GRONWALL LEMMAS: TEN OPEN PROBLEMS

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Abstract. The aim of this paper is to present ten research problems of the theory of Gronwall lemmas. These problems are in relation with: concrete Gronwall lemmas, abstract Gronwall lemmas, techniques of proof in concrete Gronwall lemmas, Picard and weakly Picard operator theory, operatorial inequations, differential inequalities, integral inequalities and applications of Gronwall lemmas.

1. Introduction

For to present our problems we need some standard notations of Nonlinear Analysis.

Let X be a nonempty set and $A : X \to X$ an operator. We denote:

$P(X) := \{Y \subset X \mid Y \neq \emptyset\}$;

$A^0 := 1_X$, $A^1 := A$, $A^{n+1} := A \circ A^n$, $n \in \mathbb{N}$ - the iterate operators of the operator A;

$F_A := \{x \in X \mid A(x) = x\}$ - the fixed point set of the operator A.

If $(X, \leq)$ is an ordered set then:

$(LF)_A := \{x \in X \mid x \leq A(x)\}$ - the lower fixed point set of A;

$(UF)_A := \{x \in X \mid x \geq A(x)\}$ - the upper fixed point set of A.

The symbol, $F_A = \{x_A^*\}$, has the following meaning: the operator A has a unique fixed point and we denote this unique fixed point by x_A^*.

In general, throughout this paper we follow the notation and terminology in I.A. Rus [19] and [20].

2. Problem 1

Let $(X, \leq)$ be an ordered set and $A : X \to X$ be an operator.

Problem 1. If $F_A = \{x_A^*\}$, in which conditions we have that:

(a) $x \in X$, $x \leq A(x) \Rightarrow x \leq x_A^*$?

(b) $x \in X$, $x \geq A(x) \Rightarrow x \geq x_A^*$?

For to present a result for this problem we need the following notion.

Definition 2.1. (I.A. Rus [19]). Let $(X, \to)$ be an L-space. By definition an operator $A : X \to X$ is a Picard operator if $F_A = \{x_A^*\}$ and $A^n(x) \to x_A^*$ as $n \to \infty$, for all $x \in X$.

We have

Lemma 2.1. (Abstract Gronwall Lemma ([19]; see also [17] and [20])). *Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ an operator. We suppose that:*

(i) A is a Picard operator ($F_A = \{x_A^*\}$);

(ii) A is an increasing operator.

Then we have:

(a) $x \in X$, $x \leq A(x) \Rightarrow x \leq x_A^*$;

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(b) $x \in X$, $x \geq A(x) \Rightarrow x \geq x_A^*$.

Remark 2.1. It is clear that in the conditions of Lemma 2.1 we have that

$$(LF)_A \leq x_A^* \leq (UF)_A.$$

Remark 2.2. The Abstract Gronwall Lemma has as particular cases some abstract Gronwall lemmas given by: K. Valeev, L. Losonczi, V.Ya. Stetsenko - M. Shaaban, V. Lakshmikantham - S. Leela - A.A. Martynyuk, I.A. Rus, M. Zima,... (see I.A. Rus [19] and [20], H. Amann [1], Sz. András [3], D. Bainov and P. Simeonov [4], A. Buică [5], H. Heikkilä and V. Lakshmikantham [10], V. Lakshmikantham and S. Leela [11], V. Lakshmikantham, S. Leela and A.A. Martynyuk [12], D.S. Mitrinović, J.E. Pečarić and A.M. Fink [14], B.G. Pachpatte [16], J. Schröder [23] and P. Ver Eecke [25]).

Remark 2.3. For a history of concrete Gronwall lemmas see T.M. Flett [8] and P. Ver Eecke [25].

3. Problem 2

Let $(X, \leq)$ be an ordered set and $A : X \to X$ be an operator.

Problem 2. If $F_A \neq \emptyset$, in which conditions there exists $\varphi : X \to F_A$ such that

(a) $x \leq A(x) \Rightarrow x \leq \varphi(x)$?

(b) $x \geq A(x) \Rightarrow x \geq \varphi(x)$?

For this problem we have the following partial result.

Lemma 3.1. (I.A. Rus [19]; see also [20]). *Let $(X, \leq, \to)$ be an ordered L-space and $A : X \to X$ be an operator. We suppose that:*

(i) A is an weakly Picard operator, i.e., $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.

(ii) A is an increasing operator.

Then we have that:

(a) $x \leq A(x) \Rightarrow x \leq A^\infty(x) := \lim\limits_{n\to\infty} A^n(x)$*;*

(b) $x \geq A(x) \Rightarrow x \geq A^\infty(x)$.

For other results for Problem 2 see I.A. Rus [20].

