Scientiae Mathematicae Japonicae Online, **e-2009**, 185–201

TALMUDIC BANKRUPTCY PROBLEM: SPECIAL AND GENERAL SOLUTIONS

Isaac Elishakoff* and André Bégin-Drolet**

Received March 21, 2008; Revised January 3, 2009

Abstract. An elementary solution is presented to the special bankruptcy problem described in the Babylonian Talmud. The presented solution pertains to the numerical data presented in the Tractate of Kethuboth 93a. General solution for any combination of data is also given. Solution of the special case appears to be suitable for inclusion in courses on economy, as well as in group study environments.

1 Introduction Modern solutions of division problems are based, primarily on the idea of proportionality, advocated by Aristotle: "A just act necessarily involves at least four terms: two persons for whom it is in fact just, and two shares in which its justice is exhibited. And there will be the same equality between the shares as between the persons, because the shares will be in the same ratio to one another as the persons what is just in this sense, then, is what is proportional, and what is unjust is what violates the proportion." Oppenheim [17] interprets the Aristotelian approach as follows: "Aristotle himself enlarged the criterion of egalitarianism to include rules which allot "equal shares to equals"; i.e. equal shares of some specified kind to all who are equal with respect to some specified characteristic. Conversely, a rule is inegalitarian when either equals are awarded unequal shares or unequals equal shares." However, "proportionality" is not necessarily synonymous with "justice". Impressive examples of fair division which do not use proportionality in the usual setting are given in the Mishna [24, 25], a 1800-year old document that forms the basis for Jewish civil, criminal, and religious law. This paper deals with a division problem discussed in the Mishna, tractate Kethuboth [28]. The particular allocations mentioned there, in the terminology of Aumann and Maschler [2] "look mysterious; but whatever they may mean, they do not fit any obvious extension of either equal or proportional division. Over two millennia, this Mishna has spawned a large literature. Many authorities disagree with it outright. Others attribute the figures to special circumstances not made explicit in the Mishna. A few have attempted direct rationalization of the figures as such, mostly with little success. One modern scholar, exasperated by his inability to make sense of the text, suggested errors in transcription. In brief, the passage is notoriously difficult." For this problem, which we have not yet introduced, Aumann and Maschler [2] provided an algorithm for solution in their breakthrough paper via the game-theoretic approach, utilizing the nucleolus concept. They presented, in their own words, "three different justifications of the solution to the bankruptsy problem that the nucleolus prescribes in terms that are independent of each other and of game theory, and that were well within the reach of the sages of the Mishna." A remarkable fact is that the non-game-theoretic approaches were identified by Aumann and Maschler [2] only after they had found the solution via game-theoretic analysis; again in their own words, "only after realizing that the numbers in the Mishna correspond to the nucleolus did we find independent rationales. Without the

2000 *Mathematics Subject Classification.* 01A07 ethnomathematics, 91B mathematical economics, 97D education and instruction in mathematics.

Key words and phrases. Bankruptcy; division problems; Talmud.

game theory, it is unlikely that we would have hit on the [non-game-theoretic] analysis." This paper provides both elementary solution of the specific case described in Talmud and general solution to the problem for arbitrary claims and arbitrary value of the estate, for the three creditor problem. As first step, it is instructive to describe yet another (albeit simpler) problem of division, also provided in the Mishna.

2 Contested Garment Problem O'Neil [16] explains : "The Babylonian Talmud is the great collection of Jewish religious and legal decisions set down during the first five centuries [of the Common Era]. It includes two kinds of teachings, the Mishna, which are short statements of the law copied down from the oral heritage of past centuries, and the Gemara, which are commentary on the Mishna by the rabbis of that time. The book [Bava Metzia] is dealing with contracts, leases, sales and found objects." It is customary to study Talmud's identical pages daily and internationally. On March 1, 2005 tens of thousands of participants joined together to celebrate an event called Siyum HaShas, associated with completion of seven-and-one-half years of study "at an inexorable page-a-day pace, known in Hebrew as Daf Yomi." Kobre [11] writes a of somewhat analogous affair: "Some years back, there were initiatives, in Chicago and elsewhere to get citizens city-wide reading the same book at the same time-it began with "To kill a Mockingbird", as I recall-thereby fostering an intellectual conversation across the social divides such as class and race. [In South Florida, some years ago, Ray Bradbury's "Fahrenheit 451", was read analogously-I.E. & A.B.D.]. The Daf Yomi program is that and much more. Not a place-it spans the globe; not a time-one can study the Talmud in person, via phone, the internet, shortwave radio; but a vehicle for bankers and bakers, athletes and aesthetes and all the rest, to come together to explore the riches of four millennia"

