of various sets constructed by the polynomial ordinary differential pencil on the whole axis. In order to give some relevant application of obtained abstract results to PDEs, let us derive some new results about R-bounded sets.

A system of numbers $\omega_1,\dots,\omega_m$ is called **p-separated** if there exists a straight line P passing through 0 such that no value of the numbers ω_j lies on it, and $\omega_1,\dots,\omega_p$ are on one side of P while $\omega_{p+1},\dots,\omega_m$ are on the other.

Consider an ordinary differential equation with constant coefficients on the whole axis

$$L_0(\lambda)u := \lambda^m u(y) + \lambda^{m-1}a_1u'(y) + \cdots + a_mu^{(m)}(y) = f(y), \quad y \in \mathbb{R}, \tag{4.1}$$

where a_k are complex numbers.

Let us enumerate the roots of the equation

$$a_m\omega^m + a_{m-1}\omega^{m-1} + \cdots + 1 = 0 \tag{4.2}$$

by ω_j, $j = 1,\dots,m$. Let numbers ω_j be p-separated.

Denote

$$\begin{aligned}\underline{\omega} &:= \min\Big\{\arg\omega_1,\dots,\arg\omega_p,\arg\omega_{p+1}+\pi,\dots,\arg\omega_m+\pi\Big\},\\ \overline{\omega} &:= \max\Big\{\arg\omega_1,\dots,\arg\omega_p,\arg\omega_{p+1}+\pi,\dots,\arg\omega_m+\pi\Big\},\end{aligned} \tag{4.3}$$

and the value $\arg\omega_j$ is chosen up to a multiple of 2π, so that $\overline{\omega} - \underline{\omega} < \pi$.

Theorem 3. *Let $m \geq 1$, $a_m \neq 0$ and the roots of equation (4.2) be p-separated.*

Then, for any $\varepsilon > 0$ and for all complex numbers $\lambda \neq 0$ satisfying $\frac{\pi}{2} - \underline{\omega} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \overline{\omega} - \varepsilon$, the operator $\mathbb{L}_0(\lambda)\ :\ u \to \mathbb{L}_0(\lambda)u := L_0(\lambda)u$ from $W_q^\ell(\mathbb{R})$ onto $W_q^{\ell-m}(\mathbb{R})$, where an integer $\ell \geq m$ and a real $q \in (1,\infty)$, is an isomorphism, and for these λ, the following estimates hold:

$$R\Big\{\lambda^{m-k}\frac{d^k}{dy^k}L_0(\lambda)^{-1}:\ \frac{\pi}{2} - \underline{\omega} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \overline{\omega} - \varepsilon\Big\}_{W_q^s(\mathbb{R})} \leq C(\varepsilon) < \infty,$$
$$k = 0,\dots,m, \quad s = 0,\dots,\ell-m. \tag{4.4}$$

Proof. The isomorphism part of the theorem follows from [12, Theorem 3.2.1]. Let us show (4.4). From (4.1) we obtain

$$\Big(\lambda^m + \lambda^{m-1}a_1(i\sigma) + \cdots + a_m(i\sigma)^m\Big)Fu = Ff,$$

where F is the Fourier transform. It is obvious that

$$\lambda^m + \lambda^{m-1}a_1(i\sigma) + \cdots + a_m(i\sigma)^m = a_m\prod_{j=1}^m(i\sigma - \omega_j\lambda). \tag{4.5}$$

Since, for $\frac{\pi}{2} - \underline{\omega} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \overline{\omega} - \varepsilon$, $\sigma \in \mathbb{R}$, we have

$$|i\sigma - \omega_j\lambda| \geq C(\varepsilon)(|\sigma| + |\lambda|), \quad j = 1,\dots,m, \tag{4.6}$$

then, for $\frac{\pi}{2} - \underline{\omega} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \overline{\omega} - \varepsilon$, $\sigma \in \mathbb{R}\backslash\{0\}$,

$$Fu = \Big(\lambda^m + \lambda^{m-1}a_1(i\sigma) + \cdots + a_m(i\sigma)^m\Big)^{-1}Ff.$$

Hence, for $k = 0,\dots,m$,

$$\begin{aligned} u^{(k)}(y) &= F^{-1}(i\sigma)^kFu \\ &= F^{-1}(i\sigma)^k\Big(\lambda^m + \lambda^{m-1}a_1(i\sigma) + \cdots + a_m(i\sigma)^m\Big)^{-1}Ff, \end{aligned} \tag{4.7}$$

where F^{-1} is the inverse Fourier transform. From (4.5) and (4.6), it follows that functions

$$T_{k,\lambda}(\sigma) := \lambda^{m-k}(i\sigma)^k\Big(\lambda^m + \lambda^{m-1}a_1(i\sigma) + \cdots + a_m(i\sigma)^m\Big)^{-1}, \quad k = 0,\dots,m,$$

for $\frac{\pi}{2} - \underline{\omega} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \overline{\omega} - \varepsilon$, are continuously differentiable in σ on $\mathbb{R}\backslash\{0\}$, and

$$|T_{k,\lambda}(\sigma)| \leq C(\varepsilon) < \infty, \quad |\sigma|\ \Big|\frac{\partial}{\partial\sigma}T_{k,\lambda}(\sigma)\Big| \leq C(\varepsilon) < \infty, \quad \sigma \in \mathbb{R}\backslash\{0\},$$

uniformly on λ in the angle $\frac{\pi}{2} - \underline{\omega} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \overline{\omega} - \varepsilon$.

On the other hand, from (4.7) it follows that

$$\lambda^{m-k}\frac{d^k}{dy^k}L_0(\lambda)^{-1}f = F^{-1}T_{k,\lambda}(\sigma)Ff, \quad k = 0,\dots,m, \tag{4.8}$$

and from [10, section 5.2, item a)], for $k = 0,\dots,m$, it follows that $R\{F^{-1}T_{k,\lambda}(\cdot)F\ :\ \frac{\pi}{2} - \underline{\omega} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \overline{\omega} - \varepsilon\}_{L_q(\mathbb{R})} \leq C(\varepsilon) < \infty$ or, by (4.8), for $k = 0,\dots,m$,

$$R\left\{\lambda^{m-k}\frac{d^k}{dy^k}L_0(\lambda)^{-1}:\ \frac{\pi}{2} - \underline{\omega} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \overline{\omega} - \varepsilon\right\}_{L_q(\mathbb{R})} \leq C(\varepsilon) < \infty,$$

i.e., (4.4) has been proved for $s = 0$.

