ON HIRATA SEPARABLE GALOIS EXTENSIONS

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Abstract. Let B be a Hirata separable and Galois extension of B^G with Galois group G of order n invertible in B for some integer n, C the center of B, and $V_B(B^G)$ the commutator subring of B^G in B. It is shown that there exist subgroups K and N of G such that K is a normal subgroup of N and one of the following three cases holds: (i) $V_B(B^K)$ is a central Galois algebra over C with Galois group K, (ii) $V_B(B^K)$ is separable C-algebra with an automorphism group induced by and isomorphic with K, and (iii) B^K is a central algebra over $V_B(B^K)$ and a Hirata separable Galois extension of B^N with Galois group N/K. More characterizations for a central Galois algebra $V_B(B^K)$ are given.

1. INTRODUCTION

The Hirata separable extension of a ring is an important generalization of Azumaya algebras. The class of Hirata separable Galois extensions of a ring has been intensively investigated ([1], [7], [9]). The purpose of the present paper is to show a classification of a Hirata separable and Galois extension B of B^G with Galois group G of order n invertible in B for some integer n. We shall show that there exist subgroups K and N of G such that K is a normal subgroup of N and one of the following three cases holds: (i) $V_B(B^K)$ is a central Galois algebra over C with Galois group K, (ii) $V_B(B^K)$ is separable C-algebra with an automorphism group induced by and isomorphic with K, and $V_B(B^K)$ and B^K have the same center, and (iii) B^K is a central algebra over $V_B(B^K)$ and a Hirata separable Galois extension of B^N with Galois group N/K. Moreover, several equivalent conditions for a central Galois algebra $V_B(B^K)$ are given by using the rank function of a projective module on the spectrum of prime ideals of a commutative ring ([2], page 27). This paper was written under the support of a Caterpillar Fellowship at Bradley University. The authors would like to thank Caterpillar Inc. for the support.

2. A CLASSIFICATION

Let B be a ring with 1, A a subring of B with the same identity 1, and C the center of B. Following the definitions and notations in [8], we call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum ba_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A, and B is called an Azumaya algebra if B is a separable algebra over its center C. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule. Let G be a finite automorphism group of B and B^G the set of elements in B fixed under each element in G. Then B is called a Galois extension of B^G with Galois

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group G if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, ..., m\}$ for some integer m such that $\sum_{i=1}^{m} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. A Hirata separable Galois extension B of B^G with Galois group G means that B is a Hirata separable extension and a Galois extension of B^G with Galois group G. A central Galois algebra is a Galois extension over its center.

Throughout this paper, we assume that B is a Hirata separable Galois extension of B^G with Galois group G of order n invertible in B for some integer n, C the center of B, and $V_B(A)$ the commutator subring of A in B for a subring A of B with the same identity 1. In this section, we shall show a classification theorem for B beginning with an important fact on the commutator subring $V_B(B^G)$ of B^G in B due to K. Sugano ([7], Theorem 6).

Lemma 2.1. ([7], Theorem 6)
Let B be a Hirata separable Galois extension of B^G with Galois group G of order n invertible in B and $K = \{g \in G \,|\, g(a) = a$ for each $a \in V_B(B^G)\}$. Then B is a Hirata separable Galois extension of B^K with Galois group K and B^K is a Hirata separable Galois extension of B^G with Galois group G/K.

By Lemma 2.1, we can show that there exists a normal series of subgroups of G leading to a classification of a Hirata separable Galois extension.

Theorem 2.2.
Let B be a Hirata separable Galois extension of B^G with Galois group G of order n invertible in B. Then there exists a chain of subgroups of G,

$$\langle 1\rangle \subset G_m \subset \cdots \subset G_i \cdots \subset G_2 \subset G_1 \subset G_0 = G,$$

such that for each $i = 1, 2, \ldots, m$, G_i is a normal subgroup of G_{i-1}, B is a Hirata separable Galois extension of B^{G_i} with Galois group G_i, B^{G_i} is a Hirata separable Galois extension of $B^{G_{i-1}}$ with Galois group G_{i-1}/G_i, and $\{g \in G_m \,|\, g(a) = a$ for each $a \in V_B(B^{G_m})\} = G_m$ or $\langle 1\rangle$.

