KIMMERLE CONJECTURE FOR THE HELD AND O'NAN SPORADIC SIMPLE GROUPS

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Abstract. Using the Luthar–Passi method, we investigate the Zassenhaus and Kimmerle conjectures for normalized unit groups of integral group rings of the Held and O'Nan sporadic simple groups. We confirm the Kimmerle conjecture for the Held simple group and also derive for both groups some extra information relevant to the classical Zassenhaus conjecture.

Let $U(\mathbb{Z}G)$ be the unit group of the integral group ring $\mathbb{Z}G$ of a finite group G. It is well known that $U(\mathbb{Z}G) = U(\mathbb{Z}) \times V(\mathbb{Z}G)$, where

$$V(\mathbb{Z}G) = \Big\{ \sum_{g \in G} \alpha_g g \in U(\mathbb{Z}G) \;\mid\; \sum_{g \in G} \alpha_g = 1,\ \alpha_g \in \mathbb{Z} \Big\}.$$

is the normalized unit group of $U(\mathbb{Z}G)$. Throughout the paper (unless stated otherwise) the unit is always normalized and not equal to the identity element of G.

One of most interesting conjectures in the theory of integral group ring is the conjecture **(ZC)** of H. Zassenhaus [27], saying that every normalized torsion unit $u \in V(\mathbb{Z}G)$ is conjugate to an element in G within the rational group algebra $\mathbb{Q}G$.

For finite simple groups, the main tool of the investigation of the Zassenhaus conjecture **(ZC)** is the Luthar–Passi method, introduced in [21] to solve this conjecture for the alternating group A_5. Later in [18] M. Hertweck applied it for some other groups using p-Brauer characters, and then extended the previous result by M. Salim [25] to confirm **(ZC)** for the alternating group A_6 in [17] (note that for larger alternating groups the problem is still open). The method also proved to be useful for groups containing non-trivial normal subgroups as well (see related results in [1, 15, 16, 18, 20, 22]).

One of the variations of **(ZC)** was formulated by W. Kimmerle in [20]. Denote by $\#(G)$ the set of all primes dividing the order of G. The Gruenberg–Kegel graph (or the prime graph) of G is the graph $\pi(G)$ with vertices labelled by the primes in $\#(G)$ and there is an edge from p to q if and only if there is an element of order pq in the group G. Then W. Kimmerle asked the following:

Conjecture (KC): Is it true that $\pi(G) = \pi(V(\mathbb{Z}G))$ for any finite group G?

It is easy to see that the Zassenhaus conjecture **(ZC)** implies the Kimmerle conjecture **(KC)**. In [20] W. Kimmerle confirmed **(KC)** for finite Frobenius and solvable groups. Recently **(KC)** was confirmed for some simple groups (see [17, 18]), including 12 of 26 sporadic simple groups (see [2, 3, 4, 5, 6, 7, 8, 10]).

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In the present paper we confirm **(KC)** for the Held sporadic simple group `He` [14, 26], using the Luthar–Passi method as the main tool. We also study the same problem for the O'Nan sporadic simple group `ON` [24], and prove the non-existence of torsion units of all orders relevant to **(KC)** except orders 33 and 57. Additionally, we derive certain information about possible torsion units in $V(\mathbb{Z}[\mathtt{He}])$ and $V(\mathbb{Z}[\mathtt{ON}])$ and their partial augmentations, which will be useful for further investigation of **(ZC)** for these groups. The development version of the GAP package LAGUNA [9] was very helpful to speed up computational work and derive purely theoretical arguments for the proof.

First we introduce some notation. Let G be a group. Let $\mathcal{C} = \{C_1, \dots, C_{nt}, \dots\}$ be the collection of all conjugacy classes of G, where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. Suppose $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ be a non-trivial unit of finite order k. Denote by

$$\nu_{nt} = \nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g,$$

the partial augmentation of u with respect to the conjugacy class C_{nt}. From the Berman–Higman Theorem (see [1]) one knows that $\nu_1 = \alpha_1 = 0$, so the sum of remaining partial augmentations for non-trivial conjugacy classes is equal to one:

$$\sum_{\substack{C_{nt} \in \mathcal{C} \\ C_{nt} \neq C_1}} \nu_{nt} = 1. \tag{1}$$

Clearly, for any character χ of G, we get that $\chi(u) = \sum \nu_{nt}\chi(h_{nt})$, where h_{nt} is a representative of a conjugacy class C_{nt}.

