THE CLASSICAL VOLTERRA OPERATOR AND SCHUR'S THEOREM

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Abstract. In this work we provide a counterexample for Schur's Theorem on triangular matrices on infinite dimensional spaces. Moreover, the counterexample provided is a compact quasinilpotent operator. Indeed, the result neither depends on the index of the chosen basis for the matrix representations nor on the upper-lower choice for the triangular matrix. As a consequence, we see the optimality of a result by Halmos on matrix representations of operators. Namely, Halmos proved that each operator can be represented by a matrix with finite columns. Finally, we 'answer' a philosophical question posed by J. B. Conway in [1, p.213].

1. Introduction

It is widely known that all linear transformations acting on finite dimensional vector spaces are continuous. These transformations are easily represented as matrices and, once we fix a basis, any $m \times n$ matrix represents a continuous linear transformation from an n dimensional vector space to an m dimensional vector space. The matrix representation provides a very fast, and intuitive, way of considering linear transformations. There are many examples of matrices that provide significant information about the associated linear transformations, e.g. diagonal, positive, real, self-adjoint, symmetric, antisymmetric, Jacobi, triangular, etc. Indeed, linear transformations that can be represented by those kinds of matrices have different properties. So much so, that many theorems have been developed to find the 'most convenient' basis for each linear transformation and to put them in the 'proper shape'. Some examples are: Jordan's Theorem, Polar Decomposition, Rational Canonical Form, Schur's Theorem, etc. Since this last result will play a central role in this work, it is convenient to state it.

Schur's Theorem. *Every (finite) square matrix is unitarily equivalent to an upper (equivalently lower) triangular matrix.*

For an extensive explanation and proof, see [6, p.79]. A straightforward consequence of Schur's Theorem is that the spectrum of a matrix can be obtained from a collection of the diagonal elements of any of its triangular representations. Furthermore, among other uses, triangular representations are used to find invariant subspaces, to solve systems of linear equations and to accelerate computations. From the point of view of general analysis, and particularly in Operator Theory, it would be useful to find extensions for these finite dimensional results to the infinite dimensional setting. Indeed, the infinite dimensional separable Hilbert space is the natural extension of the finite dimensional vector spaces. Their orthonormal bases make it possible to associate linear transformations with matrices in a natural way. All orthonormal bases in the separable Hilbert space have the same cardinality, ω_0. Nonetheless, the ordinal types of their index-sets can vary widely. Any

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totally ordered denumerable set is eligible as an index. Most published examples are: $\mathbb{N}$, $\mathbb{N}\oplus\cdots\oplus\mathbb{N}$ and $\mathbb{Z}$, where $\mathbb{N}$ is the natural numbers and $\mathbb{Z}$ is the set of all integers. Examples of these and other indexes can be found at [3, Prb. 70], [9, p. 92], [10] or [4]. As Halmos wrote in [3, Prb. 61], the use of $\mathbb{N}$ makes possible to apply standard induction and to write expression like 'the previous element' or 'the first element of the diagonal'.

One of the reasons to select an index other than the natural numbers might be to adapt the representation of the space to a given linear transformation, in such a way that its matrix will have a particularly nice structure. Again, diagonal bounded matrices are studied as multiplication operators, diagonal plus or minus one are studied as forward or backward shifts, Hankel Matrices, the Hilbert Matrix, etc. What would be the aspect of a bilateral shift if we indexed its matrix with $\mathbb{N}$?

The main difference between the finite and the infinite dimensional settings is that, in the latter, not all linear transformations are continuous, or equivalently, bounded. Actually, given an infinite basis and an infinite matrix with the same index, there is not a general way to decide on its continuity as an operator acting in the space. As Halmos states: *Not much of matrix theory carries over to infinite-dimensional spaces, and what does is not so useful, but sometimes helps* [3, p.23]. We might consider some examples.

2. Two and a Half Positive Examples

In this section we introduce some definitions and examples that will motivate our main result.

