ANDO-HIAI INEQUALITY AND A GENERALIZED FURUTA-TYPE OPERATOR FUNCTION

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Abstract. In this paper, we shall discuss generalizations of the results on Ando-Hiai inequality and a generalized Furuta-type operator function.

Firstly we shall obtain a generalization of our recent result on generalized Ando-Hiai inequality, that is, if $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$ for $A, B > 0$ and $p, r > 0$, then

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq A^{-t} \natural_{\frac{\delta+t}{s+t}} B^s$$

for $0 \leq s \leq p$, $0 \leq t \leq r$ and $-t \leq \delta \leq s$,

Secondly, as a related result to Furuta's and our recent ones, we shall show the following: Let $A, B > 0$. If $A^t \geq B^t \geq 0$ for some $t \in (0, 1]$ and $p \geq 1$, then

$$F(\lambda, \mu) = A^{-\lambda} \natural_{\frac{1-t+\lambda}{(p-t)\mu+\lambda}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\mu}.$$

satisfies $F(q, w) \geq F(r, s)$ for any $s \geq 1$, $r \geq t$, $\frac{1-t}{p-t} \leq w \leq s$ and $0 \leq q \leq r$, and also these two theorems lead Grand Furuta inequality.

Moreover we discuss further extensions of the results on these two topics.

1 Introduction Throughout this note, A and B are positive operators on a complex Hilbert space. For convenience, we denote $A \geq 0$ (resp. $A > 0$) if A is a positive (resp. strictly positive) operator.

First of all, we recall Furuta inequality [10] (cf. [2, 11, 17, 20]): If $A \geq B \geq 0$, then for each $r \geq 0$,

$$\text{(i) } (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq B^{\frac{p+r}{q}} \quad \text{and} \quad \text{(ii) } A^{\frac{p+r}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

for $p \geq 0$ and $q \geq 1$ with $(1 + r)q \geq p + r$. Furuta inequality is established as an extension of Löwner-Heinz theorem *"$A \geq B \geq 0$ ensures $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0, 1]$."* As stated in [17], when $A > 0$ and $B \geq 0$, Furuta inequality can be arranged in terms of α-power mean $\natural_{\alpha}$ for $\alpha \in [0, 1]$ introduced by Kubo-Ando [19] as $A \natural_{\alpha} B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^{\alpha} A^{\frac{1}{2}}$:

(F) $\quad A \geq B \geq 0$ with $A > 0$ implies $A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leq B \leq A$ for $p \geq 1$ and $r \geq 0$.

On the other hand, Ando and Hiai [1] have shown the following inequality (called Ando-Hiai inequality): For $A, B > 0$,

(AH) $\quad A \natural_{\alpha} B \leq I$ for $\alpha \in (0, 1)$ implies $A^r \natural_{\alpha} B^r \leq I$ for $r \geq 1$.

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By (AH), they obtained that for $A, B > 0$,

(AH') $A^{-1} \natural_{\frac{1}{p}} A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}} \le I$ implies $A^{-r} \natural_{\frac{1}{p}} (A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}})^r \le I$ for $p \ge 1$ and $r \ge 1$.

We remark that (AH') is equivalent to the main result of log majorization.

As a generalization of Furuta inequality and Ando-Hiai inequality, Furuta [12] obtained the following theorem (cf. [5, 9, 13, 15, 21, 22, 24]).

Theorem 1.A (Grand Furuta inequality [12]). *If $A \ge B \ge 0$ with $A > 0$, then for each $t \in [0,1]$ and $p \ge 1$,*

$$F(r,s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{-r}{2}} \tag{1.1}$$

is decreasing for $r \ge t$ and $s \ge 1$, and $A^{1-t+r} \ge \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$ holds for $r \ge t$ and $s \ge 1$.

We remark that (1.1) can be rewritten by using α-power mean as follows:

$$F(\lambda,\mu) = A^{-\lambda} \natural_{\frac{1-t+\lambda}{(p-t)\mu+\lambda}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\mu}. \tag{1.1'}$$

Recently, we investigate extensions of Ando-Hiai inequality in [4, 6], and the following results are obtained.