The problem opening Tractate Bava Metzia (Middle Gate) [24, 27] reads as follows: "Two [persons appearing before a court] hold a garment. One of them says, "I found it" and the other says, "I found it"; one of them says, "It is all mine," and the other says, "It is all mine;" then one shall swear that his share in it is not less than half, and the other shall swear that his share in it is not less than half, and [the value of the garment] shall then be divided [equally] between them." The ruling appears to be in accord with common sense and with the proportionality rule. Let us now consider a more challenging case (Bava Metzia 2a) [24, 27]: "If one says, "It is all mine," and the other says "Half of it is mine," he who says "It is all mine" shall swear that his share in it is not less than three quarters, and he who says "Half of it is mine" shall swear that his share in it not less than a quarter. The former then receives three quarters [of the value of the garment] and the latter receives one quarter." One would expect, if one follows proportionality-based logic that since the first litigant claims twice the amount of the second one, the court should allocate to him two-thirds of the value of the garment, and the remaining third-to the other. That is what simple mathematics would suggest: divide in proportion to the claims! But such is not the case. As Aumann an Maschler [2] note, "The principle is clear. The lesser claimant concedes half the garment to the greater one. It is only the remaining half that is at issue, this remaining half is, therefore, divided equally. Note that this is quite different from proportional division." Aumann and Maschler [2] also stress: "This explanation is explicit in the eleventh century commentary of Rabbi Shlomo Yitzhaki (Rashi)." Rashi writes: "In this case, both agree that half of the garment belongs to the first litigant; their dispute relates only to the second half. Consequently, as in the Mishna's first case, each takes an oath substantiating half his claim to the disputed article and then divide it. According, the first litigant then takes three quarters [half the garment, which was never in dispute, plus half of the disputed second half]; the second litigant takes one-fourth [half the disputed second half]." Aumann and Maschler [2] provide an additional explanation: "Alternatively, one could say that the claims total $1\frac{1}{2}$, whereas the worth of the garment is only 1; the loss is shared equally." Balinski [3] concurs with this note, when he observes "The second precept for the Talmud is: share to equalize the losses" (italics by Balinski). Young [22, p.67] articulates the contested garment rule as follows: "Let two individuals have positive claims against a common asset, where the sum of the claims exceeds (or equals) the total amount available. Each claimant's uncontested portion is the amount left over after the other claimant has either been paid all that is available (whichever is less). The contested garment rule gives each claimant his uncontested portion plus one-half of the excess over and above the sum of the uncontested portions."

3 Division Problem from Tractate Kethuboth A man passes away; his estate is sued by three contestants-his widows. One may contend that such a situation cannot arise at present, most of the world being monogamous. However, one can visualize a bankruptcy situation, in which generally unequal claims are made against the estate. The Talmud considers the case in which a man undertook to pay his first wife 100 zuzim (The zuz (pl.zuzim) was the monetary unit of that period.) in case of his death or divorce; the second wife bequeathed 200 zuz the third wife has a promissory note of 300 zuz. If the deceased left an estate of 600 zuz, the division is very simple: each of the litigants gets what her promissory note indicates. The decision-making is more complicated if the estate left is less than 600 units. The Talmud considers three cases, listed in Table I below.

Table I

Case numbers	Value of estate	Claims			Awarded shares		
		1^{st} claimant	2^{nd} claimant	3^{rd} claimant	1^{st} claimant	2^{nd} claimant	3^{rd} claimant
1	100				$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
2	200	100	200	300	50	75	75
3	300				50	100	150

The rulings appear puzzling at first glance. Although the claims are unequal, in the first case the awards are equal, while in the second case the second and third litigants get the same amount. The division seems to be totally un-Aristotelian in the first two cases, and is proportional only in the third. As Nitsan [15] notes, "The division in the second case is totally incomprehensible. Despite this fact, one can assume that one rule can explain all the divisions. Talmudic sages do not mention it explicitly; neither the interpreters succeeded in nearly two thousand years to decipher the rule that is behind the Mishnaic division." According to Malkevitch [14], "The biblical scholars could make sense of the first and last lines of the table but were puzzled by the middle line. What was the method being used here? Was there a "copying error?" Over the years had some data been changed which resulted in the second line's being erroneous?" One can try to enter the shoes of the first litigant. Let the claims court be informed that the estate equals 200 zuz and the awards set at $A_1 = 50$, $A_2 = 75$ and $A_3 = 75$. After announcing this decision, the accountant of the estate appears before the court and announces that another 100 zuzim have been found, so that the estate actually constit utes 300 zuz. The first litigant hopes that her share will increase, since the estate is now 50% larger, but in the end is disappointed-the share remains the same. One would be hard put to convince her at least in the beginning that the ruling is just! The general algorithm to the solution of this problem was pioneered by Aumann and Maschler [2] in terms of game theory. Hereinafter, a simple solution will be provided first, for the special cases described in the Talmud. We will utilize Aumann and Maschler's [2] breakthrough observation that division of the contested garment among any combination of two litigants out of three should be obtained as a particular case of the problem under study, namely, allocation of funds between three contestants. In the second part of the paper a general solution will be presented, for any combination of data.

4 An Elementary Solution of Bankruptcy Problem for Special Case of Estate of 200 zuz In this particular case, consider the "coalition" of the second and third litigants. The second claims 200 zuz, or the whole estate; the third claims 300 zuz, but has to adjust her claim to the available total of 200, so that the whole estate is contested between them, whereas the first litigant claims only half of it. The case is thus analogous to the garment-contested problem of whole against one-half. In the latter case the litigant who claimed the whole garment was awarded 3/4 of the garment, and the other 1/4 of it. Three quarters of 200 zuz for the second and third litigants combined constitute 150 zuzim. Since the second and third litigants claim equal shares, i.e. the whole available amount of 200 zuzim, they should split the 150 zuz equally, 75 zuzim for each. The first litigant is awarded $1/4 \times 200 = 50$ zuz. The "mysterious" ruling is thus shown to be justified. The following quote from Aumann and Maschler [2] is also pertinent here: "the early medieval authority Rabbi Hai Gaon (10^{th}century) did express the opinion (quoted by Rabbi Isaac Alfasi (1013-1103) in his commentary on our Mishnah in Kethuboth) should be explained on the basis of that in Bava Metzia. He did not, however, make an explicit connection, and in subsequent years, this line of attack was abandoned." The affinity between the awards in Bava Metzia [24] and Kethuboth [25] is called by Aumann and Maschler [2] as contested-garment consistency (or, for short, CG-consistency). They define: "The CG-consistent (or simply consistent) rule is that one that assigns the CG-consistent solution to each bankruptcy problem." This consistency property was discussed, albeit in a different context, by Balinski and Young [4] in their co-authored book, and summed up in the following snappy statement: "An inherent principle of any fair division is that every part of a fair division should be fair." Furthermore, they stress: "For example, one property of a fair division of an inheritance should be that no subset of heirs would want to make trades after the division is made. The principle is very general."