Remark 3.1. If $A : (X, \to) \to (X, \to)$ is a weakly Picard operator then we consider the operator A^∞ defined by $A^\infty(x) := \lim\limits_{n\to\infty} A^n(x)$. For some properties of A^∞ see [19].

4. Problem 3

Let $(X, \leq)$ be an ordered set and $A, B : X \to X$ two operators.

Problem 3. We suppose that $F_A = \{x_A^*\}$, $F_B = \{x_B^*\}$ and $A \leq B$. In which conditions we have that

$$x \leq A(x) \Rightarrow x \leq x_B^*?$$

Lemma 4.1. (Abstract Gronwall-comparison Lemma (I.A. Rus [20])). *Let $(X, \to, \leq)$ be an ordered L-space and $A, B : X \to X$ two operators. We suppose that:*

(i) A and B are Picard operators;

(ii) A is an increasing operator;

(iii) $A \leq B$.

Then:

$$x \leq A(x) \Rightarrow x \leq x_A^* \leq x_B^*.$$

Remark 4.1. If in addition the operator B is increasing, then instead of condition (iii) we can put the condition

(iii') $x = A(x) \Rightarrow x \leq B(x)$.

Remark 4.2. In order to use the Abstract Gronwall Lemma for to have a concrete Gronwall lemma we need to "determine" the fixed point x_A^*. If this is a difficult problem we choose an operator B, as in Lemma 4.1, for to have an upper bound for the solution x of the inequation, $x \leq A(x)$.

Remark 4.3. For more considerations on the Problem 3 see [20].

5. Problem 4

Let $(X, \leq)$ be an ordered set and $A, B : X \to X$ be two operators.

Problem 4. we suppose that $F_A \neq \emptyset$, $F_B \neq \emptyset$ and $A \leq B$. In which conditions there exists $\psi : X \to F_B$ such that

$$x \leq A(x) \Rightarrow x \leq \psi(x)?$$

For this problem we have:

Lemma 5.1. ([20]). *Let $(X, \to, \leq)$ be an ordered L-space and $A, B : X \to X$ two operators. We suppose that:*

(i) A and B are weakly Picard operators;

(ii) A is an increasing operator;

(iii) $A \leq B$.

Then:

(a) $x \leq y \Rightarrow A^\infty(x) \leq B^\infty(y)$;

(b) $x \leq A(x) \Rightarrow x \leq B^\infty(y)$.

For other results on the Problem 4 see [20], [4], [12], [14] and [25].

6. Problem 5

The abstract Gronwall lemmas are very usefully for to give some concrete Gronwall lemmas.

Example 6.1. Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_+$ and $\psi \in C([a, b], \mathbb{R}_+)$ be given. Then if $x \in C[a, b]$ is a solution of the inequation

$$x(t) \leq \alpha + \beta \int_a^b \psi(s) x(s) ds, \ \forall\, t \in [a, b],$$

then

$$x(t) \leq \alpha \exp\left(\beta \int_a^t \psi(s) ds\right), \ \forall\, t \in [a, b].$$

Proof. If $\alpha \geq 0$ then there is a direct proof for this well known Gronwall lemma. In what follow we shall prove that it follows from Lemma 2.1.

Let $(X, \to, \leq) := (C[a, b], \xrightarrow{\|\cdot\|_\tau}, \leq)$, where $\|\cdot\|_\tau$ is the Bielecki norm on $C[a, b]$, i.e., τ is a positive real number and

$$\|x\|_\tau := \max_{a \leq t \leq b} (|x(t)| \exp(-\tau(t - a))).$$

We consider on $X := C[a, b]$ the operator $A : X \to X$ defined by

$$A(x)(t) := \alpha + \beta \int_a^t \psi(s) x(s) ds, \quad t \in [a, b].$$

We remark that $F_A = \{x_A^*\}$, where

$$x_A^*(t) = \alpha \exp\left(\beta \int_a^t \psi(s) ds\right), \quad t \in [a, b],$$

and A is a contraction with respect to $\|\cdot\|_\tau$, with τ suitable chosen. On the other hand ($a < b$!), $x \leq y$, $x, y \in C[a, b]$ imply that $A(x) \leq A(y)$, i.e., A is an increasing operator. The proof follow from Lemma 2.1.