Theorem 1.B ([6]). *For $A, B > 0$ and $\alpha \in (0,1)$, if $A \natural_{\alpha} B \le I$, then*

$$A^r \natural_{\frac{\alpha r}{(1-\alpha)s+\alpha r}} B^s \le A \natural_{\alpha} B \le I \tag{GAH}$$

for $s \ge 1$ and $r \ge 1$.

Theorem 1.C ([4]). *For $A, B > 0$ and $\alpha \in [0,1]$, if $A \natural_{\alpha} B \le I$, then*

$$A \natural_{\alpha} B \le A^{\mu} \natural_{\frac{\alpha\mu}{(1-\alpha)\lambda+\alpha\mu}} B^{\lambda}$$

for $\mu \in [0,1]$ and $\lambda \in [0,1]$.

Very recently, as a generalization of [18, Theorem] (cf. [8]), the following theorems were shown on monotonicity of a generalized Furuta-type operator function (1.1) or (1.1').

Theorem 1.D ([14]). *Define $F(\lambda,\mu)$ as in (1.1). Let $A \ge B \ge 0$ with $A > 0$, $t \in [0,1]$ and $p \ge 1$. Then $F(\lambda,\mu)$ satisfies the following properties:*

(i) $$F(r,w) \ge F(r,1) \ge F(r,s) \ge F(r,s')$$
holds for any $s' \ge s \ge 1$, $r \ge t$ and $\frac{1-t}{p-t} \le w \le 1$.

(ii) $$F(q,s) \geq F(t,s) \geq F(r,s) \geq F(r',s)$$

holds for any $r' \geq r \geq t$, $s \geq 1$ *and* $t-1 \leq q \leq t$.

Theorem 1.E ([16]). *Define* $F(\lambda,\mu)$ *as in* (1.1'). *Let* $A \geq B \geq 0$ *with* $A > 0$, $t \in [0,1]$ *and* $p \geq 1$. *Then* $F(\lambda,\mu)$ *satisfies*

$$F(q,w) \geq F(t,1) \geq F(r,s) \geq F(r',s')$$

for any $s' \geq s \geq 1$, $r' \geq r \geq t$, $\frac{1-t}{p-t} \leq w \leq 1$ *and* $t-1 \leq q \leq t$.

We remark that the domain of Theorems 1.A, 1.D and 1.E can be expressed by the following Figure 1.

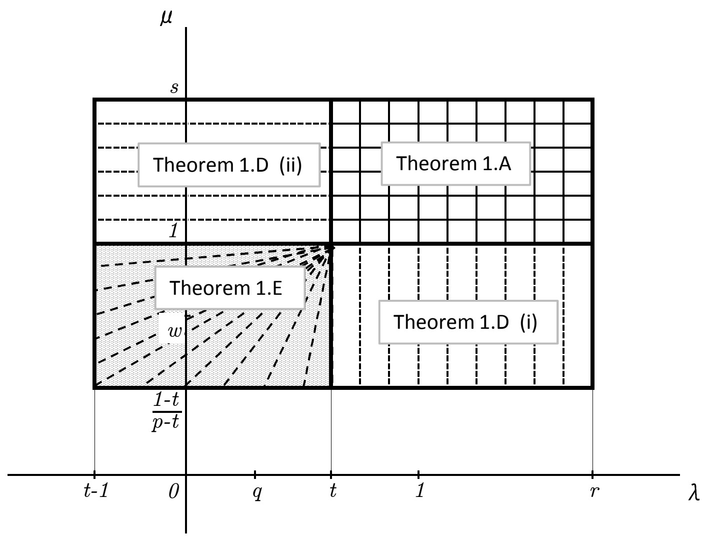

FIGURE 1

In this paper, we shall discuss generalizations of the results on Ando-Hiai inequality and a generalized Furuta-type operator function.

Firstly we shall obtain a generalization of Theorem 1.C, that is, if $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$ for $A, B > 0$ and $p, r > 0$, then

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq A^{-t} \natural_{\frac{\delta+t}{s+t}} B^s$$

for $0 \leq s \leq p$, $0 \leq t \leq r$ and $-t \leq \delta \leq s$,

Secondly, as a related result to Theorems 1.D and 1.E, we shall show the following: Let $A, B > 0$. If $A^t \geq B^t \geq 0$ for some $t \in (0,1]$ and $p \geq 1$, then (1.1') satisfies $F(q,w) \geq F(r,s)$ for any $s \geq 1$, $r \geq t$, $\frac{1-t}{p-t} \leq w \leq s$ and $0 \leq q \leq r$, and also these two theorems lead Theorem 1.A.