Observe now, that

$$\lambda^{m-k}\frac{d^{k+s}}{dy^{k+s}}L_0(\lambda)^{-1}f = \lambda^{m-k}u^{(k+s)}(y) = \lambda^{m-k}\frac{d^k}{dy^k}u^{(s)}(y) = \lambda^{m-k}\frac{d^k}{dy^k}L_0(\lambda)^{-1}f^{(s)}.$$

Then, using the definition of R-boundedness (see the Introduction), one can get (4.4) also for $s = 1, \dots, \ell - m$. □

Let us now formulate an analog of Theorem 3 for the equation

$$L_1(\lambda)u := \lambda^n u(y) + \sum_{k=1}^{n} \lambda^{n-k} a_k u^{(dk)}(y) = f(y), \quad y \in \mathbb{R}, \tag{4.9}$$

where $a_k = 0$ if dk is a non-integer number, weight $d := \frac{m}{n}$, and $a_n \neq 0$, i.e., $L_1(\lambda)$ is a polynomial operator pencil of order n and an ordinary differential operator of order m.

Theorem 4. *Let $n \geq 1$, $m \geq 1$, $a_n \neq 0$ and let the roots of the equation*

$$a_n\omega^m + a_{n-1}\omega^{d(n-1)} + \cdots + 1 = 0 \tag{4.10}$$

be p-separated.

Then, for any $\varepsilon > 0$ and for all complex numbers $\lambda \neq 0$ satisfying $\left(\frac{\pi}{2} - \underline{\omega} - 2\pi\zeta\right)d + \varepsilon < \arg\lambda < \left(\frac{3\pi}{2} - \overline{\omega} - 2\pi\zeta\right)d - \varepsilon$ for some $\zeta = 0, \dots, n-1$, where $\underline{\omega}$ and $\overline{\omega}$ are defined in (4.3) and ω_j are roots of equation (4.10), the operator $\mathbb{L}_1(\lambda) :\ u \to \mathbb{L}_1(\lambda)u := L_1(\lambda)u$ from $W_q^\ell(\mathbb{R})$ onto $W_q^{\ell-m}(\mathbb{R})$, where an integer $\ell \geq m$ and a real $q \in (1, \infty)$, is an isomorphism, and for these λ, the following estimates hold:

$$\begin{aligned} &R\Big\{\lambda^{\frac{m-k}{d}}\frac{d^k}{dy^k}L_1(\lambda)^{-1} : \Big(\frac{\pi}{2} - \underline{\omega} - 2\pi\zeta\Big)d + \varepsilon < \arg\lambda < \Big(\frac{3\pi}{2} - \overline{\omega} - 2\pi\zeta\Big)d - \varepsilon\Big\}_{W_q^s(\mathbb{R})} \\ &\qquad\leq C(\varepsilon) < \infty, \quad k = 0, \dots, m, \quad s = 0, \dots, \ell - m. \end{aligned} \tag{4.11}$$

Proof. After substituting $\lambda = \mu^d$ into the equation $L_1(\lambda)u = f(y)$ it is transformed into the equation

$$\mu^m u(y) + \sum_{k=1}^{n} \mu^{m-dk} a_k u^{(dk)}(y) = f(y), \quad y \in \mathbb{R},$$

to which we apply Theorem 3. □

Remark 2. Using, e.g., [3, Proposition 3.4] and the definition of R-boundedness (see the Introduction), one can get from (4.4) the following inequalities:

$$\begin{aligned} &R\Big\{\lambda^{m-k}L_0(\lambda)^{-1} :\ \ \frac{\pi}{2} - \underline{\omega} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \overline{\omega} - \varepsilon,\ |\lambda| \geq \lambda_0 > 0\Big\}_{W_q^s(\mathbb{R}) \to W_q^{s+k}(\mathbb{R})} \\ &\qquad\leq C(\varepsilon, \lambda_0) < \infty, \quad k = 0, \dots, m, \quad s = 0, \dots, \ell - m. \end{aligned}$$

The corresponding estimates follow from (4.11), too.

5 Application of abstract results to elliptic equations of the fourth order. In the domain $\Omega := [0,1] \times \mathbb{R}$, let us consider a boundary value problem for elliptic equations of the fourth order

$$
\begin{aligned}
L(D)u := D_x^4 u(x,y) + (aD_y^2 - 2\gamma^2) D_x^2 u(x,y) + bD_y^4 u(x,y) - a\gamma^2 D_y^2 u(x,y) \\
+ \gamma^4 u(x,y) = f(x,y), \quad (x,y) \in \Omega,
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
L_1 u &:= \alpha u(0,y) + \beta D_x u(0,y) = \varphi_1(y), \quad y \in \mathbb{R}, \\
L_2 u &:= \alpha D_x^2 u(0,y) + \beta D_x^3 u(0,y) = \varphi_2(y), \quad y \in \mathbb{R},
\end{aligned}
\tag{5.2}
$$

where a, b, α, β are complex numbers; $\gamma \in \mathbb{R}$; f and φ_k are given functions; $D_x := \frac{\partial}{\partial x}$, $D_y := \frac{\partial}{\partial y}$. By $B^s_{q,p}(\mathbb{R})$ we denote the standard Besov space, see, e.g., [11, section 2.3.1].

Theorem 5. *Let the following conditions be satisfied*:

1. $0 \neq \gamma \in \mathbb{R}$, $0 \neq b \in \mathbb{C}$, $\arg b \neq \pi$;
2. *if* $\sigma := (\sigma_1, \sigma_2) \in \mathbb{R}^2$, $\sigma \neq 0$, *then* $\sigma_1^4 + a\sigma_1^2\sigma_2^2 + b\sigma_2^4 \neq 0$;
3. *at least one of two numbers* α *and* β *is not equal to zero;* $\Re \alpha\beta^{-1} \leq 0$ *if* $\beta \neq 0$.

Then, there exists $\delta > 0$ *sufficiently small such that, for* $|a| < \delta$, *the operator* $\mathbb{L} : u \to \mathbb{L}u := (L(D)u, L_1 u, L_2 u)$ *from* $W_p^4((0,\infty); W_q^4(\mathbb{R}), L_q(\mathbb{R}))$ *into*

$$
L_p((0,\infty); L_q(\mathbb{R})) \dot{+} B_{q,p}^{4-m-\frac{1}{p}}(\mathbb{R}) \dot{+} B_{q,p}^{2-m-\frac{1}{p}}(\mathbb{R}),
$$

where $m = 0$ *if* $\beta = 0$ *and* $m = 1$ *if* $\beta \neq 0$, $q \in (1,\infty)$, $p \in (1,\infty)$, *is an isomorphism.*

Proof. Let us denote $E := L_q(\mathbb{R})$. Consider in $L_q(\mathbb{R})$ operators A_2 and A_4 which are defined by the equalities

$$
\begin{aligned}
D(A_2) &:= W_q^2(\mathbb{R}), \quad A_2 u := au''(y) - 2\gamma^2 u(y), \\
D(A_4) &:= W_q^4(\mathbb{R}), \quad A_4 u := bu''''(y) - a\gamma^2 u''(y) + \gamma^4 u(y).
\end{aligned}
$$

Then, problem (5.1)–(5.2) can be rewritten in the operator form

$$
\begin{aligned}
& u''''(x) + A_2 u''(x) + A_4 u(x) = f(x), \quad x \in [0,1], \\
& \alpha u(0) + \beta u'(0) = \varphi_1, \\
& \alpha u''(0) + \beta u'''(0) = \varphi_2,
\end{aligned}
\tag{5.3}
$$

where $u(x) := u(x,\cdot)$, $f(x) := f(x,\cdot)$ are functions with values in the Banach space $E := L_q(\mathbb{R})$ and $\varphi_k := \varphi_k(\cdot)$.