It appears instructive, following Aumann and Maschler [2], to quote from the Jerusalem Talmud: "Samuel says, the Mishna takes it that the creditors empower each other; specifically, that the third empowers the second to deal with the first. She may say to her, your claim is 100, right? Take 50 and go." It appears that the elementary solution given in this section illuminates the details of Samuel's statement. Indeed, Samuel does not explain why the first litigant should get 50 and not, say 49 or 51; had an explanation be given, the passage from Kethuboth would not be characterized, in terminology of Auman and Maschler [2] as "notoriously difficult." It is also possible that to Samuel it was clear why the first litigant should be awarded 50 zuz, but his proof was lost.

5 Solution in Terms of Singularity Functions Let us denote the awards due to the creditors as A_1, A_2 and A_3 the subscript denoting the serial number of the creditor. The creditors are numbered in correspondence to their demands D_1, D_2 and D_3, the subscript indicating the number of the associated creditor. Moreover,

$$D_1 \leq D_2 \leq D_3 \tag{1}$$

The creditors can argue their respective claim separately form a coalition with any other creditor. Let us denote by $E_{12} = A_1 + A_2$, the sum of awards obtained by the first and second creditors together; $E_{13} = A_1 + A_3$ denotes the sum of awards received by the first and the third creditors jointly, whereas $E_{23} = A_2 + A_3$ indicates the sum of awards gotten by the second and the third creditors in concert. The coalition awards E_{jk} are distributed between the creditor j and k according to the contested garment rule. Therefore, if we assume that E_{jk} are known, we can write the expression for the awards due to the first creditor as follows,

$$A_1 = \frac{1}{2}\big(\langle E_{12} - D_2\rangle + E_{12} - \langle E_{12} - D_1\rangle\big) \tag{2}$$

$$A_1 = \frac{1}{2}\big(\langle E_{13} - D_3\rangle + E_{13} - \langle E_{13} - D_1\rangle\big) \tag{3}$$

Where $\langle x - a\rangle$ is Macaulay's bracket, it equals $x - a$ if $x > a$ and zero otherwise. The former equation designates the value of the award A_1 if the coalition of the first and second creditors is formed. The latter formula indicates how the award A_1 should calculated from the coalitional award E_{13} of the first and third creditors jointly. Since singularity functions are involved we can anticipate some transition values and respective critical values of the estate at which the expressions of distribution will change.

6 Distribution of the Estate when the Latter Does Not Exceed its First Critical Value We will first concentrate on Eq. (3). Since $D_3 \geq D_1$ from eq. (1),

$$E_{13} - D_1 \geq E_{13} - D_3 \tag{4}$$

First critical value $E_{c,1}$ of the estate E is obtained when

$$E_{13} = D_1 \tag{5}$$

When

$$E = E_{c,1} \tag{6}$$

two following equalities take place simultaneously,

$$\langle E_{13} - D_1\rangle = 0 \tag{7}$$

$$\langle E_{13} - D_3\rangle = 0 \tag{8}$$

From Eq. (2) we get,

$$A_1 = \frac{1}{2}E_{13} \tag{9}$$

Since

$$E_{13} = A_1 + A_3 \tag{10}$$

we obtain in view of Eq. (9)

$$2A_1 = A_1 + A_3 \tag{11}$$

leading to the conclusion that

$$A_1 = A_3 \tag{12}$$

At this stage we do not know the value of A_2. Hence we are not in a position to determine the values of A_1 and A_3.

We resort to Eq. (2) which evaluates the share of the first creditor from the coalitional award E_{12}. Since $D_3 \geq D_2$ from eq. (1), naturally,

$$E_{13} \geq E_{12} \tag{13}$$

We established that in view of Eq. (7) that $\langle E_{13} - D_1 \rangle = 0$. Therefore, even more so $\langle E_{12} - D_1 \rangle = 0$. Therefore, $\langle E_{12} - D_2 \rangle = 0$. . Since both terms in singularity brackets in Eq.(2) vanish, we are left with

$$A_1 = \frac{1}{2} E_{12} \tag{14}$$

Since

$$E_{12} = A_1 + A_2 \tag{15}$$

we arrive at the following conclusion

$$2A_1 = A_1 + A_2 \tag{16}$$

and

$$A_1 = A_2 \tag{17}$$

by combining Eqs. (14) and (15). Thus, if the estate does not exceed the critical value Ec,1, the distribution of awards is equal, in view of Eqs. (12) and (17):

$$A_1 = A_2 = A_3 \tag{18}$$

now, since

$$A_1 + A_2 + A_3 = E \tag{19}$$

i.e. the total estate, we conclude that each creditor is awarded

$$A_1 = A_2 = A_3 = \frac{1}{3}E \tag{20}$$

This implies that

$$E_{12} = E_{13} = E_{23} = \frac{2}{3}E \qquad \text{for } \ 0 \leq E \leq E_{c,1} \tag{21}$$