Moreover we discuss further extensions of the results on these two topics.

2 Main results We can rewrite Theorem 1.C by putting $\lambda = \frac{s}{p}$, $\mu = \frac{t}{r}$ and $\alpha = \frac{r}{p+r}$ and replacing A with A^{-r} and B with B^p as follows:

Corollary 2.A ([4]). *For $A, B > 0$, $p > 0$ and $r > 0$, if $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$, then*

$$A^{-r} \natural_{\frac{r}{p+r}} B^p \leq A^{-t} \natural_{\frac{t}{s+t}} B^s$$

for $s \in [0, p]$ and $t \in [0, r]$.

Then we can obtain a generalization of Corollary 2.A.

Theorem 2.1. *For $A, B > 0$, $p > 0$ and $r > 0$, if $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$, then*

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq A^{-t} \natural_{\frac{\delta+t}{s+t}} B^s$$

for $0 \leq s \leq p$, $0 \leq t \leq r$ and $-t \leq \delta \leq s$.

Proof. Put $C = A^{\frac{r}{2}} B^p A^{\frac{r}{2}}$. Then $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$ if and only if

$$A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}} = C^{\frac{r}{p+r}}. \tag{2.1}$$

By (2.1) and Löwner-Heinz theorem, $A^{r-t} \geq C^{\frac{r-t}{p+r}}$ since $\frac{r-t}{r} \in [0, 1]$, so that we have

$$\begin{aligned} A^{-t} \natural_{\frac{\delta+t}{s+t}} B^s &= A^{\frac{-r}{2}} (A^{r-t} \natural_{\frac{\delta+t}{s+t}} (A^{\frac{r}{2}} B^s A^{\frac{r}{2}})) A^{\frac{-r}{2}} = A^{\frac{-r}{2}} (A^{r-t} \natural_{\frac{\delta+t}{s+t}} (A^r \natural_{\frac{s}{p}} C)) A^{\frac{-r}{2}} \\ &\geq A^{\frac{-r}{2}} (C^{\frac{r-t}{p+r}} \natural_{\frac{\delta+t}{s+t}} (C^{\frac{r}{p+r}} \natural_{\frac{s}{p}} C)) A^{\frac{-r}{2}} = A^{\frac{-r}{2}} C^{\frac{\delta+r}{p+r}} A^{\frac{-r}{2}} = A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \end{aligned}$$

for $0 \leq s \leq p$, $0 \leq t \leq r$ and $-t \leq \delta \leq s$. □

Remark 1. Concerning Corollary 2.A, they might expect that if $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$ for some $p > 0$ and $r > 0$, then

$$A^{-t} \natural_{\frac{t}{s+t}} B^s \leq I \quad \text{for } 0 \leq s \leq p \text{ and } 0 \leq t \leq r.$$

But it is pointed out in [23] that this conjecture does not hold, and the following counterexample in [3] plays an important role in the proof. Let

$$A = \begin{pmatrix} 17 & 7 \\ 7 & 5 \end{pmatrix}^2 \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}^2.$$

Then

$$A^2 - (AB^2A)^{\frac{1}{2}} = \begin{pmatrix} 135716.49504\ldots & 62374.58231\ldots \\ 62374.58231\ldots & 28669.17453\ldots \end{pmatrix} \geq 0$$

since eigenvalues of $A^2 - (AB^2A)^{\frac{1}{2}}$ are $164383.89711\ldots$ and $1.77246\ldots$, so that $A^{-2} \natural B^2 \leq I$. On the other hand,

$$A - (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}} = \begin{pmatrix} 309.39438\ldots & 138.04008\ldots \\ 138.04008\ldots & 60.06152\ldots \end{pmatrix} \not\geq 0$$

since eigenvalues of $A - (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}}$ are $-1.27415\ldots$ and $370.73006\ldots$, so that $A^{-1} \natural B \not\leq I$.