Let us apply Theorem 2 to problem (5.3). In fact, we have to check conditions (1), (2), and (4) of Theorem 2.

By virtue of condition (1), from, e.g., [3, Theorem 5.5] it follows that the operator $b\frac{d^4}{dy^4}$, with the domain $W_q^4(\mathbb{R})$, has a bounded H^∞-calculus in $E = L_q(\mathbb{R})$. Then, by, e.g., [3, Proposition 2.11, (iv)], the operator $\tilde{A}_4 = b\frac{d^4}{dy^4} + \gamma^4 I$, $0 \neq \gamma \in \mathbb{R}$, also has a bounded H^∞-calculus and, therefore, has bounded imaginary powers, *BIP* (see, e.g., [3, pp.50-51]), and, therefore, $\tilde{A}_4$ is R-sectorial. Moreover, the operator $\tilde{A}_4$ is also an isomorphism from $W_q^\ell(\mathbb{R})$ into $W_q^{\ell-4}(\mathbb{R})$, $\ell \geq 4$ (see Theorem 3, where $m = 4$, $a_1 = a_2 = a_3 = 0$, $a_4 = b$,

$\lambda = i\gamma \neq 0$, $\gamma \in \mathbb{R}$ - for this λ see the below considerations). Then, using a well-known interpolation inequality (see, e.g., [2, formula (1.9)]), we get, for $\ell = 4$,

$$\begin{aligned} \|a\gamma^2 u''\|_{L_q(\mathbb{R})} &= |a|\gamma^2\|u''\|_{L_q(\mathbb{R})} \le C|a|(\gamma^4\|u\|_{L_q(\mathbb{R})} + \|u\|_{W_q^4(\mathbb{R})}) \\ &\le C|a|\|\gamma^4 u + bu''''\|_{L_q(\mathbb{R})} = C|a|\|\tilde{A}_4 u\|_{L_q(\mathbb{R})}, \quad \forall u \in W_q^4(\mathbb{R}) = E(\tilde{A}_4), \end{aligned} \tag{5.4}$$

i.e., for small enough a, by [3, Proposition 4.2], the operator A_4 is also R-sectorial in E. Moreover, $A_4 = L_0(i\gamma)$, $\gamma \neq 0$ (see formula (5.6) below), i.e., A_4 is invertible in E (see the below considerations for $L_0(i\sigma)$). So, condition (1) of Theorem 2 is satisfied.

Taking now $\ell = 6$, we get

$$\begin{aligned} \|a\gamma^2 u''\|_{W_q^2(\mathbb{R})} &= |a|\gamma^2\|u''\|_{W_q^2(\mathbb{R})} \le |a|\gamma^2\|u\|_{W_q^4(\mathbb{R})} \\ &\le |a|C(\gamma^6\|u\|_{L_q(\mathbb{R})} + \|u\|_{W_q^6(\mathbb{R})}) \\ &\le |a|C\max\{\gamma^2, 1\}(\gamma^4\|u\|_{W_q^2(\mathbb{R})} + \|u\|_{W_q^6(\mathbb{R})}) \\ &\le C\|\gamma^4 u + bu''''\|_{W_q^2(\mathbb{R})} = C\|\tilde{A}_4 u\|_{W_q^2(\mathbb{R})}, \quad \forall u \in W_q^6(\mathbb{R}). \end{aligned} \tag{5.5}$$

It was mentioned above that $\tilde{A}_4$ has *BIP*. Then, by [11, Theorem 1.15.3], $E(\tilde{A}_4^{1-\frac{k}{4}}) = [L_q(\mathbb{R}), W_q^4(\mathbb{R})]_{1-\frac{k}{4}}$, $k = 1, 2, 3$. On the other hand, by virtue of [11, formula 2.4.2/(11)], $[L_q(\mathbb{R}), W_q^4(\mathbb{R})]_{1-\frac{k}{4}} = W_q^{4-k}(\mathbb{R})$, $k = 1, 2, 3$. Hence, $E(\tilde{A}_4^{1-\frac{k}{4}}) = W_q^{4-k}(\mathbb{R})$, $k = 0, \ldots, 3$. In particular, $E(\tilde{A}_4^{\frac{1}{2}}) = W_q^2(\mathbb{R})$. On the other hand, by the above isomorphism, $\tilde{A}_4$ is also invertible in E. Then,

$$\|\tilde{A}_4^{\frac{1}{2}} u\|_{L_q(\mathbb{R})} = \|u\|_{E(\tilde{A}_4^{\frac{1}{2}})} = \|u\|_{W_q^2(\mathbb{R})}, \quad \forall u \in W_q^2(\mathbb{R}).$$

Hence, from (5.5), we get

$$\begin{aligned} \|\tilde{A}_4^{\frac{1}{2}}(a\gamma^2 u'')\|_{L_q(\mathbb{R})} &= \|a\gamma^2 u''\|_{W_q^2(\mathbb{R})} \le C\|\tilde{A}_4 u\|_{W_q^2(\mathbb{R})} \\ &= C\|\tilde{A}_4^{\frac{3}{2}} u\|_{L_q(\mathbb{R})}, \quad \forall u \in W_q^6(\mathbb{R}) = E(\tilde{A}_4^{\frac{3}{2}}). \end{aligned} \tag{5.6}$$

Since $\tilde{A}_4$ has a bounded H^∞-calculus then, from inequalities (5.4) and (5.6), by virtue of N. Kalton, P. Kunstmann, and L. Weis [7, Corollary 6.5, for $\delta = \frac{1}{2}$], we get that the operator A_4, for small enough a, also has a bounded H^∞-calculus in E. Therefore, A_4 has *BIP* in E and, as above, $E(A_4^{\frac{1}{2}}) = W_q^2(\mathbb{R})$. This, in turn, implies condition (2) of Theorem 2.