Example 6.2. ([16], p. 2 or [12], p. 6 or [6], p. 146). Let $\varphi, \psi \in C([a,b], \mathbb{R}_+)$ be two functions. We suppose that φ is increasing. If $x \in C([a,b], \mathbb{R}_+)$ is a solution of the inequation

$$x(t) \leq \varphi(t) + \int_a^b \psi(s)x(s)ds, \quad t \in [a,b],$$

then

$$x(t) \leq \varphi(t) \exp\left(\int_a^t \psi(s)ds\right), \quad t \in [a,b].$$

In this case the corresponding operator, $A : (C[a,b], \xrightarrow{\|\cdot\|_\tau}, \leq) \to (C[a,b], \xrightarrow{\|\cdot\|_\tau}, \leq)$ is defined by

$$A(x)(t) := \varphi(t) + \int_a^t \psi(s)x(s)ds, \quad t \in [a,b]$$

and is an increasing Picard operator, but the function

$$t \mapsto \varphi(t) \exp\left(\int_a^t \psi(s)ds\right)$$

is not a fixed point of the operator A. So, this result is not a consequence of Lemma 2.1.

In fact from Lemma 2.1 we have a better result, namely,

$$x(t) \leq \varphi(t) + \int_a^t \left(\varphi(s)\psi(s) \exp\left(\int_s^t \psi(\xi)d\xi\right)\right) ds.$$

By the above considerations we have

Problem 5. In which Gronwall lemmas the upper bounds are fixed points of the corresponding operator A?

7. Problem 6

Problem 6. If the answer in Problem 5 is yes, which of them are a consequence of some abstract Gronwall lemmas?

Example 7.1. Specific Gronwall lemma ([22], p. 5 or [6], p. 147). Let $x \in C([a,b], \mathbb{R}_+)$ be such that

$$x(t) \leq \delta_2(t-a) + \delta_1 \int_a^t x(s)ds + \delta_3, \ \forall \ t \in [a,b]$$

where $\delta_1 > 0$, $\delta_2 \geq 0$ and $\delta_3 \geq 0$ are real numbers. Then

$$x(t) \leq \left(\frac{\delta_2}{\delta_1} + \delta_3\right) \exp(\delta_1(t-a)) - \frac{\delta_2}{\delta_1}, \ \forall \ t \in [a,b].$$

In this case we have:

(a) $A(x)(t) := \delta_2(t-a) + \delta_1 \int_a^t x(s)ds + \delta_3$, $t \in [a,b]$;

(b) $F_A = \{x_A^*\}$, where

$$x_A^*(t) = \left(\frac{\delta_2}{\delta_1} + \delta_3\right) \exp(\delta_1(t-a)) - \frac{\delta_2}{\delta_1};$$

(c) the operator

$$A : (C[a,b], \xrightarrow{\|\cdot\|_\tau}, \leq) \to (C[a,b], \xrightarrow{\|\cdot\|_\tau}, \leq)$$

is an increasing Picard operator, for τ suitable chosen.

So, the specific Gronwall lemma is a consequence of Lemma 2.1.

More general we have ([7], [9], [10], [16], [17],...):

Theorem 7.1. *Let $x \in C([a,b])$ be such that*

$$x(t) \le g(t) + \int_a^t K(t,s,x(s))ds, \quad t \in [a,b]$$

where:

(i) $g \in C[a,b]$, $K \in C([a,b]\times[a,b]\times\mathbb{R})$;

(ii) there exists $L_K > 0$ such that

$$|K(t,s,u) - K(t,s,v)| \le L_K|u-v|, \ \forall\ t,s \in [a,b], \ \forall\ u,v \in \mathbb{R};$$

(iii) $K(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ is increasing for all $t,s \in [a,b]$.

Then we have that:

(a) the equation

$$x(t) = g(t) + \int_a^t K(t,s,x(s))ds, \quad t \in [a,b]$$

has in $C[a,b]$ a unique solution, x^*;

(b) $x(t) \le x^*(t)$, $\forall\ t \in [a,b]$.

Theorem 7.2. *Let $x \in C[a,b]$ be such that*

$$x(t) \le g(t) + \int_a^b K(t,s,x(s))ds, \quad t \in [a,b],$$

where g and K are as in Theorem 7.1 and in addition we suppose that:

(iv) $L_K(b-a) < 1$.

Then we have that:

(a) the equation

$$x(t) = g(t) + \int_a^t K(t,s,x(s))ds, \quad t \in [a,b]$$

has in $C[a,b]$ a unique solution, x^*;

(b) $x(t) \le x^*(t)$, $\forall\ t \in [a,b]$.

8. Problem 7

Problem 7. If the answer in Problem 6 is negative, in which of them the corresponding operator A is such that

$$F_A = F_{A^n} = \{x^*\},$$

i.e., A is a Bessaga operator?

If yes, the problem is to find " $\to$ " and " $\le$ " on X such that

$$A : (X, \to, \le) \to (X, \to, \le)$$

is an increasing Picard operator.

In this case we have that

$$x \le A(x) \Rightarrow x \le x^*$$

and the upper bound x^* is the best upper bound for the set $(LF)_A$ of the lower fixed point set of A.