The main results are the following.

Theorem 1. *Let G be the Held sporadic simple group* `He`. *Let $\mathfrak{P}(u)$ be the tuple of partial augmentations of a torsion unit $u \in V(\mathbb{Z}G)$ of order $|u|$, corresponding to all non-central conjugacy classes of the group G, that is*

$$\begin{aligned}\mathfrak{P}(u) = (&\nu_{2a},\ \nu_{2b}, \nu_{3a},\ \nu_{3b},\ \nu_{4a},\ \nu_{4b},\ \nu_{4c},\ \nu_{5a},\ \nu_{6a},\ \nu_{6b},\ \nu_{7a},\ \nu_{7b},\ \nu_{7c},\\ &\nu_{7d}, \nu_{7e},\ \nu_{8a},\ \nu_{10a},\ \nu_{12a},\ \nu_{12b},\ \nu_{14a},\ \nu_{14b},\ \nu_{14c},\ \nu_{14d},\\ &\nu_{15a},\ \nu_{17a},\ \nu_{17b},\ \ \nu_{21a},\ \nu_{21b},\ \nu_{21c},\ \nu_{21d},\ \nu_{28a},\ \nu_{28b}) \in \mathbb{Z}^{32}.\end{aligned}$$

The following properties hold.

- (i) *There is no elements of orders* 34, 35, 51, 85 *and* 119 *in $V(\mathbb{Z}G)$. Equivalently, if $|u| \notin \{20, 24, 30, 40, 42, 56, 60, 84, 120, 168\}$, then $|u|$ coincides with the order of some $g \in G$.*
- (ii) *If $|u| = 2$, the tuple of the partial augmentations of u belongs to the set*
$$\big\{\ \mathfrak{P}(u)\ \mid\ -6 \le \nu_{2a} \le 6,\ \nu_{2a} + \nu_{2b} = 1, \quad \nu_{kx} = 0,\ kx \notin \{2a, 2b\}\ \big\}.$$
- (iii) *If $|u| = 5$, then u is rationally conjugate to some $g \in G$.*
- (iv) *If $|u| = 3$, the tuple of the partial augmentations of u belongs to the set*
$$\big\{\ \mathfrak{P}(u)\ \mid\ -4 \le \nu_{3a} \le 5,\ \nu_{3a} + \nu_{3b} = 1, \quad \nu_{kx} = 0,\ kx \notin \{3a, 3b\}\ \big\}.$$
- (v) *If $|u| = 17$, the tuple of the partial augmentations of u belongs to the set*
$$\big\{\ \mathfrak{P}(u)\ \mid\ -14 \le \nu_{17a} \le 15,\ \nu_{17a} + \nu_{17b} = 1,\ \nu_{kx} = 0,\ kx \notin \{17a, 17b\}\ \big\}.$$

Corollary. *If G is the Held sporadic simple group, then $\pi(G)=\pi(V(\mathbb{Z}G))$.*

Theorem 2. *Let G be the O'Nan sporadic simple group* `ON`*. Let $\mathfrak{P}(u)$ be the tuple of partial augmentations of a torsion unit $u\in V(\mathbb{Z}G)$ of order $|u|$, corresponding to all non-central conjugacy classes of the group G, that is*

$$\begin{aligned}\mathfrak{P}(u)=(&\nu_{2a},\nu_{3a},\ \nu_{4a},\ \nu_{4b},\ \nu_{5a},\ \nu_{6a},\ \nu_{7a},\ \nu_{7b},\ \nu_{8a},\ \nu_{8b},\ \nu_{10a},\\ &\nu_{11a},\ \nu_{12a},\ \nu_{14a},\ \nu_{15a},\ \nu_{15b},\ \nu_{16a},\ \nu_{16b},\ \nu_{16c},\ \nu_{16d},\\ &\nu_{19a},\ \nu_{19b},\ \nu_{19c},\ \nu_{20a},\ \nu_{20b},\ \nu_{28a},\ \nu_{28b},\ \nu_{31a},\ \nu_{31b})\in\mathbb{Z}^{29}.\end{aligned}$$

The following properties hold.

(i) *There is no elements of orders* 21, 22, 35, 38, 55, 62, 77, 93, 95, 133, 155, 209, 217, 341 *and* 589 *in $V(\mathbb{Z}G)$. Equivalently, if*

$$|u|\not\in\{24,30,33,40,48,56,57,60,80,112,120,240\},$$

then $|u|$ coincides with the order of some $g\in G$.