The critical value$E_{c,1}$ of the estate E is obtained from Eq. (5) and Eq. (21)

$$E_{c,1} = \frac{3}{2}D_1 \tag{22}$$

Thus, we established the first critical value of the estate, namely ($E_{c,1} = \frac{3}{2}D_1$) as well as the attendant distribution:

$$E_{12} = E_{13} = E_{23} = \frac{2}{3}E \qquad \text{for } \ 0 \leq E \leq (3/2)D_1 \tag{23}$$

7 Establishment of the Second Critical Value of the Estate and Attendant Distribution Consider now the case when the available estate exceeds the first critical value $E_{c,1} = (3/2)D_1$. In these circumstances,

$$E_{13} > D_1 \tag{24}$$

Hence

$$\langle E_{13} - D_1 \rangle = E_{13} - D_1 \tag{25}$$

and

$$\langle E_{12} - D_1 \rangle = E_{12} - D_1 \tag{26}$$

The Eqs. (2) and (3) reduce to:

$$A_1 = \frac{1}{2}\big(\langle E_{12} - D_2 \rangle + D_1\big) = \frac{1}{2}\big(\langle E_{13} - D_3 \rangle + D_1\big) \tag{27}$$

Another critical value Ec,2 of the estate is found from the condition that the following equalities take place simultaneously

$$\langle E_{13} - D_3 \rangle = 0, \quad \text{or} \quad E_{13} \leq D_3 \tag{28}$$

$$\langle E_{12} - D_2 \rangle = 0, \quad \text{or} \quad E_{12} \leq D_2 \tag{29}$$

If the estate satisfies the inequality

$$E_{c,1} \leq E \leq E_{c,2} \tag{30}$$

The expressions in singularity brackets in Eq. (27) vanish, and

$$A_1 = \frac{1}{2} D_1 \tag{31}$$

Eq. (29) signifies that

$$E_{12} = A_1 + A_2 \leq D_2 \tag{32}$$

resulting in

$$A_2 \leq D_2 - A_1 \tag{33}$$

Or, in view of Eq. (31)

$$A_2 \leq D_2 - \frac{1}{2} D_1 \tag{34}$$

The award to the third creditor is obtained from the demand

$$E_{13} - D_3 \leq 0 \tag{35}$$

stemming from Eq. (28). Then

$$A_3 = E_{13} - A_1 \leq D_2 - A_1 = D_3 - \frac{1}{2} D_1 \tag{36}$$

To summarize partially, in the region defined by Eq. (30)

$$A_1 = \frac{1}{2} D_1, \quad A_2 \leq D_2 - \frac{1}{2} D_1, \quad A_3 \leq D_3 - \frac{1}{2} D_1 \tag{37}$$

The critical value $E_{c,2}$ of the estate E is obtained by observing the sum of the maximum possible awards specified in Eq. (37). When inequalities in Eq. (37) turn into equalities, the critical value is obtained. Therefore,

$$E_{c,2} = \frac{1}{2}(2D_2 + 2D_3 - D_1) \tag{38}$$

Consider now the case when the following inequality holds

$$E \geq E_{c,2} \tag{39}$$

Then

$$\langle E_{13} - D_3 \rangle \geq 0, \quad \langle E_{12} - D_2 \rangle \geq 0 \tag{40}$$

The Eqs. (2) and (3) for A_1 become

$$A_1 = \frac{1}{2}(E_{12} - D_2 + D_1) = \frac{1}{2}(E_{13} - D_3 + D_1) \tag{41}$$

Substituing Eqs. (10) and (15) in Eq. (40) and taking into account Eq (19) we arrive at

$$A_1 = \frac{1}{3}(E - D_2 - D_3 + 2D_1) \tag{42}$$

To derive the expression forA_2, we resort to the general expressions for A_2,

$$A_2 = \frac{1}{2}\big(\langle E_{12} - D_1 \rangle + E_{12} - \langle E_{12} - D_2 \rangle\big) \tag{43}$$

$$A_2 = \frac{1}{2}\big(\langle E_{23} - D_3 \rangle + E_{23} - \langle E_{23} - D_2 \rangle\big) \tag{44}$$

The Eq. (43) is compatible with Eq. (1), and represents the second creditor's share from the joint allocation E_{12} of the first and second creditors. Likewise, Eq. (44) corresponds to Eq. (2) and represents a share of the second creditor form the joint allocation E_{23} of the second and third creditors.