Remark 2. [7, Theorem 4] can be expressed as a special case of Theorem 2.1 as follows: For $A, B > 0$, $p > 0$ and $r \geq 0$, suppose $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$. Then

(i) for each $\delta \in [0,p]$, $A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq A^{-r} \natural_{\frac{\delta+r}{s+r}} B^s$ for $\delta \leq s \leq p$,

(ii) for each $t \in [0,r]$, $A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq A^{-t} \natural_{\frac{\delta+t}{p+t}} B^p$ for $-t \leq \delta \leq p$.

In fact, by putting $t = r$ and $\delta \geq 0$ in Theorem 2.1, we can get (i). Similarly, by putting $s = p$ in Theorem 2.1, we can get (ii).

Next we shall obtain the following Theorem 2.2 related to Theorems 1.D and 1.E.

Theorem 2.2. *Let $A, B > 0$. Define $F(\lambda,\mu)$ as in (1.1'). If $A^t \geq B^t \geq 0$ for some $t \in (0,1]$ and $p \geq 1$, then $F(\lambda,\mu)$ satisfies*

$$F(q,w) \geq F(r,s)$$

for any $s \geq 1$, $r \geq t$, $\frac{1-t}{p-t} \leq w \leq s$ and $0 \leq q \leq r$.

Proof. Put $D = (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1}{p-t}}$, then $A^t \geq B^t$ if and only if

$$I \geq A^{\frac{-t}{2}} B^t A^{\frac{-t}{2}} = A^{\frac{-t}{2}} (A^{\frac{t}{2}} D^{p-t} A^{\frac{t}{2}})^{\frac{t}{p}} A^{\frac{-t}{2}} = A^{-t} \natural_{\frac{t}{p}} D^{p-t}. \tag{2.2}$$

Applying Theorem 1.B to (2.2), we have

$$I \geq A^{-r} \natural_{\frac{\frac{t}{p}\cdot\frac{r}{t}}{(1-\frac{t}{p})s+\frac{t}{p}\cdot\frac{r}{t}}} D^{(p-t)s} = A^{-r} \natural_{\frac{r}{(p-t)s+r}} D^{(p-t)s}. \tag{2.3}$$

for $s \geq 1$ and $r \geq t$. By Theorem 2.1, (2.3) ensures

$$F(r,s) = A^{-r} \natural_{\frac{1-t+r}{(p-t)s+r}} D^{(p-t)s} \leq A^{-q} \natural_{\frac{1-t+q}{(p-t)w+q}} D^{(p-t)w} = F(q,w)$$

for $0 \leq (p-t)w \leq (p-t)s$, $0 \leq q \leq r$ and $-q \leq 1-t \leq (p-t)w$. Therefore $F(q,w) \geq F(r,s)$ holds for any $s \geq 1$, $r \geq t$, $\frac{1-t}{p-t} \leq w \leq s$ and $0 \leq q \leq r$. □

At the end of this section, we shall show that Theorems 1.B and 2.1 lead Theorem 1.A via Theorem 2.2.

Proof of Theorem 1.A. We may assume that B is invertible. When $t \in (0,1]$, $A \geq B > 0$ ensures $A^t \geq B^t$ by Löwner-Heinz theorem. Therefore, for $t \in (0,1]$ and $p \geq 1$,

$$\begin{aligned} F(r,s) &= A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} A^{\frac{-r}{2}} \leq F(q,w) \\ &\leq F(t,1) = A^{\frac{-t}{2}} B A^{\frac{-t}{2}} \leq A^{1-t}. \end{aligned}$$

for any $s \geq q \geq 1$, $r \geq w \geq t$ by Theorem 2.2.

When $t = 0$, it is obtained in [6] that (F) follows from (AH), and also $A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leq A$ leads $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$ for $p \geq 1$ and $r \geq 0$ by Löwner-Heinz theorem. Therefore desired result is obtained immediately by Theorem 2.1. □

3 Further extensions In this section, firstly we shall show the following proposition.

Proposition 3.1. *For $A, B > 0$ and $p \geq 0$, $r \geq 0$ such that $p + r \neq 0$,*

(i) *if $A^{-r} \natural_{\frac{\delta_0+r}{p+r}} B^p \leq A^{\delta_0}$ for some δ_0 with $-r < \delta_0 \leq p$, then*

$$A^{-r} \natural_{\frac{\delta_1+r}{p+r}} B^p \leq A^{\delta_1}$$

for $-r < \delta_1 \leq \delta_0$.