(ii) *If $|u|\in\{2,3,5,11\}$, then u is rationally conjugate to some $g\in G$.*

(iii) *If $|u|=7$, the tuple of the partial augmentations of u belongs to the set*

$$\{\ \mathfrak{P}(u)\ \mid\ -3\le\nu_{7a}\le 22,\ \nu_{7a}+\nu_{7b}=1,\quad \nu_{kx}=0,\ kx\not\in\{7a,7b\}\ \}.$$

(iv) *If $|u|=31$, the tuple of the partial augmentations of u belongs to the set*

$$\{\ \mathfrak{P}(u)\ \mid\ -39\le\nu_{31a}\le 40,\ \nu_{31a}+\nu_{31b}=1,\quad \nu_{kx}=0,\ kx\not\in\{31a,31b\}\ \}.$$

(v) *If $|u|=33$, the tuple of the partial augmentations of u belongs to the set*

$$\{\ \mathfrak{P}(u)\ \mid\ (\nu_{3a},\nu_{11a})=(12,-11),\quad \nu_{kx}=0,\ kx\not\in\{3a,11a\}\ \}.$$

(vi) *If $|u|=57$, then tuple of the partial augmentations of u belongs to the set*

$$\begin{aligned}\{\ \mathfrak{P}(u)\ \mid\ \nu_{3a}=-18,\quad &\nu_{19a}+\nu_{19b}+\nu_{19c}=19,\\ &\nu_{kx}=0,\quad kx\not\in\{3a,19a,19b,19c\}\ \}.\end{aligned}$$

For the proof we will need the following results. The first one relates the solution of the Zassenhaus conjecture to vanishing of partial augmentations of torsion units.

Proposition 1 (see [21] and Theorem 2.5 in [23])**.** *Let $u\in V(\mathbb{Z}G)$ be of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g\in G$ if and only if for each d dividing k there is precisely one conjugacy class C with partial augmentation $\varepsilon_C(u^d)\neq 0$.*

The next result yields that several partial augmentations are zero.

Proposition 2 (see [15], Proposition 3.1; [18], Proposition 2.2)**.** *Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$. If $x\in G$ and its p-part, for some prime p, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u)=0$.*

The main restriction on the partial augmentations is given by the following result.

Proposition 3 (see [18, 21])**.** *Let either $p=0$ or p is a prime divisor of $|G|$. Suppose that $u\in V(\mathbb{Z}G)$ has finite order k and assume that k and p are coprime when $p\neq 0$. If z is a complex primitive k-th root of unity and χ is either a classical character or a p-Brauer character of G then, for every integer l, the number*

$$\mu_l(u,\chi,p)=\tfrac{1}{k}\textstyle\sum_{d|k}Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi(u^d)z^{-dl}\}\tag{2}$$

is an integer such that $0\le\mu_l(u,\chi,p)\le deg(\chi)$.

Note that if $p = 0$, we will use the notation $\mu_l(u, \chi, *)$ for $\mu_l(u, \chi, 0)$.

When p and q are two primes such that G contains no element of order pq, and u is a normalized torsion unit of order pq, Proposition 3 may be reformulated for as follows. Let ν_k be the sum of partial augmentations of u with respect all conjugacy classes of elements of order k in G, i.e. $\nu_2 = \nu_{2a} + \nu_{2b}$, etc. Then by (1) and Proposition 2 we obtain that $\nu_p + \nu_q = 1$ and $\nu_k = 0$ for $k \notin \{p, q\}$. For each character χ of G (an ordinary character or a Brauer character in characteristic not dividing pq) that is constant on all elements of orders p and on all elements of order q, we have $\chi(u) = \nu_p \chi(C_p) + \nu_q \chi(C_q)$, where $\chi(C_t)$ denote the value of the character χ on any element of order t from G.