Proof. Let $G_0 = G$. If $\{g \in G_0 \,|\, g(a) = a$ for each $a \in V_B(B^{G_0})\} = G_0$ or $\langle 1\rangle$, then $m = 0$; and we are done. Otherwise, let $G_1 = \{g \in G_0 \,|\, g(a) = a$ for each $a \in V_B(B^{G_0})\}$, that is, G_1 is a proper subgroup of G. Then, since the order n of G_0 is invertible in B, B is a Hirata separable Galois extension of B^{G_1} with Galois group G_1, and B^{G_1} is a Hirata separable Galois extension of B^{G_0} with Galois group G_0/G_1 by Lemma 2.1. Similarly, by repeating the above argument for the Hirata separable Galois extension B of B^{G_1} with Galois group G_1, we have a normal subgroup G_2 of G_1 where $G_2 = \{g \in G \,|\, g(a) = a$ for each $a \in V_B(B^{G_1})\}$. Since G is a finite group, the above process terminates in m-steps for some integer m. Thus we have a sequence of subgroups of G,

$$\langle 1\rangle \subset G_m \subset \cdots \subset G_i \cdots \subset G_2 \subset G_1 \subset G_0 = G,$$

such that for each $i = 1, 2, \ldots, m$, G_i is a normal subgroup of G_{i-1}, B is a Hirata separable Galois extension of B^{G_i} with Galois group G_i, B^{G_i} is a Hirata separable Galois extension of $B^{G_{i-1}}$ with Galois group G_{i-1}/G_i, and $\{g \in G_m \,|\, g(a) = a$ for each $a \in V_B(B^{G_m})\} = G_m$ or $\langle 1\rangle$.

Next is a classification of a Hirata separable Galois extension by using Theorem 2.2.

Theorem 2.3.
Let B be a Hirata separable Galois extension of B^G with Galois group G of order n invertible in B. Then there exist subgroups K and N of G such that K is a normal subgroup of N and one of the following three cases holds: (i) $V_B(B^K)$ is a central Galois algebra over C with Galois group K; (ii) $V_B(B^K)$ is separable C-algebra with an automorphism group induced by and isomorphic with K, and $V_B(B^K)$ and B^K have the same center; and (iii) B^K is a central algebra over $V_B(B^K)$ (i.e., the center of B^K is $V_B(B^K)$) and a Hirata separable Galois extension of B^N with Galois group N/K.

Proof. By Theorem 2.2, there exists a sequence of subgroups of G,

$$\langle 1\rangle \subset G_m \subset \cdots \subset G_i \cdots \subset G_2 \subset G_1 \subset G_0 = G,$$

such that for each $i = 1, 2, \ldots, m$, G_i is a normal subgroup of G_{i-1}, B is a Hirata separable Galois extension of B^{G_i} with Galois group G_i, B^{G_i} is a Hirata separable Galois extension of $B^{G_{i-1}}$ with Galois group G_{i-1}/G_i, and $\{g \in G_m \,|\, g(a) = a \text{ for each } a \in V_B(B^{G_m})\} = G_m$ or $\langle 1\rangle$. Let $K = G_m$ and $N = G_{m-1}$ where $m \geq 1$. Then K is a normal subgroup of N. By Theorem 2.2, $\{g \in G_m \,|\, g(a) = a$ for each $a \in V_B(B^{G_m})\} = \langle 1\rangle$ or G_m. When it is $\langle 1\rangle$ we have two cases: (a) $B = B^K \cdot V_B(B^K)$ and (b) $B \supset B^K \cdot V_B(B^K)$. For case (a), $V_B(B^K)$ is a central Galois algebra over C with Galois group K ([7], Theorem 6(3)), so (i) holds. For case (b), since $\{g \in K \,|\, g(a) = a$ for each $a \in V_B(B^K)\} = \langle 1\rangle$, the restriction of K to $V_B(B^K)$ is isomorphic to K, Moreover, since the order of K is a unit in B, $V_B(B^K)$ is separable C-algebra ([7], Proposition 4(3)). Also since $V_B(V_B(B^K) = B^K$ ([7], Proposition 4(1)), we can check that $V_B(B^K)$ and B^K have the same center. Hence (ii) holds. Next we discuss the possibility that $\{g \in G_m \,|\, g(a) = a$ for each $a \in V_B(B^{G_m})\} = G_m$ $(= K)$ as given by Theorem 2.2. In this case, $V_B(B^K) \subseteq B^K$. Hence $V_B(B^K) = V_{B^K}(B^K) =$ the center of B^K. Thus B^K is a central algebra over $V_B(B^K)$. Moreover, B^K is also a Hirata separable Galois extension of B^N with Galois group N/K by Theorem 2.2 again. This implies (iii).