(ii) *if $A^{-r} \natural_{\frac{\delta_0+r}{p+r}} B^p \leq B^{\delta_0}$ for some δ_0 with $-r \leq \delta_0 < p$, then*

$$A^{-r} \natural_{\frac{\delta_2+r}{p+r}} B^p \leq B^{\delta_2}$$

for $\delta_0 \leq \delta_2 \leq p$.

Proof. We can easily obtain (i) as follows:

$$A^{-r} \natural_{\frac{\delta_1+r}{p+r}} B^p = A^{-r} \natural_{\frac{\delta_1+r}{\delta_0+r}} (A^{-r} \natural_{\frac{\delta_0+r}{p+r}} B^p) \leq A^{-r} \natural_{\frac{\delta_1+r}{\delta_0+r}} A^{\delta_0} = A^{\delta_1}.$$

Similarly, we can obtain (ii) as follows:

$$\begin{aligned} A^{-r} \natural_{\frac{\delta_2+r}{p+r}} B^p &= B^p \natural_{\frac{-\delta_2+p}{p+r}} A^{-r} = B^p \natural_{\frac{-\delta_2+p}{-\delta_0+p}} (B^p \natural_{\frac{-\delta_0+p}{p+r}} A^{-r}) \\ &= B^p \natural_{\frac{-\delta_2+p}{-\delta_0+p}} (A^{-r} \natural_{\frac{\delta_0+r}{p+r}} B^p) \leq B^p \natural_{\frac{-\delta_2+p}{-\delta_0+p}} B^{\delta_0} = B^{\delta_2}. \end{aligned}$$

Hence the proof is complete. □

By considering Proposition 3.1, we can get further extensions of Theorem 2.1.

Theorem 3.2. *For $A, B > 0$ and $p \geq 0$, $r \geq 0$ such that $p + r \neq 0$,*

(i) *if $A^{-r} \natural_{\frac{\delta_0+r}{p+r}} B^p \leq A^{\delta_0}$ for some δ_0 with $-r < \delta_0 \leq 0$, then*

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq A^{-t} \natural_{\frac{\delta+t}{p+t}} B^p$$

for $-\delta_0 \leq t \leq r$ and $-t \leq \delta \leq p$.

(ii) *if $A^{-r} \natural_{\frac{\delta_0+r}{p+r}} B^p \leq B^{\delta_0}$ for some δ_0 with $0 \leq \delta_0 < p$, then*

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq A^{-r} \natural_{\frac{\delta+r}{s+r}} B^s$$

for $\delta_0 \leq s \leq p$ and $-r \leq \delta \leq s$.

Theorem 3.3. *For $A, B > 0$ and $p \geq 0$, $r \geq 0$ such that $p + r \neq 0$,*

(i) *if* $A^{-r} \natural_{\frac{\delta_0+r}{p+r}} B^p \le A^{\delta_0}$ *for some* δ_0 *with* $0 \le \delta_0 \le p$*, then*

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \le A^{-t} \natural_{\frac{\delta+t}{s+t}} B^s$$

for $0 \le s \le p$, $-\delta_0 \le t \le r$ *and* $-t \le \delta \le s$.

(ii) *if* $A^{-r} \natural_{\frac{\delta_0+r}{p+r}} B^p \le B^{\delta_0}$ *for some* δ_0 *with* $-r \le \delta_0 \le 0$*, then*

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \le A^{-t} \natural_{\frac{\delta+t}{s+t}} B^s$$

for $\delta_0 \le s \le p$, $0 \le t \le r$ *and* $-t \le \delta \le s$.

We remark that the assumption of Theorem 3.2 is weaker than that of Theorem 2.1, and also the assumption of Theorem 3.3 is stronger than that of Theorem 2.1 by Proposition 3.1. By putting $\delta_0 = 0$ in (i) or (ii) of Theorem 3.3, we have Theorem 2.1.