8 Establishment of the Values $\mathbf{A_2}$ and $\mathbf{A_3}$ for $\mathbf{E_{c,1} \leq E \leq E_{c,2}}$. Once the value of A_3 is known the problem is reduced to the distribution of the amount $E_{23} = E - A_1$ between second and third litigants. For $E_{c,1} \leq E \leq E_{c,2}$, we have, from eq.31 $A_1 = 1/2D_1$. We then express E_{23} in terms of D_1, namely, $E_{23} = E - 1/2D_1$. We observe that the awards A_2 and A_3 are obtainable from the contested garment rule

$$A_2 = \frac{1}{2}\big(\langle E_{23} - D_3 \rangle + E_{23} - \langle E_{23} - D_2 \rangle\big), \quad A_3 = \frac{1}{2}\big(\langle E_{23} - D_2 \rangle + E_{23} - \langle E_{23} - D_3 \rangle\big) \tag{45}$$

Since $D_3 \geq D_2$, the following inequality holds

$$E_{23} - D_2 \geq E_{23} - D_3 \tag{46}$$

The critical value $E_{c,3}$ of the estate E is obtained when

$$E_{23} = D_2 = E - \frac{1}{2}D_1 \tag{47}$$

since then the expression $\langle E_{23} - D_2 \rangle$ vanishes. Thus,

$$E_{c,3} = D_2 + \frac{1}{2}D_1 \tag{48}$$

Let the estate E be less than this value, $E_{23} - D_3 \le 0$. Then $\langle E_{23} - D_2 \rangle = 0$ and $\langle E_{23} - D_3 \rangle = 0$ Substituting these into Eq. (45), we get

$$A_2 = A_3 = \frac{1}{2} E_{23} = \frac{1}{2}\left(E - \frac{D_1}{2}\right) = \frac{1}{4}(2E - D_1) \tag{49}$$

After this critical value $E_{c,3}$ is reached, the inequality $E_{23} - D_2 \ge 0$ is holding, implying that $\langle E_{23} - D_2 \rangle = E_{23} - D_2 \ge 0$. Therefore,

$$A_2 = \frac{1}{2}\big(\langle E_{23} - D_3 \rangle + D_2\big), \quad A_3 = \frac{1}{2}\big(2E_{23} - D_2 - \langle E_{23} - D_3 \rangle\big) \tag{50}$$

The latter expression signifies that there is yet another critical value $E_{c,4}$ of the estate E, reached when $E_{23} = D_3$, i.e. $D_3 = E - 1/2D_1$, leading to

$$E_{c,4} = D_3 + \frac{1}{2} D_1 \tag{51}$$

Prior to reaching this critical value, in the range $E_{c,3} \le E \le E_{c,4}$, we have

$$A_2 = \frac{1}{2} D_2, \quad A_3 = \frac{1}{2}(2E_{23} - D_2) = \frac{1}{2}(2E - D_1 - D_2) \tag{52}$$

Beyond this critical value, for $E \ge E_{c,4}$

$$A_2 = \frac{1}{2}\big(E_{23} - D_3 + D_2\big) = \frac{1}{2}\big(2E - D_1 - 2D_3 + 2D_2\big)$$

$$A_3 = \frac{1}{2}\big(2E_{23} - D_2 - E_{23} + D_3\big) = \frac{1}{2}\big(E_{23} - D_2 + D_3\big) = \frac{1}{4}\big(2E - D_1 - 2D_2 + 2D_3\big) \tag{53}$$

9 Establishment of the Values $\mathbf{A_2}$ and $\mathbf{A_3}$ for $\mathbf{E \ge E_{c,2}}$ For the joint award to the second and the third claimants we have

$$E_{23} = E - A_1 = E - \frac{1}{3}(E + 2D_1 - D_2 - D_3) = \frac{1}{3}(2E - 2D_1 + D_2 + D_3) \tag{54}$$

Eq. (45) yields, in view of the inequality $D_3 \ge D_2 \ge E_{23}$, the following value for A_2:

$$A_2 = \frac{1}{2}(E_{23} - D_3 + D_2) = \frac{1}{2}(2E - 2D_1 + D_2 + D_3 - 3D_3 + 3D_2)$$

$$\Longrightarrow A_2 = \frac{1}{3}(E + 2D_2 - D_1 - D_3) \tag{55}$$

From Eq. (45), in view of inequality $D_3 \ge D_2 \ge E_{23}$, we get

$$A_3 = \frac{1}{2}(E_{23} - D_2 + D_3) = \frac{1}{2}(2E - 2D_1 + D_2 + D_3 - 3D_2 + 3D_3)$$

$$\Longrightarrow A_3 = \frac{1}{3}(E + 2D_3 - D_1 - D_2) \tag{56}$$

10 Summary of Results In the previous section, 4 critical estate values have been identified. It makes instructive to order them in the ascending order. Thereinafter the critical values will be supplied with a subscript in the form of the Roman numeral:

$$
\begin{aligned}
E_I &= E_{c,1} = \frac{3}{2}D_1 \\
E_{II} &= E_{c,3} = D_2 + \frac{1}{2}D_1 \\
E_{III} &= E_{c,4} = D_3 + \frac{1}{2}D_1 \\
E_{IV} &= E_{c,2} = D_2 + D_3 - \frac{1}{2}D_1
\end{aligned}
\tag{57}
$$

The general formulas for the awards are given below. The first contestant is awarded:

$$
\begin{array}{ll}
A_1 = \frac{1}{3}E & \text{if } E \le E_I \\
A_1 = \frac{1}{2}D_1 & \text{if } E_I \le E \le E_{IV} \\
A_1 = \frac{1}{3}(E - D_2 - D_3 + 2D_1) & \text{if } E \ge E_{IV}
\end{array}
\tag{58}
$$