From the Proposition 3 we obtain that the values

$$\begin{aligned} \mu_l(u, \chi) = \tfrac{1}{pq}\big(\ \chi(1) &+ Tr_{\mathbb{Q}(z^p)/\mathbb{Q}}\{\chi(u^p) z^{-pl}\} \\ &+ Tr_{\mathbb{Q}(z^q)/\mathbb{Q}}\{\chi(u^q) z^{-ql}\} + Tr_{\mathbb{Q}(z)/\mathbb{Q}}\{\chi(u) z^{-l}\} \ \big) \end{aligned} \tag{3}$$

are nonnegative integers. It follows that if χ has the specified property, then

$$\mu_l(u, \chi) = \tfrac{1}{pq} \left(m_1 + \nu_p m_p + \nu_q m_q \right), \tag{4}$$

where

$$\begin{aligned} m_1 &= \chi(1) + \chi(C_q) Tr_{\mathbb{Q}(z^p)/\mathbb{Q}}(z^{-pl}) + \chi(C_p) Tr_{\mathbb{Q}(z^q)/\mathbb{Q}}(z^{-ql}), \\ m_p &= \chi(C_p) Tr_{\mathbb{Q}(z)/\mathbb{Q}}(z^{-l}), \qquad m_q = \chi(C_q) Tr_{\mathbb{Q}(z)/\mathbb{Q}}(z^{-l}). \end{aligned} \tag{5}$$

Finally, we shall use the well-known bound for orders of torsion units.

Proposition 4 (see [11])**.** *The order of a torsion element $u \in V(\mathbb{Z}G)$ is a divisor of the exponent of G.*

Proof of Theorem 1. Throughout the proof we denote by G the Held sporadic simple group `He`. It is well known [12, 13] that $|G| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$ and $exp(G) = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 17$. The character table of G, as well as the Brauer character tables for $p \in \{2, 3, 5, 7, 17\}$ can be found by the computational algebra system GAP [13], which derives its data from [12, 19]. Throughout the paper we will use the notation of GAP Character Table Library for the characters and conjugacy classes of the group `He`.

It is known that `He` possesses elements of orders 2, 3, 4, 5, 6, 7, 8, 10, 12, 14, 15, 17, 21 and 28. From this we can derive that to deal with **(KC)**, we need to show that $V(\mathbb{Z}G)$ has no units of orders 34, 35, 51, 85 and 119. Therefore, there are also no units of orders divisible by any number from this list. Since by Proposition 4, the order of each torsion unit divides the exponent of G, the only remaining opportunities for the order are 20, 24, 30, 40, 42, 56, 60, 84, 120 and 168, explaining formulation of part (i) of Theorem 1.

We will begin with orders that do not appear in G, and will give a detailed proof for the order 35. The proof for the other cases can be derived similarly from the table below, which contains the data describing the constraints on partial augmentations ν_p and ν_q for possible orders pq (including the order 35 as well) accordingly to (3)–(5).

If u is a unit of order 35, then $\nu_5 + \nu_7 = 1$. Consider 2-Brauer characters $\xi = \chi_1 + \chi_2 + \chi_3 + \chi_6 + \chi_8$ and $\tau = \chi_6 + \chi_7 + \chi_8 + \chi_9$, which are encoded in the table as $\xi = (1, 2, 3, 6, 8)_{[2]}$ and $\tau = (6, 7, 8, 9)_{[2]}$ respectively. These characters are constant of elements of order 5 and elements of order 7: $\xi(C_5) = 4$, $\xi(C_7) = 0$, $\tau(C_5) = -8$ and $\tau(C_7) = 5$. Now we obtain the system of inequalities

$$\mu_0(u, \xi, 2) = \tfrac{1}{35}(96\nu_5 + 1045) \geq 0; \quad \mu_7(u, \xi, 2) = \tfrac{1}{35}(-24\nu_5 + 1025) \geq 0;$$
$$\mu_0(u, \tau, 2) = \tfrac{1}{35}(-192\nu_5 + 120\nu_7 + 3090) \geq 0,$$

which has no nonnegative integral solution (ν_5, ν_7) such that all $\mu_i(u, \chi_j, 2)$ are non-negative integers.

The data for orders 34, 51, 85 and 119 are given in the following table.

$\|u\|$	p	q	$\xi,\ \tau$	$\xi(C_p)$	$\xi(C_q)$	l	m_1	m_p	m_q
34	2	17	$\xi = (7,8,9,12)_{[3]}$	69	0	0	5322	1104	0
						2	5322	-69	0
						17	5184	-1104	0
35	5	7	$\xi = (1,2,3,6,8)_{[2]}$	4	0	0	1045	96	0
			$\xi = (1,2,3,6,8)_{[2]}$	4	0	7	1025	-24	0
			$\tau = (6,7,8,9)_{[2]}$	-8	5	0	3090	-192	120
51	3	17	$\xi = (2,4,5)_{[*]}$	6	0	0	369	192	0
			$\xi = (2,4,5)_{[*]}$	6	0	17	351	-96	0
			$\tau = (24,28,33)_{[*]}$	-7	0	1	5299	-7	0
85	5	17	$\xi = (2)_{[*]}$	1	0	0	55	64	0
						17	50	-16	0
119	7	17	$\xi = (1,9,15)_{[*]}$	2	0	0	7560	192	0
			$\tau = (2,3,14)_{[*]}$	-5	0	0	4424	-480	0

Therefore, part (i) of Theorem 1 is proved. Now it remains to consider elements of orders covered by parts (ii)–(v) of Theorem 1.