Proof of Theorem 3.2. Put $C = A^{\frac{r}{2}} B^p A^{\frac{r}{2}}$. Then $A^{-r} \natural_{\frac{\delta_0+r}{p+r}} B^p \le A^{\delta_0}$ if and only if

$$A^{\delta_0+r} \ge (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{\delta_0+r}{p+r}} = C^{\frac{\delta_0+r}{p+r}}. \tag{3.1}$$

Proof of (i). By (3.1) and Löwner-Heinz theorem, $A^{r-t} \ge C^{\frac{r-t}{p+r}}$ since $\frac{r-t}{\delta_0+r} \in [0,1]$, so that we have

$$\begin{aligned} A^{-t} \natural_{\frac{\delta+t}{p+t}} B^p &= A^{\frac{-r}{2}} (A^{r-t} \natural_{\frac{\delta+t}{p+t}} (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})) A^{\frac{-r}{2}} = A^{\frac{-r}{2}} (A^{r-t} \natural_{\frac{\delta+t}{p+t}} C) A^{\frac{-r}{2}} \\ &\ge A^{\frac{-r}{2}} (C^{\frac{r-t}{p+r}} \natural_{\frac{\delta+t}{p+t}} C) A^{\frac{-r}{2}} = A^{\frac{-r}{2}} C^{\frac{\delta+r}{p+r}} A^{\frac{-r}{2}} = A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \end{aligned}$$

for $-\delta_0 \le t \le r$ and $-t \le \delta \le p$.

Proof of (ii). By replacing A with B^{-1} and B with A^{-1} in (i), we obtain the following: For $A, B > 0$ and $p \ge 0$, $r \ge 0$ such that $p + r \ne 0$,

$$\begin{aligned} &\text{if } B^{-r} \natural_{\frac{\delta_0+r}{p+r}} A^p \ge B^{\delta_0} \text{ for some } \delta_0 \text{ with } -r < \delta_0 \le 0, \\ &\text{then } B^{-r} \natural_{\frac{\delta+r}{p+r}} A^p \ge B^{-t} \natural_{\frac{\delta+t}{p+t}} A^p \end{aligned} \tag{3.2}$$

for $-\delta_0 \le t \le r$ and $-t \le \delta \le p$.

On the other hand, $A^{-r} \natural_{\frac{\delta_0+r}{p+r}} B^p \le B^{\delta_0}$ if and only if

$$B^{-p} \natural_{\frac{-\delta_0+p}{r+p}} A^r \ge B^{-\delta_0}. \tag{3.3}$$

Applying (3.2) to (3.3) for $-p < -\delta_0 \le 0$, we have

$$B^{-p} \natural_{\frac{-\delta+p}{r+p}} A^r \ge B^{-s} \natural_{\frac{-\delta+s}{r+s}} A^r \tag{3.4}$$

for $-(-\delta_0) \le s \le p$ and $-s \le -\delta \le r$, and also (3.4) if and only if

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \le A^{-r} \natural_{\frac{\delta+t}{s+r}} B^s$$

for $\delta_0 \le s \le p$ and $-r \le \delta \le s$.

Hence the proof of Theorem 3.2 is complete. □

Proof of Theorem 3.3. Put $C = A^{\frac{r}{2}} B^p A^{\frac{r}{2}}$. Then $A^{-r} \natural_{\frac{\delta_0+r}{p+r}} B^p \le A^{\delta_0}$ if and only if

$$A^{\delta_0+r} \ge (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{\delta_0+r}{p+r}} = C^{\frac{\delta_0+r}{p+r}}. \tag{3.5}$$

Proof of (i). By (3.5) and Löwner-Heinz theorem, $A^{r-t} \ge C^{\frac{r-t}{p+r}}$ since $\frac{r-t}{\delta_0+r} \in [0,1]$ and $A^r \ge C^{\frac{r}{p+r}}$ since $\frac{r}{\delta_0+r} \in [0,1]$, so that we have

$$\begin{aligned} A^{-t} \natural_{\frac{\delta+t}{s+t}} B^s &= A^{\frac{-r}{2}} (A^{r-t} \natural_{\frac{\delta+t}{s+t}} (A^{\frac{r}{2}} B^s A^{\frac{r}{2}})) A^{\frac{-r}{2}} = A^{\frac{-r}{2}} (A^{r-t} \natural_{\frac{\delta+t}{s+t}} (A^r \natural_{\frac{s}{p}} C)) A^{\frac{-r}{2}} \\ &\ge A^{\frac{-r}{2}} (C^{\frac{r-t}{p+r}} \natural_{\frac{\delta+t}{s+t}} (C^{\frac{r}{p+r}} \natural_{\frac{s}{p}} C)) A^{\frac{-r}{2}} = A^{\frac{-r}{2}} C^{\frac{\delta+r}{p+r}} A^{\frac{-r}{2}} = A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \end{aligned}$$

for $0 \le s \le p$, $-\delta_0 \le t \le r$ and $-t \le \delta \le s$.