The second claimant's award is:

$$
\begin{array}{ll}
A_2 = \frac{1}{3}E & \text{if } E \le E_I \\
A_2 = \frac{1}{4}(2E - D_1) & \text{if } E_I \le E \le E_{II} \\
A_2 = \frac{1}{2}D_2 & \text{if } E_{II} \le E \le E_{III} \\
A_2 = \frac{1}{2}(2E - D_1 - 2D_3 + 2D_2) & \text{if } E_{III} \le E \le E_{IV} \\
A_2 = \frac{1}{3}(E - D_1 - D_3 + 2D_2) & \text{if } E \ge E_{IV}
\end{array}
\tag{59}
$$

The third litigant's award constitutes:

$$
\begin{array}{ll}
A_3 = \frac{1}{3}E & \text{if } E \le E_I \\
A_3 = \frac{1}{4}(2E - D_1) & \text{if } E_I \le E \le E_{II} \\
A_3 = \frac{1}{2}(2E - D_1 - D_2) & \text{if } E_{II} \le E \le E_{III} \\
A_3 = \frac{1}{2}(2E - D_1 - 2D_2 + 2D_3) & \text{if } E_{III} \le E \le E_{IV} \\
A_3 = \frac{1}{3}(E - D_1 - D_2 + 2D_3) & \text{if } E \ge E_{IV}
\end{array}
\tag{60}
$$

11 Specific Examples Consider first the example from the Kethuboth 93a:

$$D_1 = 100, \quad D_2 = 200, \quad D_3 = 300$$

The critical values are:

$$E_I = 150, \quad E_{II} = 250, \quad E_{III} = 350, \quad E_{IV} = 450$$

For $E = 100$, we observe that $E < E_I$. Therefore,

$$A_1 = A_2 = A_3 = \frac{1}{3}E = 33\frac{1}{3}$$

For $E = 200$, we observe that $E_I < E < E_{II}$. Therefore,

$$A_1 = \frac{1}{2}100 = 50, \quad A_2 = \frac{1}{4}(2 \cdot 200 - 100) = 75, \quad A_3 = \frac{1}{4}(2 \cdot 200 - 100) = 75$$

For $E = 300$, we observe that $E_{II} < E < E_{III}$. Thus,

$$A_1 = \frac{1}{2}100 = 50, \quad A_2 = \frac{1}{2}200 = 100, \quad A_3 = \frac{1}{2}(2 \cdot 300 - 100 - 200) = 150$$

Table II in the Appendix A lists values of the awards when the estate varies between 0 to 600 zuz, by increments of 10 zuz.

Consider now the numerical data kindly communicated to us by Professor Velleman [20]:

$$D_1 = 30, \quad D_2 = 60, D_3 = 90$$

In this case the critical values are:

$$E_I = 45, \quad E_{II} = 75, \quad E_{III} = 105, \quad E_{IV} = 135$$

The specific value of the estate suggested by Professor Velleman [20] equals 150, or$E > E_{IV}$. The distribution equals:

$$\begin{aligned} A_1 &= \frac{1}{3}(150 - 60 - 90 + 2 \cdot 30) = 20 \\ A_2 &= \frac{1}{3}(150 - 30 - 90 + 2 \cdot 60) = 50 \\ A_3 &= \frac{1}{3}(150 - 30 - 60 + 2 \cdot 90) = 80 \end{aligned}$$

It is easy to check that the "coalitional" award $A_{12} = A_1 + A_2 = 70$ is contested-garment consistent. Indeed, these claimants demand in total $30 + 60 = 90$. Their joint loss is $90 - 70 = 20$ which is divided equally. Hence the first claimant gets $30 - 10 = 20$, while the second receives $60 - 10 = 50$. Likewise, one can show that the distribution is pairwise contested-garment consistent.

Consider another specific case with $E = 60$, then $E_I < E < E_{II}$. Then:

$$\begin{aligned} A_1 &= \frac{1}{2}30 = 15 \\ A_2 &= \frac{1}{4}(2 \cdot 60 - 30) = 22\frac{1}{2} \\ A_3 &= \frac{1}{4}(2 \cdot 60 - 30) = 22\frac{1}{2} \end{aligned}$$

For this particular case one doesn't need the general formulation. One can resort to a mere elementary reasoning as was done in Section 4. The second claimant demands the whole estate. Likewise, the third claimant demands the whole estate. Together too, since there is not more available than the entire estate, they demand the whole estate, while the first claimant demands the half of it. Hence, according to the contested garment case, the first claimant is awarded three quarters of the estate, or half of his demand, i.e. 15, whereas the second and third claimants are jointly awarded 45. This amount they share equally, since each of them has the same claim on this amount. Thus each of them ought to be awarded $22\frac{1}{2}$. Table III in the Appendix B lists the values of the awards when the estate varies between 0 to 180 zuz, by increments of 10 zuz.

12 Discussion Referring to the problem of division in Kethuboth [25] Balinski [3] asks in his recent paper: "why the curious division when the total estate is worth 200 zuz?" and answers: "This enigma was finally answered in 1985, after twenty centuries of debate, via an esoteric concept in the theory of games. The situation described by Table I was modeled as a game and various known concepts of a "solution" were tried: the "nucleolus" agreed!" Our interest in this problem was prompted by Aumann and Maschler's [2] pioneering paper. In his interview with Aumann, Hart [9] says (see p.23 there): "religion dictates certain rules of behavior. These rules, first of all, are not well defined. They are interpreted by human beings. Second, these rules may be justified in a rational way. Like in your work with Michael Maschler, where you give a game-theoretic interpretation of a passage from the Talmud that nobody could understand and suddenly everything became crystal clear." It is hoped that the present elementary discourse will magnify the understanding of the division problem in Kethuboth [25, 28].