• Let $|u| = 5$. Since there is only one conjugacy class in G consisting of elements or order 5, the Proposition 2 yields immediately that for units of order 5 that there is precisely one conjugacy class with non-zero partial augmentation and by Proposition 1 part (iii) of Theorem 1 is proved.

• Let u be an involution. By (1) and Proposition 2 we get $\nu_{2a} + \nu_{2b} = 1$. Applying Proposition 3 to the character χ_2 with $\chi_2(2a) = 11$, $\chi_2(2b) = 3$, we obtain

$$\mu_0(u, \chi_2, *) = \tfrac{1}{2}(11\nu_{2a} + 3\nu_{2b} + 51) \geq 0;$$
$$\mu_1(u, \chi_2, *) = \tfrac{1}{2}(-11\nu_{2a} - 3\nu_{2b} + 51) \geq 0.$$

From the requirement that all $\mu_i(u, \chi_j, *)$ must be non-negative integers it can be deduced that (ν_{2a}, ν_{2b}) satisfies the conditions of part (ii) of Theorem 1.

• Let u has order 3. By (1) and Proposition 2 we obtain that $\nu_{3a} + \nu_{3b} = 1$. Then using Proposition 3 for the character χ_2, we get the system

$$\mu_0(u, \chi_2, *) = 4\nu_{3a} + 17 \geq 0; \qquad \mu_1(u, \chi_2, *) = -2\nu_{3a} + 17 \geq 0;$$
$$\mu_0(u, \chi_4, 2) = \tfrac{1}{3}(-14\nu_{3a} + 4\nu_{3b} + 101) \geq 0,$$

that has only 10 integer solutions (ν_{3a}, ν_{3b}) listed in the part (iv) of Theorem 1, such that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

• Let u of order 17. By (1) and Proposition 2 we get $\nu_{17a} + \nu_{17b} = 1$. Applying (2) to the ordinary character χ_7 with $\chi_7(17a) = \frac{1-\sqrt{17}}{2}$ and $\chi_7(17b) = \frac{1+\sqrt{17}}{2}$, and to the 2-Brauer character χ_6 with $\chi_6(17a) = \frac{-1+\sqrt{17}}{2}$ and $\chi_6(17b) = \frac{-1-\sqrt{17}}{2}$, and putting $t = 9\nu_{17a} - 8\nu_{17b}$, we obtain the system of inequalities

$$\mu_1(u, \chi_7, *) = \tfrac{1}{17}(-t + 1029) \geq 0; \quad \mu_1(u, \chi_6, 2) = \tfrac{1}{17}(t + 246) \geq 0;$$
$$\mu_3(u, \chi_6, 2) = \tfrac{1}{17}(-8\nu_{17a} + 9\nu_{17b} + 246) \geq 0,$$

that has 30 integer solutions (ν_{17a}, ν_{17b}) listed in part (v) of Theorem 1, such that all $\mu_i(u, \chi_j, *)$ are non-negative integers. □

Proof of Theorem 2. Throughout the proof we denote by G the O'Nan sporadic simple group `ON` of order $|G| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$ and $exp(G) = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 31$. Besides ordinary character tables, for G also Brauer characters tables are known for $p \in \{2, 3, 5, 7, 11, 19, 31\}$ (see [12, 13, 19]). As before, we use the GAP notation for the characters and conjugacy classes of G.

It is known that G possesses elements of orders 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14, 15, 16, 19, 20, 28 and 31. The Kimmerle conjecture requires us to consider possible units of orders 21, 22, 33, 35, 38, 55, 57, 62, 77, 93, 95, 133, 155, 209, 217, 341 and 589. As in the proof of Theorem 1, below we give the table containing the data describing the constraints on partial augmentations ν_p and ν_q accordingly to (3)–(5) for all these orders, except order 22. From this table parts (i), (v) and (vi) of Theorem 2 are derived in the same way as in the proof of Theorem 1, except orders 22 and 35 which will be treated separately. Since we are not able to prove the non-existence of units of orders 33 and 57, and also do not consider orders with more than two prime factors, the condition $\exp(G) \equiv 0 \pmod{|u|}$ (see Proposition 4) results in listing orders 24, 30, 33, 40, 48, 56, 57, 60, 80, 112, 120 and 240 in the "exclusive" part (i) of Theorem 2.