Proof of (ii). By replacing A with B^{-1} and B with A^{-1} in (i), we obtain the following: For $A, B > 0$ and $p \ge 0$, $r \ge 0$ such that $p + r \ne 0$,

$$\begin{aligned} &\text{if } B^{-r} \natural_{\frac{\delta_0+r}{p+r}} A^p \ge B^{\delta_0} \text{ for some } \delta_0 \text{ with } 0 \le \delta_0 \le p, \\ &\text{then } B^{-r} \natural_{\frac{\delta+r}{p+r}} A^p \ge B^{-t} \natural_{\frac{\delta+t}{s+t}} A^s \end{aligned} \tag{3.6}$$

for $0 \le s \le p$, $-\delta_0 \le t \le r$ and $-t \le \delta \le s$.

On the other hand, $A^{-r} \natural_{\frac{\delta_0+r}{p+r}} B^p \le B^{\delta_0}$ if and only if

$$B^{-p} \natural_{\frac{-\delta_0+p}{r+p}} A^r \ge B^{-\delta_0}. \tag{3.7}$$

Applying (3.6) to (3.7) for $0 \le -\delta_0 \le r$, we have

$$B^{-p} \natural_{\frac{-\delta+p}{r+p}} A^r \ge B^{-s} \natural_{\frac{-\delta+s}{t+s}} A^t \tag{3.8}$$

for $0 \le t \le r$, $-(-\delta_0) \le s \le p$ and $-s \le -\delta \le t$, and also (3.8) if and only if

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \le A^{-t} \natural_{\frac{\delta+t}{s+t}} B^s$$

for $\delta_0 \le s \le p$, $0 \le t \le r$ and $-t \le \delta \le s$.

Hence the proof of Theorem 3.3 is complete. □

Next, by using Theorem 3.3, we shall give a direct proof of the following Theorem 3.4 combined Theorems 1.D and 1.E. We remark that Theorem 2.2 is a variant of Theorem 3.4 since Theorem 2.2 is the result for $A^t \ge B^t \ge 0$ for $t \in (0,1]$ and this assumption is weaker than that of Theorem 3.4 (i.e., $A \ge B \ge 0$).

Theorem 3.4. *Define $F(\lambda, \mu)$ as in* (1.1'). *If $A \ge B \ge 0$ with $A > 0$, $t \in [0,1]$ and $p \ge 1$, then $F(\lambda, \mu)$ satisfies*

$$F(q, w) \ge F(r, s)$$

for any $s \ge 1$, $r \ge t$, $\frac{1-t}{p-t} \le w \le s$ and $t - 1 \le q \le r$.

Proof. We may also assume that B is invertible.

Put $D = (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^{\frac{1}{p-t}}$. By Theorem 1.A, $A \geq B > 0$ ensures

$$A^{1-t+r} \geq (A^{\frac{r}{2}} D^{(p-t)s} A^{\frac{r}{2}})^{\frac{1-t+r}{(p-t)s+r}}, \quad \text{that is,} \quad A^{-r} \natural_{\frac{1-t+r}{(p-t)s+r}} D^{(p-t)s} \leq A^{1-t} \tag{3.9}$$

for $t \in [0,1]$, $p \geq 1$, $s \geq 1$ and $r \geq t$. By (i) of Theorem 3.3, for $0 \leq 1-t \leq (p-t)s$ and $r \geq 0$, (3.9) ensures

$$F(r,s) = A^{-r} \natural_{\frac{1-t+r}{(p-t)s+r}} D^{(p-t)s} \leq A^{-q} \natural_{\frac{1-t+q}{(p-t)w+q}} D^{(p-t)w} = F(q,w)$$

for $0 \leq (p-t)w \leq (p-t)s$, $-(1-t) \leq q \leq r$ and $-q \leq 1-t \leq (p-t)w$. Therefore $F(q,w) \geq F(r,s)$ holds for any $s \geq 1$, $r \geq t$, $\frac{1-t}{p-t} \leq w \leq s$ and $t-1 \leq q \leq r$. □

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