In describing the impact of their study, Aumann and Maschler [2] noted: "It is hoped that the research will be of interest in two spheres-in the study of the Talmud, and in game theory." Analogous hope is humbly shared by the authors of the present contribution. Thomas Schelling, co-laureate of the Nobel Prize in economics with Aumann, noted in the interview Steelman [19] that "The greatest advance in mathematics is the equal sign." It appears that another similarly valid statement can be made: "The greatest advance in interpersonal harmony is the equity concept." According to Balinski [3], the great French novelist Victor Hugo stated: "The material world depends on equilibria, the moral world on equity." The elementary solution of the special Talmudic case presented herein can be directly incorporated in courses on equity, and directed at a wide audience.

We know about two basic means of distribution: The first one can loosely be called 'socialistic' and is associated with egalitarianism (everybody gets the same amount irrespective of the amount of the claim; we can be remined the verse of Russian poet Vladimir Mayakovsky: "All that is yours, is mine, except the tooth brush!"). The other one can roughly be called 'capitalistic', and is associated with proportional distribution (everybody gets a proportional share of the estate for any size of the estate). The amazing fact that stems from the Talmudic division problem is that both are the two facets of the more general three-faceted phenomenon, and can be derived from the unified principle. The third facet is neither 'socialistic' nor 'capitalistic.' It appears appropriate to conclude this paper by another quote from Aumann's [1] study: "It's like "Alice in Wonderland". The game theory provides the key to the garden, which Alice had such great difficulty in obtaining. Once in the garden, though, Alice can discard the key; the garden can be enjoyed without it."

This study derives both on elementary explanation of the particular numerical example in the Talmud, as well as provides with the general solution, without resorting to the game theory on the principle "more that half is like the whole" (see Ref.[2], P.204). Aumann and Maschler [2] provide an algorithm for any number of claimants; we furnish straightforward solution of the original problem, apparently reconstructing the thinking process of the author of the Mishna Rabbi Nathan.

13 Acknowledgements Thanks are due to Mr. Christopher Brewster for his contribution to section 4, to Ing. Arie Aloni for bringing to our attention the paper by O. Nitsan, to Ing. Eliezer Goldberg for discussion, and Ing. Demetris Pentaras for kindly typing the preliminary versions of the manuscript. We are grateful to Professor Daniel J. Velleman of the Amherst College for suggesting a numerical example for consideration.

Appendix A

Table II: Values of the Awards as a Function of the Estate (Example from Ketuboth 93a)

Estate	Award (1)	Award (2)	Award (3)	Estate	Award (1)	Award (2)	Award (3)
0	0.00	0.00	0.00	300	50.00	100.00	150.00
10	3.33	3.33	3.33	310	50.00	100.00	160.00
20	6.67	6.67	6.67	320	50.00	100.00	170.00
30	10.00	10.00	10.00	330	50.00	100.00	180.00
40	13.33	13.33	13.33	340	50.00	100.00	190.00
50	16.67	16.67	16.67	350	50.00	100.00	200.00
60	20.00	20.00	20.00	360	50.00	105.00	205.00
70	23.33	23.33	23.33	370	50.00	110.00	210.00
80	26.67	26.67	26.67	380	50.00	115.00	215.00
90	30.00	30.00	30.00	390	50.00	120.00	220.00
100	33.33	33.33	33.33	400	50.00	125.00	225.00
110	36.67	36.67	36.67	410	50.00	130.00	230.00
120	40.00	40.00	40.00	420	50.00	135.00	235.00
130	43.33	43.33	43.33	430	50.00	140.00	240.00
140	46.67	46.67	46.67	440	50.00	145.00	245.00
150	50.00	50.00	50.00	450	50.00	150.00	250.00
160	50.00	55.00	55.00	460	53.33	153.33	253.33
170	50.00	60.00	60.00	470	56.67	156.67	256.67
180	50.00	65.00	65.00	480	60.00	160.00	260.00
190	50.00	70.00	70.00	490	63.33	163.33	263.33
200	50.00	75.00	75.00	500	66.67	166.67	266.67
210	50.00	80.00	80.00	510	70.00	170.00	270.00
220	50.00	85.00	85.00	520	73.33	173.33	273.33
230	50.00	90.00	90.00	530	76.67	176.67	276.67
240	50.00	95.00	95.00	540	80.00	180.00	280.00
250	50.00	100.00	100.00	550	83.33	183.33	283.33
260	50.00	100.00	110.00	560	86.67	186.67	286.67
270	50.00	100.00	120.00	570	90.00	190.00	290.00
280	50.00	100.00	130.00	580	93.33	193.33	293.33
290	50.00	100.00	140.00	590	96.67	196.67	296.67
300	50.00	100.00	150.00	600	100.00	200.00	300.00

Appendix B

Table III: Values of the Awards as a Function of the Estate (Professor Velleman's data)