In order to check condition (4) of Theorem 2, denote by

$$L_0(\lambda) := \lambda^4 I + \lambda^2 a\frac{d^2}{dy^2} + b\frac{d^4}{dy^4}. \tag{5.7}$$

Then, the operator pencil corresponding to the equation in (5.3), has the form

$$\begin{aligned} L(\lambda) &:= \lambda^4 I + \lambda^2 A_2 + A_4 = \lambda^4 I + \lambda^2\left(a\frac{d^2}{dy^2} - 2\gamma^2 I\right) \\ &\quad + b\frac{d^4}{dy^4} - a\gamma^2\frac{d^2}{dy^2} + \gamma^4 I = \lambda^4 I + \lambda^2 a\frac{d^2}{dy^2} + b\frac{d^4}{dy^4} \\ &\quad - 2\lambda^2\gamma^2 I - a\gamma^2\frac{d^2}{dy^2} + \gamma^4 I = L_0(\lambda) - 2\lambda^2\gamma^2 I - a\gamma^2\frac{d^2}{dy^2} + \gamma^4 I. \end{aligned}$$

From this, it can be easily seen that for $\lambda = i\sigma$, $\sigma \in \mathbb{R}$,

$$L(i\sigma) = L_0(i\sqrt{\sigma^2 + \gamma^2}). \tag{5.8}$$

Further, by virtue of condition (2), the equation

$$1 + a\omega^2 + b\omega^4 = 0 \tag{5.9}$$

does not have real roots. Therefore, if roots of (5.9), ω_1 and ω_2, are situated in the upper-half complex plane (including the case $\omega_1 = \omega_2$), then roots $\omega_3 = -\omega_1$ and $\omega_4 = -\omega_2$ are situated in the lower-half complex plane. Therefore, from (4.3), $0 < \underline{\omega} \leq \overline{\omega} < \pi$, i.e., the angle of Theorem 3 contains $\lambda = i\sigma$, $\sigma > 0$. Changing the numeration of the roots of (5.9) in such a way that ω_1 and ω_2 are now in the lower-half complex plane and ω_3, ω_4 are in the upper-half complex plane, we get, from (4.3), $\pi < \underline{\omega} \leq \overline{\omega} < 2\pi$. So, the angle of Theorem 3 contains also $\lambda = i\sigma$, $\sigma < 0$. Hence, the angle of Theorem 3 contains $\lambda = i\sigma$, $\sigma \in \mathbb{R}\backslash\{0\}$ and, by Theorem 3, we get that $L_0(i\sigma)$, which is defined by (5.7), is invertible for $0 \neq \sigma \in \mathbb{R}$ and, for integers $s \geq 0$,

$$R\Big\{\sigma^{4-k}\frac{d^k}{dy^k}L_0(i\sigma)^{-1} :\ \sigma \in \mathbb{R}\backslash\{0\}\Big\}_{W_q^s(\mathbb{R})} \leq C < \infty, \quad k = 0, \dots, 4. \tag{5.10}$$

First, it means that $L(i\sigma)$ (see (5.8)) is invertible for $\sigma \in \mathbb{R}$ (remind that $0 \neq \gamma \in \mathbb{R}$). Using now (5.8), (5.10), [3, Proposition 3.4], and the contraction principle of Kahane (see, e.g., [3, Lemma 3.5]), we get, for integers $s \geq 0$,

$$\begin{aligned}
&R\Big\{\sigma^{4-k}\frac{d^k}{dy^k}L(i\sigma)^{-1} :\ \sigma \in \mathbb{R}\Big\}_{W_q^s(\mathbb{R})} \\
&\quad = R\Big\{\frac{\sigma^{4-k}}{(\sqrt{\sigma^2+\gamma^2})^{4-k}}(\sqrt{\sigma^2+\gamma^2})^{4-k}\frac{d^k}{dy^k}L_0(i\sqrt{\sigma^2+\gamma^2})^{-1} :\ \sigma \in \mathbb{R}\Big\}_{W_q^s(\mathbb{R})} \\
&\quad \leq R\Big\{\frac{\sigma^{4-k}}{(\sqrt{\sigma^2+\gamma^2})^{4-k}}I :\ \sigma \in \mathbb{R}\Big\}_{W_q^s(\mathbb{R})} \cdot R\Big\{(\sqrt{\sigma^2+\gamma^2})^{4-k}\frac{d^k}{dy^k}L_0(i\sqrt{\sigma^2+\gamma^2})^{-1} : \\
&\sigma \in \mathbb{R}\Big\}_{W_q^s(\mathbb{R})} \leq 1 \cdot C < \infty, \quad k = 0, \dots, 4.
\end{aligned} \tag{5.11}$$

Therefore, the first and the third inequalities in condition (4) of Theorem 2 follow from (5.11) under $k = s = 0$ and $k = 0, s = 2$ (remind that $E_2 := E(A_4^{\frac{1}{2}}) = W_q^2(\mathbb{R})$), respectively. In order to get the second and the fourth inequalities in condition (4) of Theorem 2, let us observe that again, by (5.8), (5.10), [3, Proposition 3.4], and [3, Lemma 3.5], we get, for integers $s \geq 0$,

$$R\Big\{A_4 L(i\sigma)^{-1}:\ \sigma\in\mathbb{R}\Big\}_{W_q^s(\mathbb{R})} = R\Big\{b\frac{d^4}{dy^4}L(i\sigma)^{-1} - a\gamma^2\frac{d^2}{dy^2}L(i\sigma)^{-1}$$
$$+\gamma^4 L(i\sigma)^{-1}:\ \sigma\in\mathbb{R}\Big\}_{W_q^s(\mathbb{R})} \leq |b| R\Big\{\frac{d^4}{dy^4}L_0(i\sqrt{\sigma^2+\gamma^2})^{-1}:\ \sigma\in\mathbb{R}\Big\}_{W_q^s(\mathbb{R})}$$
$$+|a|R\Big\{\gamma^2\frac{d^2}{dy^2}L_0(i\sqrt{\sigma^2+\gamma^2})^{-1}:\ \sigma\in\mathbb{R}\Big\}_{W_q^s(\mathbb{R})}$$
$$+R\Big\{\gamma^4 L_0(i\sqrt{\sigma^2+\gamma^2})^{-1}:\ \sigma\in\mathbb{R}\Big\}_{W_q^s(\mathbb{R})}$$
$$\leq |b| R\Big\{\frac{d^4}{dy^4}L_0(i\sqrt{\sigma^2+\gamma^2})^{-1}:\ \sigma\in\mathbb{R}\Big\}_{W_q^s(\mathbb{R})}$$
$$+|a|R\Big\{\frac{\gamma^2}{\sigma^2+\gamma^2}(\sigma^2+\gamma^2)\frac{d^2}{dy^2}L_0(i\sqrt{\sigma^2+\gamma^2})^{-1}:\ \sigma\in\mathbb{R}\Big\}_{W_q^s(\mathbb{R})}$$
$$+R\Big\{\frac{\gamma^4}{(\sigma^2+\gamma^2)^2}(\sigma^2+\gamma^2)^2 L_0(i\sqrt{\sigma^2+\gamma^2})^{-1}:\ \sigma\in\mathbb{R}\Big\}_{W_q^s(\mathbb{R})} < \infty.$$

Finally, interpolation spaces of Theorem 2 are equal, by, e.g., [11, formula 2.4.2/(16)], to Besov spaces, $(W_q^4(\mathbb{R}), L_q(\mathbb{R}))_{\theta,p} = B_{q,p}^{4(1-\theta)}(\mathbb{R})$. □