$\|u\|$	p	q	$\xi,\ \tau$	$\xi(C_p)$	$\xi(C_q)$	l	m_1	m_p	m_q
						0	98493	312	0
21	3	7	$(1,3,9,10)_{[5]}$	26	0	1	98415	26	0
						7	98415	-156	0
			$\xi=(3)_{[7]}$	6	0	0	1233	120	0
33	3	11	$\xi=(3)_{[7]}$	6	0	11	1215	-60	0
			$\tau=(23)_{[7]}$	4	0	0	143382	80	0
35	5	7	$\xi=(1,3)_{[3]}$	-2	0	0	335	-48	0
						7	345	12	0
						0	70358	4806	0
38	2	19	$\xi=(2,7,8)_{[*]}$	267	0	1	69824	267	0
						19	69824	-4806	0
						0	415	80	0
55	5	11	$\xi=(1,2)_{[7]}$	2	0	5	415	-8	0
						11	405	-20	0
			$\xi=(1,2,8,9)_{[7]}$	18	0	0	21924	648	0
57	3	19	$\xi=(1,2,8,9)_{[7]}$	18	0	0	36369	-972	0
			$\tau=(23)_{[7]}$	4	0	1	143370	4	0
						0	150	-150	0
62	2	31	$\xi=(1,2)_{[3]}$	-5	0	2	150	5	0
						31	160	150	0
77	7	11	$\xi=(1,3)_{[3]}$	0	2	0	363	0	120
						11	363	0	-20
						0	26799	-1380	0
93	3	31	$\xi=(1,4,5)_{[2]}$	23	0	3	26799	-46	0
						31	26730	-690	0

$\vert u\vert$	p	q	$\xi,\ \tau$	$\xi(C_p)$	$\xi(C_q)$	l	m_1	m_p	m_q
95	5	19	$\xi=(1,2)_{[3]}$	0	3	0	209	0	216
						19	209	0	-54
133	7	19	$\xi=(1,3)_{[3]}$	0	1	0	361	0	108
						19	361	0	-18
155	5	31	$\xi=(2,3)_{[3]}$	-4	0	0	480	-480	0
						5	480	16	0
						31	500	120	0
209	11	19	$\xi=(2,5)_{[3]}$	0	3	0	703	0	540
						19	703	0	-54
217	7	31	$\xi=(1,3)_{[3]}$	0	2	0	403	0	360
						31	403	0	-60
341	11	31	$\xi=(1,2)_{[3]}$	1	0	0	165	300	0
						31	154	-30	0
589	19	31	$\xi=(1,2)_{[3]}$	3	0	0	209	1620	0
						31	152	-90	0

• Let u be a torsion unit of order 22. By (1) and Proposition 2 we have that $\nu_{2a}+\nu_{11a}=1$. Then using Proposition 3 for the ordinary character χ_2 and 3-Brauer character χ_2 and putting $t=64\nu_{2a}-\nu_{11a}$, we obtain the system

$$\mu_0(u,\chi_2,3)=\tfrac{1}{22}(-60\nu_{2a}+148)\geq 0; \qquad \mu_{11}(u,\chi_2,3)=\tfrac{1}{22}(60\nu_{2a}+160)\geq 0;$$
$$\mu_1(u,\chi_2,*)=\tfrac{1}{22}(t+10881)\geq 0; \qquad \mu_{11}(u,\chi_2,*)=\tfrac{5}{11}(-t+1087)\geq 0;$$
$$\mu_0(u,\chi_2,*)=\tfrac{1}{22}(10t+10998)\geq 0,$$

which has no integral solutions such that all $\mu_i(u,\chi_j,*)$ are non-negative integers.
• Let $|u|=35$. By (1) and Proposition 2, $\nu_{5a}+\nu_{7a}+\nu_{7b}=1$. Also, from the table above we have that $\nu_{5a}=-20$. This restriction greatly facilitates the next step, when using Proposition 3 for the ordinary character χ_2 with $\chi_2(7a)=17$, $\chi_2(7b)=3$ and putting $t=17\nu_{7a}+3\nu_{7b}$, we obtain two incompatible constraints

$$\mu_0(u,\chi_2,*)=\tfrac{1}{35}(24t+\alpha_1)\geq 0; \qquad \mu_7(u,\chi_2,*)=\tfrac{1}{35}(-6t+\alpha_2)\geq 0,$$

where the values of α_1 and α_2 parametrised by $\chi(u^5)=k_1\chi(7a)+k_2\chi(7b)$ are given in the following table.