Estate	Awarded (1)	Awarded (2)	Awarded (3)
0	0.00	0.00	0.00
5	1.67	1.67	1.67
10	3.33	3.33	3.33
15	5.00	5.00	5.00
20	6.67	6.67	6.67
25	8.33	8.33	8.33
30	10.00	10.00	10.00
35	11.67	11.67	11.67
40	13.33	13.33	13.33
45	15.00	15.00	15.00
50	15.00	17.50	17.50
55	15.00	20.00	20.00
60	15.00	22.50	22.50
65	15.00	27.50	27.50
75	15.00	30.00	30.00
80	15.00	30.00	35.00
85	15.00	30.00	40.00
90	15.00	30.00	50.00
100	15.00	30.00	55.00
105	15.00	30.00	60.00
110	15.00	32.50	62.50
115	15.00	35.00	65.00
120	15.00	37.50	67.50
125	15.00	40.00	70.00
130	15.00	42.50	72.50
135	15.00	45.00	75.00
140	16.67	46.67	76.67
145	18.33	48.33	78.33
150	20.00	50.00	80.00
155	21.67	51.67	81.67
160	23.33	53.33	83.33
165	25.00	55.00	85.00
170	26.67	56.67	86.67
175	28.33	58.33	88.33
180	30.00	60.00	90.00

References

[1] R.J. Aumann, Game Theory in the Talmud, Research Bulletin Series on Jewish Law and Economics, Bar Ilan University, 2002.

[2] R.Y. Aumann and M. Maschler, Game theoretic analysis of a bankruptcy problem from the Talmud, J. Econ. Theory, 36 (1985).

[3] M. L. Balinski, What is just?, The Mathematical Association of America Monthly, 112 (2005) 502-511.

[4] M. L. Balinski and H. P. Young, Fair Representation: Meeting the Ideal of One Man, One Vote, pp. 44, 141, Yale University Press, New Haven, 1982.

[5] J. Benoit, The nucleolus is contested-garment consistent: a direct proof, J. Econ. Theory, 77 (1997) 192-196.

[6] J. Bohannon, Two honored for the theory and practice of game theory, Science, 310, No. 5746 (2005) 212.

[7] A. Brandenburger, H. W. Stuart Jr. and B. J. Nalebuff, A Bankruptcy Problem from the Talmud, Harvard Business Online, Jan. 26, (1995).

[8] Y. Chun, The equal-law principle for bargain problems, Economics Letters, 26 (1988) 103-106.

[9] S. Hart, An interview with Robert Aumann, http://www.ma.huji.ac.il/ hart/abs/aumann .html, in Macroeconomic Dynamics, 9, 2005 683-740.

[10] T. P. Hill, Mathematical devices for getting a fair share, American Scientist, 88, 4 (July/August 2000) 316-324.

[11] E. Kobre, Full cycle: celebrating 2711 days of text study, Forward (2005) Feb. 25.

[12] S. E. Landsburg, Let the Rabbi split the pie: Talmudic wisdom applied to bankruptsy, Slate, posted April 10 (1998).

[13] W.H. Macaulay, Note on the deflection of beams, The Messenger of Mathematics, 48 (1919) 129-130.

[14] J. Malkevitch, Resolving bankruptsy claims, http://www.ams.org/featurecolumn/archive/ bankruptsy.html.

[15] O. Nitsan, One woman gets 75 zuz, second woman gets 50 zuz: and this is scientific, Haaretz (2002), July 9 (in Hebrew).

[16] B. O'Neill, A problem of rights arbitration from the Talmud, Math. Social Sciences, 2 (1982) 245-271.

[17] F. E. Oppenheim, Egalitarianism as a descriptive concept, Americal Philosophical Quarterly, 7 (1990) 143-152.

[18] L. P. Pojman and R. Westmoreland, Equality: Selected Readings, Oxford University Press, New York, 1997.

[19] A. Steelman, Interview, Thomas Schelling, Region Focus (2005) 26-41.

[20] D.J. Velleman, Private communication

[21] M. Yaari and M. Bar-Hillel, On dividing justly, Social Choice and Welfare, 1 (1984) 1-24.

[22] H. Young, Equity, Princeton University Press, 1994.

[23] H. Young, On dividing an amount according to individual claims or liabilities, Mathem. of Operations Research, 12 (1987) 398-414.

[24] Talmud Bavli, Schottenstein Edition, Tractate Bava Metzia (ed. Rabbi Hersh Goldwurm), Vol. 1, Mesorah Publications, Ltd., New York, 1992.

[25] Mishnah, Seder Nashim, Vol. 1(b) Tractate Kesubos (Rabbi Tzvi Zev Arem, ed., translation and anthological commentary by Rabbi Avrohom Yoseif Rosenberg), Artscroll Mishnah Series, Mesora Publications, Ltd., New York, pp. 204-209, 1984.

[26] Mishnah, Seder Nezikin, Vol. 1(b) Tractate Bava Metzia (Rabbi Tzvi Zev Arem, ed., translation and anthologiacal commentary by Rabbi Avrohom Yoseif Rosenberg), Artscroll Mishnah Series, Mesora Publications, Ltd., New York, pp. 7-12, 1984.

[27] Talmud (Steinsaltz edition), Vol. 1, Tractate Bava Metzia, Part 1, New York, 1984.

[28] http://www.come-and-hear.com/babamezia/

[29] http://www.come-and-hear.com/kethuboth/

*Department of Mechanical Engineering, Department of Mathematical Sciences, Florida atlantic university, Boca raton, FL 33431-0991, USA

E-mail ; elishako@fau.edu

**Département de Génie Mécanique, Université Laval, Québec, Qc, Canada, G1K 7P4

E-mail ; andre.begin-drolet.1@ulaval.ca