6 Application of abstract results to elliptic and quasi-elliptic equations. A case of the whole space $\mathbb{R}^n$ for y-variable. In the domain $\Omega := [0,\infty)\times\mathbb{R}^n$, $n\geq 1$, consider a boundary value problem for elliptic ($m=2$) and quasi-elliptic ($m\neq 2$ is natural) equations

$$L(D)u := D_x^4 u(x,y) + \sum_{|\alpha|=2m} a_\alpha(y) D_y^\alpha u(x,y) + \nu u(x,y) = f(x,y), \quad (x,y)\in\Omega, \tag{6.1}$$

$$\begin{aligned} L_1 u &:= \alpha u(0,y) + \beta D_x u(0,y) = \varphi_1(y), \quad y\in\mathbb{R}^n,\\ L_2 u &:= \alpha D_x^2 u(0,y) + \beta D_x^3 u(0,y) = \varphi_2(y), \quad y\in\mathbb{R}^n, \end{aligned} \tag{6.2}$$

where $\nu > 0$, α and β are complex numbers, f and φ_k are given functions, $D_x := \frac{\partial}{\partial x}$, $D_y^\alpha := D^\alpha := D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, $D_j := -i\frac{\partial}{\partial y_j}$, $|\alpha| = \alpha_1+\cdots+\alpha_n$.

Let $Au(y) := \sum_{|\alpha|=2m} a_\alpha(y) D^\alpha u(y)$ be an (M,ω_0)-elliptic operator (see, e.g., [7, p.790]) with complex-valued Hölder continuous coefficients $a_\alpha \in C^\gamma(\mathbb{R}^n)$, $|\alpha| = 2m$, for some $\gamma > 0$.

By $B_{q,p}^s(\mathbb{R}^n)$ we denote the standard Besov space, see, e.g., [11, section 2.3.1].

Theorem 6. *Let at least one of two numbers α and β be not equal to zero; $\Re\alpha\beta^{-1}\leq 0$ if $\beta\neq 0$.*

Then, there exists $\nu > 0$ sufficiently large such that the operator $\mathbb{L}: u \to \mathbb{L}u := (L(D)u, L_1 u, L_2 u)$ from $W_p^4((0,\infty); W_q^{2m}(\mathbb{R}^n), L_q(\mathbb{R}^n))$ into

$$L_p((0,\infty); L_q(\mathbb{R}^n)) \dot{+} B_{q,p}^{2m-\frac{m\ell}{2}-\frac{m}{2p}}(\mathbb{R}^n) \dot{+} B_{q,p}^{m-\frac{m\ell}{2}-\frac{m}{2p}}(\mathbb{R}^n),$$

where $\ell = 0$ if $\beta = 0$ and $\ell = 1$ if $\beta\neq 0$, $q\in(1,\infty)$, $p\in(1,\infty)$, is an isomorphism.

Proof. Let us denote $E := L_q(\mathbb{R}^n)$. Consider in E an operator A_4 which is defined by the equalities

$$D(A_4) := W_q^{2m}(\mathbb{R}^n), \quad A_4u := Au(y) + \nu u(y),$$

where $\nu > 0$ is sufficiently large. Then, problem (6.1)–(6.2) can be rewritten in the operator form

$$\begin{aligned} &u''''(x) + A_4u(x) = f(x), \quad x \in [0, \infty), \\ &\alpha u(0) + \beta u'(0) = \varphi_1, \\ &\alpha u''(0) + \beta u'''(0) = \varphi_2, \end{aligned} \tag{6.3}$$

where $u(x) := u(x, \cdot)$, $f(x) := f(x, \cdot)$ are functions with values in the Banach space $E := L_q(G)$ and $\varphi_k := \varphi_k(\cdot)$.

Let us apply Theorem 2 to problem (6.3). We have to check conditions (1), (2), and (4) of Theorem 2. From [7, Proposition 9.5] it follows, for some $\nu > 0$ sufficiently large, that the operator A_4 is invertible and has a bounded H^∞-calculus in $L_q(\mathbb{R}^n)$. Hence, A_4 has *BIP* and, therefore, is an R-sectorial operator in $L_q(\mathbb{R}^n)$, since $L_q(\mathbb{R}^n)$ is a *UMD* Banach space (see, e.g., [3, pp.50-51]). Therefore, condition (1) of Theorem 2 is satisfied. Condition (2) of Theorem 2 is obvious since $A_2 = 0$. Let us now check condition (4) of Theorem 2.

Since A_4 has *BIP* in $L_q(\mathbb{R}^n)$, then, by virtue of [11, Theorem 1.15.3], $E(A_4^{1-\frac{k}{2m}}) = [L_q(\mathbb{R}^n), W_q^{2m}(\mathbb{R}^n)]_{1-\frac{k}{2m}}$, $k = 1, \ldots, 2m-1$. On the other hand, by [11, formula 2.4.2/(11)], $[L_q(\mathbb{R}^n), W_q^{2m}(\mathbb{R}^n)]_{1-\frac{k}{2m}} = W_q^{2m-k}(\mathbb{R}^n)$, $k = 1, \ldots, 2m-1$. Therefore, $E(A_4^{1-\frac{k}{2m}}) = W_q^{2m-k}(\mathbb{R}^n)$, $k = 1, \ldots, 2m-1$. In particular, $E_2 := E(A_4^{\frac{1}{2}}) = W_q^m(\mathbb{R}^n)$.

Further, if $L(\lambda) = \lambda^4 I + A_4$ then $L(i\sigma)^{-1} = -R(-\sigma^4, A_4)$, $\sigma \in \mathbb{R}$, and for A_4 we have already checked condition (1) of Theorem 2. Therefore,

$$R\{\sigma^4 L(i\sigma)^{-1} : \sigma \in \mathbb{R}\}_{L_q(\mathbb{R}^n)} < \infty. \tag{6.4}$$

Since $A_4L(i\sigma)^{-1} = I - \sigma^4 L(i\sigma)^{-1}$ then, using, e.g., [3, Proposition 3.4], we get

$$R\{A_4 L(i\sigma)^{-1} : \sigma \in \mathbb{R}\}_{L_q(\mathbb{R}^n)} < \infty. \tag{6.5}$$

So, (6.4) and (6.5) are two first inequalities in condition (4) of Theorem 2.

By [7, Proposition 9.5], for some $\nu > 0$ sufficiently large, the operator A_4 is invertible and has a bounded H^∞-calculus in $W_q^m(\mathbb{R}^n)$. Hence, A_4 has *BIP* in $W_q^m(\mathbb{R}^n)$ and, therefore, is an R-sectorial operator in $W_q^m(\mathbb{R}^n)$, since $W_q^m(\mathbb{R}^n)$ is a *UMD* Banach space (see, e.g., [3, pp.50-51]). So, taking into account that $L(i\sigma)^{-1} = -R(-\sigma^4, A_4)$, we get

$$R\{\sigma^4 L(i\sigma)^{-1} : \sigma \in \mathbb{R}\}_{W_q^m(\mathbb{R}^n)} < \infty. \tag{6.6}$$

As above, from (6.6) we get

$$R\{A_4 L(i\sigma)^{-1} : \sigma \in \mathbb{R}\}_{W_q^m(\mathbb{R}^n)} < \infty. \tag{6.7}$$

Inequalities (6.6) and (6.7) are two last inequalities in condition (4) of Theorem 2.