An invariant subspace for an operator T, is a non-trivial closed vector space that is mapped into itself by T. One of the most successful extensions from finite to infinite dimensions, within the field of Operator Theory, regards the existence of invariant subspaces for compact operators, Lomonosov [7]. Actually, Lomonosov proved more, showing that all the operators in the double commutant of a compact operator have a non-trivial invariant subspace in common. The general question, whether a general bounded operator has a non-trivial invariant subspace remains open, see [9] or [2].

Another interesting extension is the Ringrose Theorem for compact operators, see [9] or [10]. One of its simplest applications consists of a corollary that provides a way of computing the spectrum of a compact operator.

Ringrose's Corollary. Let T be a compact operator acting on a separable Hilbert space and let $\{T_{i,j}\}_{i,j\in I\times I}$ be an upper or lower triangular matrix representation of T. Then, the spectrum of T, $\sigma(T)$, consists of the set

$$\{T_{i,i}:\ i\in I\}\cup\{0\}.$$

This result parallels the calculation method for the spectrum of a finite dimensional linear transformation mentioned above; see comments to Schur's Theorem above. In fact, in order to apply this result to a given compact operator, T, we must find a triangular matrix representation for it. Equivalently, we must find an orthonormal bases of the space that produces an upper or lower triangular matrix for T. It is remarkable that since in this result there are no restrictions on the index-set, I, we can choose any totaly ordered denumerable set to be the index-set of the matrix representation of T. The choices of the index-set and the upper or lower triangularity, are grades of freedom to be used along with the choice of the orthonormal vectors when constructing the triangularizing bases for the operator T.

Returning to Halmos, we consider the following result:

Theorem 2.1. *Every operator acting on a separable Hilbert space can be represented by a* $(\mathbb{N}\times\mathbb{N})$ *matrix which has finite columns.*

For an extensive explanation and proof see [3, Prob. 44]. This result is similar to Schur's Theorem but the only index-set considered is $\mathbb{N}$ and it only relates to upper triangular matrices. Those two constraints might be suitable for some operators but as soon as an operator has no eigenvectors, its matrix will fail to be upper triangular. It would seem sensible to think that by weakening these two constraints, admitting any denumerable index-set and both upper or lower triangular matrices, then it should be possible to improve the result, at least for compact operators. Of course, the general result lies far beyond the invariant subspace problem, but compact operators have proved their convenience for supporting extensions of finite dimensional results. We might try to prove the following result: *Every compact operator acting on an infinite dimensional separable Hilbert space can be represented by an upper or lower triangular matrix.* We will show this to be untrue.
Remark 1: It is worth mentioning that if an operator is cyclic, then there is a matrix representation with only one non-zero diagonal below the main one, see [3, Prob. 167].

3. Schur's Theorem and the classical Volterra operator

A well known representation of the infinite dimensional separable Hilbert space is the space $L^2[0,1]$ formed by all complex functions, supported on the unit segment, with squared integrable modulus. The inner product of this space is defined by:

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\,\mathrm{d}t, \qquad \text{for any } f \text{ and } g \text{ in } L^2[0,1].$$

Hereafter, a general orthonormal basis in $L^2[0,1]$ will be denoted by $\{e_i\}_{i\in I}$, where I is a totally ordered denumerable set. Acting on this space, we define the classical Volterra operator, V, as,

$$(Vf)(x) = \int_0^x f(t)\,\mathrm{d}t, \qquad \text{for each } f \text{ in } L^2[0,1].$$

Introduced by V. Volterra, this is one of the first studied operators which still generates significant interest; see, for instance, works by A. Montes and S. Shkarin [8], G. Herzog and A. Weber [5], and S. Shkarin [12].