(k_1,k_2)	(α_1,α_2)	(k_1,k_2)	(α_1,α_2)
(1,0)	(11522,10927)	(11,-10)	(12362,11767)
(0,1)	(11438,10843)	(10,-9)	(12278,11683)
(22,-21)	(13286, 12691)	(9,-8)	(12194,11599)
(21,-20)	(13202,12607)	(8,-7)	(12110,11515)
(20,-19)	(13118,12523)	(7,-6)	(12026,11431)
(19,-18)	(13034,12439)	(6,-5)	(11942,11347)
(18,-17)	(12950,12355)	(5,-4)	(11858,11263)
(17,-16)	(12866,12271)	(4,-3)	(11774,11179)
(16,-15)	(12782,12187)	(3,-2)	(11690,11095)
(15,-14)	(12698,12103)	(2,-1)	(11606,11011)
(14,-13)	(12614,12019)	(-1,2)	(11354,10759)
(13,-12)	(12530,11935)	(-2,3)	(11270,10675)
(12,-11)	(12446, 11851)	(-3,4)	(11186, 10591)

Now it remains to consider elements of orders that appear in the group G.
• Let $|u|\in\{2,3,5,11\}$. Since there is only one conjugacy class in G consisting of elements or each of these orders, the Proposition 2 yields immediately that for such units there is

precisely one conjugacy class with non-zero partial augmentation and by Proposition 1 part (ii) of Theorem 2 is proved.

• Let u has order 7. By (1) and Proposition 2 we obtain that $\nu_{7a}+\nu_{7b}=1$. Put $t=-17\nu_{7a}-3\nu_{7b}$. Then using Proposition 3 for the ordinary character χ_2 with values $\chi_2(7a)=17$, $\chi_2(7b)=3$ and 3-Brauer character χ_2 with $\chi_2(7a)=7$, $\chi_2(7b)=0$ we get the system of inequalities

$$\begin{aligned} \mu_0(u,\chi_2,*) &= \tfrac{1}{7}(-6t+10944)\geq 0; \qquad & \mu_1(u,\chi_2,*) &= \tfrac{1}{7}(t+10944)\geq 0;\\ \mu_0(u,\chi_2,3) &= \tfrac{1}{7}(42\nu_{7a}+154)\geq 0; \qquad & \mu_1(u,\chi_2,3) &= \tfrac{1}{7}(-7\nu_{7a}+154)\geq 0, \end{aligned}$$

that has 26 integer solutions (ν_{7a},ν_{7b}) listed in the part (iii) of Theorem 2, such that all $\mu_i(u,\chi_j,*)$ are non-negative integers.

• Let $|u|=31$. By (1) and Proposition 2 we obtain that $\nu_{31b}=1-\nu_{31a}$. Proposition 3 for the 7-Brauer character χ_3 with $\chi_3(31a)=\overline{\chi_3(31b)}=\frac{-7+\sqrt{-31}}{2}$ yields

$$\mu_1(u,\chi_3,7)=\tfrac{1}{31}(31\nu_{31a}+1209)\geq 0; \quad \mu_3(u,\chi_3,7)=\tfrac{1}{31}(-31\nu_{31a}+1240)\geq 0.$$

This system of inequalities has 80 integer solutions (ν_{31a},ν_{31b}) listed in the part (iv) of Theorem 2, such that all $\mu_i(u,\chi_j,*)$ are non-negative integers. □

References

[1] V. A. Artamonov and A. A. Bovdi. Integral group rings: groups of invertible elements and classical K-theory. In *Algebra. Topology. Geometry, Vol. 27 (Russian)*, Itogi Nauki i Tekhniki, pages 3–43, 232. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989. Translated in J. Soviet Math. **57** (1991), no. 2, 2931–2958.

[2] V. Bovdi, E. Jespers, and A. Konovalov. Torsion units in integral group rings of Janko simple groups. *Preprint*, pages 1–30, submitted, 2007. (E-print `arXiv:math/0608441v3`).