It remains only to observe that, by virtue of, e.g., [11, formula 2.4.2/(16)], $(W_q^{2m}(\mathbb{R}^n), L_q(\mathbb{R}^n))_{\theta,p} = B_{q,p}^{2m(1-\theta)}(\mathbb{R}^n)$. Then, $(W_q^{2m}(\mathbb{R}^n), L_q(\mathbb{R}^n))_{\frac{\ell}{4}+\frac{1}{4p},p} = B_{q,p}^{2m-\frac{m\ell}{2}-\frac{m}{2p}}(\mathbb{R}^n)$ and $(W_q^{2m}(\mathbb{R}^n), L_q(\mathbb{R}^n))_{\frac{\ell+2}{4}+\frac{1}{4p},p} = B_{q,p}^{m-\frac{m\ell}{2}-\frac{m}{2p}}(\mathbb{R}^n)$. ∎

Remark 3. Using standard perturbation arguments (see the arguments in the resolvent decomposition in, e.g., [3, Propositions 4.2 and 4.3]), [9, Lemma 10], [7, Proposition 9.5], and the calculations in the proof of Theorem 6, one can prove Theorem 6 for more general equations than (6.1), namely, for

$$L(D)u := D_x^4 u(x,y) + \sum_{|\alpha|\le m} b_\alpha(y) D_x^2 D_y^\alpha u(x,y) + \sum_{|\alpha|\le 2m} a_\alpha(y) D_y^\alpha u(x,y) + \nu u(x,y) = f(x,y), \quad (x,y)\in\Omega,$$

where $b_\alpha \in BUC^m(\mathbb{R}^n)$, $\sup_{y\in\mathbb{R}^n} |D_y^\beta b_\alpha(y)|$, for all $|\alpha|, |\beta| \le m$, are sufficiently small, and $\nu > 0$ is, as previous, sufficiently large, even maybe larger than ν in (6.1).

7 Application of abstract results to elliptic and quasi-elliptic equations. A case of a bounded domain G for y-variable. In the cylindrical domain $\Omega := [0,1]\times G$, where $G \subset \mathbb{R}^n$, $n \ge 2$, is a bounded domain with an $(n-1)$-dimensional boundary $\partial G \in C^{2m}$, which locally admits rectification, let us consider a principally non-local boundary value problem for elliptic ($m = 2$) and quasi-elliptic ($m \ne 2$ is natural) equations

$$L(D)u := D_x^4 u(x,y) + \sum_{|\alpha|=2m} a_\alpha(y) D_y^\alpha u(x,y) + \sum_{k=0}^{3} B_k(x) D_x^k u(x,\cdot)|_y = f(x,y), \quad (x,y)\in\Omega, \tag{7.1}$$

$$L_k u := \gamma_k D_x^{m_k} u(0,y) + \delta_k D_x^{m_k} u(1,y) + \sum_{j=0}^{m_k-1}\sum_{s=1}^{N_{kj}} T_{kjs} D_x^j u(x_{kjs},\cdot)|_y = \varphi_k(y), \quad y\in G, \quad k=1,\dots,4, \tag{7.2}$$

$$B_\ell u := \sum_{|\beta|\le p_\ell} b_{\ell\beta}(y') D_y^\beta u(x,y') = 0, \quad (x,y')\in[0,1]\times\partial G,\ \ell=1,\dots,m, \tag{7.3}$$

where $0 \le m_1, m_2 \le 1$, $m_3 = m_1 + 2$, $m_4 = m_2 + 2$, $p_\ell \le 2m - 1$; γ_k and δ_k are complex numbers, $x_{kjs} \in [0,1]$; f and φ_k are given functions; $B_k(x)$, for any $x \in [0,1]$, and T_{kjs} are, generally speaking, unbounded operators in $L_q(G)$, $1 < q < \infty$; $D_x := \frac{\partial}{\partial x}$, $D_y^\alpha := D^\alpha := D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, $D_j := -i\frac{\partial}{\partial y_j}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Let $Au(y) := \sum_{|\alpha|=2m} a_\alpha(y) D^\alpha u(y)$ be an (M,ω_0)-elliptic operator (see, e.g., [7, p.790]) with complex-valued coefficients $a_\alpha \in C^\gamma(\overline{G})$, $|\alpha| = 2m$, and complex-valued coefficients of the boundary conditions B_ℓ, $b_{\ell\beta} \in C^{2m-p_j+\gamma}(\overline{G})$, where $\gamma \in (0,1)$ (the continuation of the coefficients from ∂G into G is possible without loss of generality). We assume that $(A, B_1, \dots, B_m)$ satisfies the Lopatinskii-Shapiro condition (see, e.g., [3, p.100]) at every point $y' \in \partial G$.

By $B_{q,p}^s(G)$ we denote the standard Besov space and by $H_q^s(G)$ - the standard Bessel potential space, see, e.g., [11, section 4.2.1]. If $s = 0, 1, 2, \dots$, then $H_q^s(G)$ coincides with Sobolev spaces $W_q^s(G)$.

Theorem 7. *Let, in addition to the above, the following conditions be satisfied*:

1.

$$\begin{vmatrix} \gamma_1(-1)^{m_1} & 0 & \delta_1 & 0 \\ 0 & \gamma_3(-1)^{m_1} & 0 & \delta_3 \\ \gamma_2(-1)^{m_2} & 0 & \delta_2 & 0 \\ 0 & \gamma_4(-1)^{m_2} & 0 & \delta_4 \end{vmatrix} \neq 0;$$

for $m_1 \neq m_2,\ \gamma_k = \gamma_{k+2},\ \delta_k = \delta_{k+2},\ k = 1, 2;$

2. for any $\varepsilon > 0$ *and for almost all* $x \in [0,1]$, $k = 0, \ldots, 3$,

$$\|B_k(x)u\|_{L_q(G)} \leq \varepsilon \|u\|_{H_q^{2m-\frac{mk}{2}}(G)} + C(\varepsilon)\|u\|_{L_q(G)}, \quad u \in H_q^{2m-\frac{mk}{2}}(G);$$

for $u \in H_q^{2m-\frac{mk}{2}}(G)$, *the function* $x \to B_k(x)u$ *from* $[0,1]$ *into* $L_q(G)$ *is measurable for some* $1 < q < \infty$*;*