The classical Volterra operator is a non-trivial example of compact operator and its spectrum reduces to the singleton $\{0\}$. That is, the operator $V - \lambda I$ is always invertible, except for $\lambda = 0$, where I stands for the identity operator, see [1, pp. 44 and 212]. The computation of the adjoint of this operator is particulary simple since it only involves a straightforward application of Fubini's Theorem, see [11]. The adjoint of V, denoted V^*, is defined as

$$(V^*f)(x) = \int_x^1 f(t)\,\mathrm{d}t, \qquad \text{for each } f \text{ in } L^2[0,1].$$

A remarkably simple, useful and neat identity found by Halmos, see [3, Prob. 188]:

$$V + V^* = P, \tag{3}$$

where P is the orthogonal projection onto constant functions. This is easily proved by replacing the operators by their explicit expressions.
We can now state and prove our result:

Theorem 3.1. *The classical Volterra operator does not have any possible upper or lower triangular matrix representations.*

Proof. Suppose there is an orthonormal basis, $\{e_i\}_{i\in I}$, of the space $L^2[0,1]$, such that V has an upper or lower triangular matrix. Applying Halmos' formula for the Volterra operator

(3), to the computation of a general entry, $V_{i,j}$, of the corresponding triangular matrix we get:

$$\begin{aligned} V_{i,j} &= \langle Ve_j, e_i\rangle \\ &= \langle Pe_j, e_i\rangle - \langle V^*e_j, e_i\rangle \\ &= \langle P^2e_j, e_i\rangle - \overline{\langle e_j, Ve_i\rangle} \\ &= \langle Pe_j, Pe_i\rangle - \overline{\langle Ve_i, e_j\rangle} \\ &= \langle Pe_j, Pe_i\rangle - \overline{V}_{j,i} \qquad \text{for each } (i,j) \text{ in } I\times I. \end{aligned}$$

Letting $i = j$,

$$\begin{aligned} V_{i,i} &= \langle Pe_i, Pe_i\rangle - \overline{V}_{i,i} \\ &= \|Pe_i\|^2 - \overline{V}_{i,i} \qquad \text{for each } i \text{ in } I. \end{aligned}$$

Solving for $\|Pe_i\|^2$, we obtain

$$\begin{aligned} \|Pe_i\|^2 &= V_{i,i} + \overline{V}_{i,i} \\ &= 2\Re(V_{i,i}), \qquad \text{for each } i \text{ in } I, \end{aligned}$$

where $\Re$ stands for the real part. Since for each i in I we have that $V_{i,i}$ are the diagonal entries of a triangular matrix associated to the compact operator V, we can apply the Ringrose Colloray. Therefore, for each i in I, we have that $V_{i,i}$ belongs to the spectrum of V, which is the set $\{0\}$. This means that $\|Pe_i\| = 0$ for all i in I, or equivalently, that all the elements in the base $\{e_i\}_{i\in I}$ are orthogonal to the constant function 1, which is a contradiction. □

Remark 2: Notice that neither the index-set nor the upper-lower nature of the matrix have any role in the proof of the above result.
Remark 3: In the last result, we can actually replace V by any compact quasinilpotent operator T acting on a Hilbert space such that the self-adjoint operator $S = T + T^*$ is non-zero and non-negative. Indeed, in the latter case $S = R^2$ were R is also self-adjoint and non-zero. Therefore, the same argument runs just replacing S by R^2 (instead of replacing P by P^2). The contradiction comes from the fact that a bounded operator (R in this case), annihilating an orthonormal basis, is 0. The case $S \leqslant 0$ is not different, thus more examples could be provided. Nonetheless, the method does not apply to V^2 since the operator $V^2 + (V^*)^2$ has both positive and negative eigenvalues.

4. Consequences

The first consequence is that Schur's Theorem on triangular representations does not extend to general operators acting on infinite dimensional separable Hilbert spaces. Indeed, since the Volterra operator is compact, Schur's Theorem will not even extend to the set of compact operators. Moreover, the result is still not true if we look for the triangular matrix representations in the larger set of all upper or lower triangular matrices indexed by any totally ordered countable set.

We also can look again at Halmos' result on matrices with finite columns (see Theorem 2.1) to realize that it is optimal.

Finally, we can provide an 'answer' to a 'question' posed by J. B. Conway in one of his books [1, p.213], where he asked: *Is there any analogy between V_k for a Volterra kernel k and a lower triangular matrix?.* While we cannot provide a general answer, we can assert that for the most outstanding member of the Volterra kernel family, the classical one that we denote V, there is no possible triangular matrix representation.

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