3. EQUIVALENT CONDITIONS

Let B be an Azumaya C-algebra with a finite automorphism group G and $J_g = \{a \in B \,|\, ax = g(x)a$ for each $x \in B\}$ for a $g \in G$. It is well known that J_g is a rank 1 projective C-module such that $J_g \cdot J_h = J_{gh} \cong J_g \otimes_C J_h$ for all $g, h \in G$ ([6], Lemma 5), and that B is a central Galois algebra over C with Galois group G if and only if $B = \oplus \sum_{g\in G} J_g$ ([3], Theorem 1 and [5], Theorem 1). These properties are generalized to a Hirata separable Galois extension B of B^G with Galois group G; that is, $\text{rank}_C(J_g) = 1$ where C is the center of B, $J_g \cdot J_h = J_{gh} \cong J_g \otimes_C J_h$ for all $g, h \in G$, and $V_B(B^G)$ is a central Galois algebra with Galois group G/K if and only if the center of $V_B(B^G)$ is $\oplus \sum_{g\in K} J_g$ where $K = \{g \in G \,|\, g(a) = a$ for each $a \in V_B(B^G)\}$ ([7], Theorem 6(3)). In this section, we shall give a different proof for the above equivalent condition in [7] and then derive more equivalent conditions for a central Galois algebra $V_B(B^G)$ with Galois group G/K in terms of the rank of a projective module over a commutative ring. We begin with the equivalent condition for a central Galois algebra $V_B(B^G)$ with a different proof from Theorem 6 in [7].

Proposition 3.1.
Let B be a Hirata separable Galois extension of B^G with Galois group G of order n invertible in B and $K = \{g \in G \,|\, g(a) = a$ for each $a \in V_B(B^G)\}$. Then, $V_B(B^G)$ is a central Galois algebra with Galois group G/K if and only if the center of $V_B(B^G)$ is $\oplus \sum_{g\in K} J_g$.

Proof. ($\Longleftarrow$) Let $J'_g = \{a \in V_B(B^G) \,|\, ax = g(x)a$ for each $x \in V_B(B^G)\}$ for a $g \in G$ and C' the center of $V_B(B^G)$. Noting that $V_B(B^G) = \oplus \sum_{g\in G} J_g$ ([5], Proposition 1), we have that $J_g \subseteq J'_g$ for each $g \in G$. We claim that $V_B(B^G) = \oplus \sum_{\overline{g}\in G/K} J'_g$. In fact, since for each $h \in K$, $J'_{gh} = \{a \in V_B(B^G) \,|\, ax = gh(x)a$ for each $x \in V_B(B^G)\} = \{a \in V_B(B^G) \,|\, ax = g(x)a$ for each $x \in V_B(B^G)\} = J'_g$, $J_{gh} \subseteq J'_{gh} = J'_g$ for each $h \in K$. Hence $\oplus \sum_{h\in K} J_{gh} \subseteq J'_g$. By hypothesis, $C' = \oplus \sum_{g\in K} J_g$, so $\oplus \sum_{h\in K} J_{gh} \cong \oplus \sum_{h\in K}(J_g \otimes_C J_h) \cong J_g \otimes_C (\oplus \sum_{h\in K} J_g) \cong J_g \otimes_C C'$ which is a rank 1 projective C'-module (for J_g is a rank 1 projective C-module). On the other hand, since the order of G is a unit in B, $V_B(B^G)$ is separable C-algebra ([7], Proposition 4(3)). Hence $V_B(B^G)$ is Azumaya C'-algebra. Thus J'_g is a rank 1 projective C'-module for each $g \in G$. Therefore $V_B(B^G) = \oplus \sum_{g\in G} J_g = \oplus \sum_{\overline{g}\in G/K} \sum_{h\in K} J_{gh} \subseteq \oplus \sum_{\overline{g}\in G/K} J'_g \subseteq V_B(B^G)$; and so $V_B(B^G) = \oplus \sum_{\overline{g}\in G/K} J'_g$. Hence $V_B(B^G)$ is a central Galois algebra with Galois group G/K ([3], Theorem 1).

($\Longrightarrow$) Since $V_B(B^G)$ is a central Galois algebra over C' with Galois group G/K, $\text{rank}_{C'}(V_B(B^G)) = |G/K|$, the order of G/K. By hypothesis, B is a Hirata separable Galois extension of B^G with Galois group G. Hence each J_g is a projective C-module of rank 1 ([7], Theorem 2) and $\text{rank}_C(V_B(B^G)) = |G|$, the order of G ([7], Proposition 4(2)). Noting that $V_B(B^G) = \oplus \sum_{g\in G} J_g$, we conclude that $\text{rank}_C(\oplus \sum_{h\in K} J_h) = |K|$. On the other hand, since $|G| = \text{rank}_C(V_B(B^G)) = \text{rank}_{C'}(V_B(B^G))\cdot\text{rank}_C(C') = |G/K|\cdot\text{rank}_C(C')$, we have that $\text{rank}_C(C') = |K|$. Thus $\text{rank}_C(\oplus \sum_{h\in K} J_h) = |K| = \text{rank}_C(C')$. Noting that $\oplus \sum_{h\in K} J_h \subseteq J'_1 = C'$ as a direct summand, we conclude that $C' = \oplus \sum_{h\in K} J_h$.