3. $1 < p < \infty$; *operators* T_{kjs} *from* $B_{q,p}^{2m-\frac{jm}{2}-\frac{m}{2p}}(G; B_\ell u = 0, p_\ell < 2m - \frac{jm}{2} - \frac{m}{2p} - \frac{1}{q})$, *if all* $p_\ell \neq 2m - \frac{jm}{2} - \frac{m}{2p} - \frac{1}{q}$, [4] *into* $B_{q,p}^{2m-\frac{m_k m}{2}-\frac{m}{2p}}(G; B_\ell u = 0, p_\ell < 2m - \frac{m_k m}{2} - \frac{m}{2p} - \frac{1}{q})$, *if all* $p_\ell \neq 2m - \frac{m_k m}{2} - \frac{m}{2p} - \frac{1}{q}$ *(a similar footnote as above takes place), are compact.*

Then, the operator $\mathbb{L} : u \to \mathbb{L}u := \big(L(D)u, L_1u, L_2u, L_3u, L_4u\big)$ *from* $W_p^4((0,1); W_q^{2m}(G; B_\ell u = 0, \ell = 1, \ldots, m), L_q(G))$ *into* $L_p((0,1); L_q(G)) \dot{+}_{k=1}^{4} B_{q,p}^{2m-\frac{m_k m}{2}-\frac{m}{2p}}(G; B_\ell u = 0, p_\ell < 2m - \frac{m_k m}{2} - \frac{m}{2p} - \frac{1}{q})$ *is bounded and Fredholm.*

Proof. Let us denote $E := L_q(G)$. Consider in E an operator A_4 which is defined by the equalities

$$D(A_4) := W_q^{2m}(G; B_\ell u = 0, \ell = 1, \ldots, m), \quad A_4 u := Au(y) + \nu u(y),$$

where $\nu > 0$ is sufficiently large. Then, problem (7.1)–(7.3) can be rewritten in the operator form

$$\begin{aligned} &u''''(x) + A_4 u(x) + \sum_{k=0}^{3} M_k(x) u^{(k)}(x) = f(x), \quad x \in [0,1], \\ &\gamma_k u^{(m_k)}(0) + \delta_k u^{(m_k)}(1) + \sum_{j=0}^{m_k-1} \sum_{s=1}^{N_{kj}} T_{kjs} u^{(j)}(x_{kjs}) = \varphi_k, \quad k = 1, \ldots, 4, \end{aligned} \tag{7.4}$$

where $M_0(x) = B_0(x) - \nu I$, $M_k(x) = B_k(x)$, $k = 1, 2, 3$; $u(x) := u(x, \cdot)$, $f(x) := f(x, \cdot)$ are functions with values in the Banach space $E := L_q(G)$ and $\varphi_k := \varphi_k(\cdot)$.

Let us apply Theorem 1 to problem (7.4). The operator A_4 is an isomorphism from $W_q^{4+s}(G)$ into $W_q^s(G)$, for any integer $s \geq 0$ (see, e.g., [12, Theorem 4.2.2/1]). R-sectoriality of A_4 follows from [3, Theorem 8.2]. Therefore, condition (1) of Theorem 1 is satisfied.

[4]If $p = q$ and there exists ℓ such that $p_\ell = 2m - \frac{jm}{2} - \frac{m}{2p} - \frac{1}{q}$ then one should take, for this ℓ, $B_\ell u \in \tilde{B}_{p,p}^{\frac{1}{p}}(G)$ instead of $B_\ell u = 0$ (see, e.g., [11, formula 4.3.3/(8)]), where $\tilde{B}_{q,p}^s(G) = \{u \mid u \in B_{q,p}^s(\mathbb{R}^n),\ \mathrm{supp}(u) \subset \overline{G}\}$.

Condition (2) of Theorem 1 follows from, e.g., [11, Theorem 3.2.5]. Condition (3) of Theorem 1 is obvious since $A_2 = 0$.

Let us now check condition (4) of Theorem 1. From [7, Proposition 9.8] it follows that the operator A_4 has a bounded H^∞-calculus in $L_q(G)$, therefore, A_4 has *BIP* in $L_q(G)$. Then, by [11, Theorem 1.15.3], $E(A_4^{1-\frac{k}{2m}}) = [L_q(G), W_q^{2m}(G; B_\ell u = 0, \ell = 1, \ldots, m)]_{1-\frac{k}{2m}}$, $k = 1, \ldots, 2m-1$. On the other hand, by virtue of [11, Theorem 4.3.3], $[L_q(G), W_q^{2m}(G; B_\ell u = 0, \ell = 1, \ldots, m)]_{1-\frac{k}{2m}} = W_q^{2m-k}(G; B_\ell u = 0, p_\ell < 2m-k)$, $k = 1, \ldots, 2m-1$. Hence, $E(A_4^{1-\frac{k}{2m}}) = W_q^{2m-k}(G; B_\ell u = 0, p_\ell < 2m-k)$, $k = 0, \ldots, 2m-1$. In particular, $E_2 := E(A_4^{\frac{1}{2}}) = W_q^m(G; B_\ell u = 0, p_\ell < m)$.

Further, if $L_0(\lambda) = \lambda^4 I + A_4$ then $L_0(i\sigma)^{-1} = -R(-\sigma^4, A_4)$, $\sigma \in \mathbb{R}$, and for A_4 we have already checked condition (1) of Theorem 1. Therefore,

$$R\{\sigma^4 L_0(i\sigma)^{-1} : \sigma \in \mathbb{R}\}_{L_q(G)} < \infty. \tag{7.5}$$

Since $A_4 L_0(i\sigma)^{-1} = I - \sigma^4 L_0(i\sigma)^{-1}$ then, using, e.g., [3, Proposition 3.4], we get

$$R\{A_4 L_0(i\sigma)^{-1} : \sigma \in \mathbb{R}\}_{L_q(G)} < \infty. \tag{7.6}$$

So, (7.5) and (7.6) are two first inequalities in condition (4) of Theorem 1.

It was mentioned above that the operator A_4 has a bounded H^∞-calculus in $L_q(G)$, therefore, A_4 has also a bounded H^∞-calculus in the domain of fractional powers of A_4, i.e., in $E(A_4^{1-\frac{k}{2m}}) = W_q^{2m-k}(G; B_\ell u = 0, p_\ell < 2m-k)$, $k = 1, \ldots, 2m-1$. In particular, A_4 has a bounded H^∞-calculus in $E_2 := E(A_4^{\frac{1}{2}}) = W_q^m(G; B_\ell u = 0, p_\ell < m)$. This implies that A_4 is an R-sectorial operator in E_2. So, taking into account that $L_0(i\sigma)^{-1} = -R(-\sigma^4, A_4)$, we get

$$R\{\sigma^4 L_0(i\sigma)^{-1} : \sigma \in \mathbb{R}\}_{E_2} < \infty. \tag{7.7}$$

As above, from (7.7) we get

$$R\{A_4 L_0(i\sigma)^{-1} : \sigma \in \mathbb{R}\}_{E_2} < \infty. \tag{7.8}$$

Inequalities (7.7) and (7.8) are two last inequalities in condition (4) of Theorem 1.