9. Problem 8

Let $(X, \leq)$ be an ordered set and $A : X \to X$ an operator with $F_A = \{x^*\}$. In the case in which we have some difficulties to determine x^* the following problem arises.

Problem 8. Find some upper fixed points of the operator A, or some upper bounds for x^*.

Example 9.1. ([18]). We have the following result

Theorem 9.1. *Let $F : [a,b] \times C([a,b], \mathbb{R}_+) \to \mathbb{R}_+$ be a functional. We suppose that:*

(i) $F(\cdot, x) \in C^1[a,b]$, for all $x \in C([a,b], \mathbb{R}_+)$;

(ii) there exists $\phi \in C([a,b] \times \mathbb{R}_+)$ such that

$$(F(\cdot,x))'(t)\Big|_{x=x(t)} = \phi(t, x(t)), \quad t \in [a,b];$$

(iii) $\phi(t, \cdot)$ is increasing for all $t \in [a,b]$;

(iv) $(F(\cdot,x))'(t) \leq \phi(t, x(t))$ for all $t \in [a,b]$;

(v) there exists $\alpha \in \mathbb{R}_+$ such that

$$F(a, x) = \alpha, \text{ for all } x \in C([a,b], \mathbb{R}_+).$$

If $x \in C([a,b], \mathbb{R}_+)$ is a solution of the inequation

$$x(t) \leq F(t, x), \ \forall\, t \in [a,b],$$

and $y \in C^1[a,b]$ is the maximal solution of the Cauchy problem

$$y'(t) = \phi(t, y(t)), \quad t \in [a,b],$$
$$y(a) = \alpha,$$

then $x \leq y$.

Remark 9.1. For the differential inequalities and for the relations between differential inequalities and integral inequalities see J. Szarski [24], W. Walter [26], H. Amann [1], D. Bainov and P. Simeobov [4], V. Lakshmikantham and S. Leela [11], N. Lungu [13], A. Buică [5], D.S. Mitrinović, J.E. Pečarić and A.M. Fink [14] and the references therein.

10. The last two problems

With the above considerations in mind look to the following problems:

Problem 9. Give new concrete and abstract Gronwall lemmas.

Problem 10. Give new applications of the Gronwall lemmas.

In what follow we shall give an application of Gronwall lemma in Example 6.2, to study the Hyers-Ulam-Rassias stability of an integral equation.

Let $(\mathbb{B}, |\cdot|)$ be a real or complex Banach space, $K : [a,b] \times [a,b] \times \mathbb{B} \to \mathbb{B}$ and $g : [a,b] \to \mathbb{B}$. We consider the following nonlinear Volterra integral equation

$$x(t) = g(t) + \int_a^t K(t, s, x(s))ds, \quad t \in [a,b]. \tag{10.1}$$

We have the following stability result of Hyers-Ulam-Rassias type ([21]) for the equation (10.1).

Theorem 10.1. *We suppose that:*

(i) $K \in C([a,b] \times [a,b] \times \mathbb{B}, \mathbb{B})$, $g \in C([a,b], \mathbb{B})$ and $\varphi \in C([a,b], \mathbb{R}_+)$;

(ii) there exists $L_K > 0$ such that

$$|K(t,s,u) - K(t,s,v)| \leq L_K |u - v|$$

for all $t, s \in [a,b]$ and $u, v \in \mathbb{B}$;

(iii) φ is an increasing function.

Then:

(a) The equation (10.1) has in $C([a,b],\mathbb{B})$ a unique solution, x^*;
(b) If $x \in C([a,b],\mathbb{B})$ is such that

$$\left|x(t) - g(t) - \int_a^t K(t,s,x(s))ds\right| \leq \varphi(t),\ \forall\ t \in [a,b],$$

then

$$|x(t) - x^*(t)| \leq e^{L_K(b-a)}\varphi(t),\ \forall\ t \in [a,b],$$

Proof. (a) It is a well known result ([7], [9], [17],...).
(b) We have

$$\begin{aligned}|x(t) - x^*(t)| &\leq \left|x(t) - g(t) - \int_a^t K(t,s,x(s))ds\right| \\ &+ \int_a^t |K(t,s,x(s)) - K(t,s,x^*(s))|ds \\ &\leq \varphi(t) + L_K \int_a^t |x(s) - x^*(s)|ds.\end{aligned}$$

From the Gronwall lemma in Example 6.2 it follows that

$$|x(t) - x^*(t)| \leq \varphi(t)e^{L_K(b-a)},\ \forall\ t \in [a,b].$$

Remark 10.1. For some applications of the Gronwall lemmas see [2]-[7], [9]-[26] and the references therein.

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