From the proof of Proposition 3.1, we derive two equivalent conditions for a central Galois algebra $V_B(B^G)$ in terms of the rank of a projective module over a commutative ring.

Theorem 3.2.
By keeping the notations of Proposition 3.1, the following statements are equivalent:
(1) $V_B(B^G)$ is a central Galois algebra over C' with Galois group G/K;
(2) $J'_g = \oplus \sum_{h\in K} J_{gh}$ for each $g \in G$; and
(3) $\text{rank}_C(C') = |K|$, the order of K.

Proof. (1) $\Longrightarrow$ (2) By the proof of Proposition 3.1,

$$V_B(B^G) = \oplus \sum_{\overline{g}\in G/K} \sum_{h\in K} J_{gh} \subseteq \oplus \sum_{\overline{g}\in G/K} J'_g \subseteq V_B(B^G)$$

such that $\oplus \sum_{h\in K} J_{gh} \subseteq J'_g$ for each $g \in G$. Hence $J'_g = \oplus \sum_{h\in K} J_{gh}$ for each $g \in G$.

(2) $\Longrightarrow$ (1) Taking $g = 1$, we have $C' = J'_1 = \oplus \sum_{h\in K} J_h$. Hence, by Proposition 3.1, $V_B(B^G)$ is a central Galois algebra over C' with Galois group G/K.

(1) $\Longrightarrow$ (3) By the proof of the necessity of Proposition 3.1, $\text{rank}_C(C') = |K|$.

(3) $\Longrightarrow$ (1) By the proof of the necessity of Proposition 3.1 again, that $\text{rank}_C(C') = |K|$ implies that $C' = \oplus \sum_{h\in K} J_h$. Thus $V_B(B^G)$ is a central Galois algebra over C' with Galois group G/K by Proposition 3.1.

We conclude the present paper with three examples of a Hirata separable Galois extension of B^G with Galois group G of order n invertible in B such that K as given in Theorem 2.3 is $\langle 1\rangle$, G, and a proper subgroup of G, respectively.

Example 1.
Let $B = R[i,j,k]$, the real quaternion algebra over R and $G = \{1,\, g_i,\, g_j,\, g_k\}$ where $g_i(x) = ixi^{-1}$, $g_j(x) = jxj^{-1}$, and $g_k(x) = kxk^{-1}$ for all x in B. Then,

(1) B is a central Galois algebra over R with a G-Galois system: $\{a_1 = 1,\, a_2 = i,\, a_3 = j,\, a_4 = k,\, b_1 = \frac{1}{4},\, b_2 = -\frac{1}{4}i,\, b_3 = -\frac{1}{4}j,\, b_4 = -\frac{1}{4}k, \}$;

(2) $B^G = R$;

(3) B is a Hirata separable extension of R because B is an Azumaya R-algebra;

(4) By (1)-(3), B is a Hirata separable Galois extension of B^G with Galois group G of order 4 invertible in B;

(5) Since $V_B(B^G) = V_B(R) = B$, $K = \langle 1 \rangle$ as given in Theorem 2.3.

Example 2.
Let $B = R[i,j,k]$, the real quaternion algebra over R and $G = \{1,\, g_i\}$ where $g_i(x) = ixi^{-1}$ for all x in B. Then, (1) B is a Galois extension of $R[i]$ with Galois group G;

(2) B is a Hirata separable extension of $R[i]$ by Theorem 1 in [4]. Thus, B is a Hirata separable Galois extension of B^G with Galois group G of order 2 invertible in B;

(3) $V_B(B^G) = V_B(R[i]) = R[i] = B^G$, so $K = G$ as given in Theorem 2.3.

Example 3.
Let $F \subset L$ be a Galois field extension with a Galois group G such that G has a proper center Z, $B = L * G$, the skew group ring of G over L, and $\overline{G} = \{\overline{g} \mid g \in G\}$, the inner automorphism group of B induced by the elements of G. Then,

(1) B is a Galois extension of $B^{\overline{G}}$ with Galois group $\overline{G}$ isomorphic with G (for L is a Galois extension of F with Galois group G);

(2) B is a Hirata separable extension of $B^{\overline{G}}$ ([7], Corollary 3);

(3) $V_B(B^{\overline{G}}) = \oplus \sum_{\overline{g} \in \overline{G}} J_{\overline{g}} = \oplus \sum_{\overline{g} \in \overline{G}} C\overline{g}$ ([5], Proposition 1) where C is the center of B;

(4) $(\overline{G})_1 = \overline{Z}$, a proper subgroup of $\overline{G}$;

(5) $(\overline{G})_2 = (\overline{G})_1$ (for $V_B(B^{\overline{Z}}) = \oplus \sum_{\overline{g} \in \overline{Z}} C\overline{g}$). Thus $K = (\overline{G})_1 = \overline{Z}$ as given in Theorem 2.3.

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