Condition (5) of Theorem 1 is just condition (1). Condition (6) of Theorem 1 follows from condition (2) since, as above, using [11, Theorems 1.15.3 and 4.3.3], one can see that $E(A_4^{1-\frac{k}{4}}) \subset H_q^{2m-\frac{mk}{2}}(G)$, $k = 0, \ldots, 3$. Condition (7) of Theorem 1 follows from condition (3), in view of [11, Theorem 4.3.3]. □

Consider now, in the domain $\Omega := [0, \infty) \times G$, the following boundary value problem

$$L(D)u := D_x^4 u(x,y) + \sum_{|\alpha|=2m} a_\alpha(y) D_y^\alpha u(x,y) = f(x,y), \quad (x,y) \in \Omega,$$

$$L_1 u := \alpha u(0,y) + \beta D_x u(0,y) = \varphi_1(y), \quad y \in G,$$
$$L_2 u := \alpha D_x^2 u(0,y) + \beta D_x^3 u(0,y) = \varphi_2(y), \quad y \in G,$$

$$B_\ell u := \sum_{|\beta| \le p_\ell} b_{\ell\beta}(y') D_y^\beta u(x,y') = 0, \quad (x,y') \in [0,\infty) \times \partial G, \ \ell = 1, \ldots, m,$$

where α, β are complex numbers and all other data as previously.

Theorem 8. *Let at least one of two numbers α and β be not equal to zero; $\Re\alpha\beta^{-1} \le 0$ if $\beta \ne 0$.*

Then, the operator $\mathbb{L} : u \to \mathbb{L}u := (L(D)u, L_1u, L_2u)$ from $W_p^4((0,\infty); W_q^{2m}(G;\ B_\ell u = 0, \ell = 1,\dots,m), L_q(G))$ into $L_p((0,\infty); L_q(G)) \dot{+} B_{q,p}^{2m-\frac{sm}{2}-\frac{m}{2p}}(G; B_\ell u = 0, p_\ell < 2m - \frac{sm}{2} - \frac{m}{2p} - \frac{1}{q}) \dot{+} B_{q,p}^{m-\frac{sm}{2}-\frac{m}{2p}}(G; B_\ell u = 0, p_\ell < m - \frac{sm}{2} - \frac{m}{2p} - \frac{1}{q})$, if all $p_\ell \ne 2m - \frac{sm}{2} - \frac{m}{2p} - \frac{1}{q}$ and $\ne m - \frac{sm}{2} - \frac{m}{2p} - \frac{1}{q}$,[5] where $s = 0$ if $\beta = 0$ and $s = 1$ if $\beta \ne 0$, $q \in (1,\infty)$, $p \in (1,\infty)$, is an isomorphism.

Proof. The proof is the same as that of Theorem 7. We only apply Theorem 2 instead of Theorem 1. □

Examples of the operators B_k and T_{kjs} satisfying conditions of Theorem 7.

One can take for $B_k(x)$ some differential-integral operators in $L_q(G)$, where the differential part is of order $\le 2m - \frac{mk}{2} - 1$ and the integral part contains integrals of the function and its derivatives with respect to $y \in G$ up to order $\le 2m - \frac{mk}{2} - 1$. Indeed, in this case, the operators $B_k(x)$ are bounded from $H_q^{2m-\frac{mk}{2}-1}(G)$ into $L_q(G)$. On the other hand, the embedding $H_q^{2m-\frac{mk}{2}}(G) \subset H_q^{2m-\frac{mk}{2}-1}(G)$ is compact (see, e.g., [11, Theorem 4.10.2]). Therefore, the operators $B_k(x)$ are compact from $H_q^{2m-\frac{mk}{2}}(G)$ into $L_q(G)$, i.e., by [12, Lemma 1.2.7/3], condition (2) of Theorem 7 is satisfied.

The first simple example of T_{kjs} is $(T_{kjs}u)(y) := \gamma_{kjs}u(y)$, where $\gamma_{kjs} \in \mathbb{C}$. Indeed, since the embeddings $B_{q,p}^{2m-\frac{jm}{2}-\frac{m}{2p}}(G) \subset B_{q,p}^{2m-\frac{m_k m}{2}-\frac{m}{2p}}(G)$, $j = 0,\dots,m_k - 1$, are compact (see, e.g., [11, Theorem 4.10.2]) then condition (3) of Theorem 7 is satisfied.

Let us now take another model example of $(T_{kjs}u)(y) := \int_G T_{kjs}(x,y)u(x)dx$, where $T_{kjs}(x,y) \in L_{t'}(G \times G)$, $\frac{1}{t'} + \frac{1}{t} = 1$, $t = \min\{q, q'\}$, $\frac{1}{q'} + \frac{1}{q} = 1$, and $T_{kjs}(x,y)$ are $2m$-times continuously differentiable with respect to $y \in G$, and all these derivatives also belong to $L_{t'}(G \times G)$. Since the operators T_{kjs} from $H_q^{2m-\frac{jm}{2}}(G)$ into $H_q^{2m-\frac{m_k m}{2}}(G)$ and from $H_q^{m-\frac{jm}{2}}(G)$ into $H_q^{2m-\frac{m_k m}{2}}(G)$, $j = 0,\dots,m_k-1$, are bounded then, by virtue of [11, Theorem 1.3.3/(a)], the operators T_{kjs} from $(H_q^{2m-\frac{jm}{2}}(G), H_q^{m-\frac{jm}{2}}(G))_{\frac{1}{2p},p}$ into $(H_q^{2m-\frac{m_k m}{2}}(G), H_q^{2m-\frac{m_k m}{2}}(G))_{\frac{1}{2p},p}$ are also bounded. On the other hand, by [11, Theorem 4.3.1, formula 2.4.2/(14)], $(H_q^{2m-\frac{jm}{2}}(G), H_q^{m-\frac{jm}{2}}(G))_{\frac{1}{2p},p} = B_{q,p}^{2m-\frac{jm}{2}-\frac{m}{2p}}(G)$ and $(H_q^{2m-\frac{m_k m}{2}}(G), H_q^{2m-\frac{m_k m}{2}}(G))_{\frac{1}{2p},p} = B_{q,p}^{2m-\frac{m_k m}{2}}(G)$. Therefore, the operators T_{kjs} from $B_{q,p}^{2m-\frac{jm}{2}-\frac{m}{2p}}(G)$ into $B_{q,p}^{2m-\frac{m_k m}{2}}(G)$ are bounded. Taking into account that the embedding $B_{q,p}^{2m-\frac{m_k m}{2}}(G) \subset B_{q,p}^{2m-\frac{m_k m}{2}-\frac{m}{2p}}(G)$ is compact (see [11, Theorem 4.10.2]), we get that the operators T_{kjs} from $B_{q,p}^{2m-\frac{jm}{2}-\frac{m}{2p}}(G)$ into $B_{q,p}^{2m-\frac{m_k m}{2}-\frac{m}{2p}}(G)$ are compact, i.e., condition (3) of Theorem 7 is satisfied.

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