[3] V. Bovdi and A. Konovalov. Integral group ring of the first Mathieu simple group. In *Groups St. Andrews 2005. Vol. 1*, volume 339 of *London Math. Soc. Lecture Note Ser.*, pages 237–245. Cambridge Univ. Press, Cambridge, 2007.

[4] V. Bovdi and A. Konovalov. Integral group ring of the McLaughlin simple group. *Algebra Discrete Math.*, (2):43–53, 2007.

[5] V. Bovdi and A. Konovalov. Integral group ring of the Mathieu simple group M_{23}. *Comm. Algebra*, 36(7):2670–2680, 2008.

[6] V. Bovdi and A. Konovalov. Integral group ring of Rudvalis simple group. *Ukraïn. Mat. Zh.*, 61(1):3–13, 2009.

[7] V. Bovdi, A. Konovalov, and S. Linton. Torsion units in integral group ring of the Mathieu simple group M_{22}. *LMS J. Comput. Math.*, 11:28–39, 2008.

[8] V. Bovdi, A. Konovalov, and E.N. Marcos. Integral group ring of the Suzuki sporadic simple group. *Publ. Math. Debrecen*, 72(3-4):487–503, 2008.

[9] V. Bovdi, A. Konovalov, R. Rossmanith, and Cs. Schneider. *LAGUNA – Lie AlGebras and UNits of group Algebras, Version 3.4*, 2007. (`http://www.cs.st-andrews.ac.uk/~alexk/laguna.htm`).

[10] V. Bovdi, A. Konovalov, and S. Siciliano. Integral group ring of the Mathieu simple group M_{12}. *Rend. Circ. Mat. Palermo (2)*, 56(1):125–136, 2007.

[11] J.A. Cohn and D. Livingstone. On the structure of group algebras. I. *Canad. J. Math.*, 17:583–593, 1965.

[12] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of Finite Groups*. Oxford University Press, Eynsham, 1985. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

[13] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.4.10*, 2007. (`http://www.gap-system.org`).

[14] D. Held. The simple groups related to M_{24}. *J. Algebra*, 13:253–296, 1969.

[15] M. Hertweck. On the torsion units of some integral group rings. *Algebra Colloq.*, 13(2):329–348, 2006.

[16] M. Hertweck. Torsion units in integral group rings of certain metabelian groups. *Proc. Edinb. Math. Soc.*, 51(2):363–385, 2008.

[17] M. Hertweck. Zassenhaus conjecture for A_6. *Proc. Indian Acad. Sci. Math. Sci.*, 118(2):189–195, 2008.

[18] M. Hertweck. Partial augmentations and Brauer character values of torsion units in group rings. *Comm. Algebra*, pages 1–16, to appear, 2007. (E-print `arXiv:math.RA/0612429v2`).

[19] C. Jansen, K. Lux, R. Parker, and R. Wilson. *An Atlas of Brauer Characters*, volume 11 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1995. Appendix 2 by T. Breuer and S. Norton, Oxford Science Publications.

[20] W. Kimmerle. On the prime graph of the unit group of integral group rings of finite groups. In *Groups, rings and algebras*, volume 420 of *Contemp. Math.*, pages 215–228. Amer. Math. Soc., Providence, RI, 2006.

[21] I. S. Luthar and I. B. S. Passi. Zassenhaus conjecture for A_5. *Proc. Indian Acad. Sci. Math. Sci.*, 99(1):1–5, 1989.

[22] I. S. Luthar and P. Trama. Zassenhaus conjecture for S_5. *Comm. Algebra*, 19(8):2353–2362, 1991.

[23] Z. Marciniak, J. Ritter, S. K. Sehgal, and A. Weiss. Torsion units in integral group rings of some metabelian groups. II. *J. Number Theory*, 25(3):340–352, 1987.

[24] M. E. O'Nan. Some evidence for the existence of a new simple group. *Proc. London Math. Soc. (3)*, 32(3):421–479, 1976.

[25] M. A. M. Salim. Torsion units in the integral group ring of the alternating group of degree 6. *Comm. Algebra*, 35(12):4198–4204, 2007.

[26] G. Stroth. A construction of the Held simple group. *J. Algebra*, 234(2):707–742, 2000. Special issue in honor of Helmut Wielandt.

[27] H. Zassenhaus. On the torsion units of finite group rings. In *Studies in mathematics (in honor of A. Almeida Costa) (Portuguese)*, pages 119–126. Instituto de Alta Cultura, Lisbon, 1974.

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