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CERTAIN HYPERGEOMETRIC MATRIX FUNCTIONS

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Abstract. This paper deals with the study of a new kind of hypergeometric matrix function, say, p-hypergeometric function of the single complex variable z. The integral form and the composition of p-hypergeometric matrix function are investigated.

1 Introduction Special matrix functions appear in the study of statistics [2] and Lie groups theory [3]. Hypergeometric matrix function

$$F(A,B;C;z)=\sum_{n=0}^{\infty}\frac{(A)_n(B)_n[(C)_n]^{-1}}{n!}z^n,$$

has been recently introduced by L. Jódar and J.C. Cortés in [5]. They prove that this function is convergent for $|z|<1$ if $C+nI$ is invertible for all integer $n\geq 0$. Moreover, if A, B, C are positive stable matrices in $\mathbb{C}^{N\times N}$ such that $m(C)>M(A)+M(B)$, the series is absolutely convergent for $|z|=1$ where,

$$M(A)=\max\{Re(z):z\in\sigma(A)\},$$
$$M(B)=\max\{Re(z):z\in\sigma(B)\},$$

and

$$m(C)=\min\{Re(z):z\in\sigma(C)\},$$

where $\sigma(A)$ is the set of all eigenvalues of A.

Also, they give an integral representation of $F(A,B;C;z)$ for $|z|<1$ in the form

$$F(A,B;C;z)=\left(\int_0^1(1-tz)^{-A}t^{B-1}(1-t)^{C-B-I}dt\right)\Gamma^{-1}(B)\Gamma^{-1}(C-B)\Gamma(C),$$

where B, C and $C-B$ are positive stable matrices and $BC=CB$.

2 p-Hypergeometric Matrix Function We introduce and study a new kind of hypergeometric matrix function with parameter p that is the p-hypergeometric matrix function

$${}^pF(A,B;C;z)=\sum_{n\geq 0}\frac{(A)_n(B)_n[(C)_n]^{-1}}{(pn)!}z^n=\sum_{n\geq 0}U_nz^n;\ p=2,3,\dots,$$

of matrices coefficients of the single complex variable z. Note that if $p=1$ one obtains the hypergeometric matrix function $F(A,B;C;z)$.

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The radius of convergence of it is the subject of this section. For this purpose we recall relation (17) of [6] in the form

$$\left\|(C+nI)^{-1}\right\| = \left\|\frac{1}{n}\left(\frac{C}{n}+I\right)^{-1}\right\| = \frac{1}{n}\left\|\left(\frac{C}{n}+I\right)^{-1}\right\| \leq \frac{1}{n-\|C\|}, n > \|C\|. \tag{2.1}$$

Denote

$$\gamma(n) = \| C^{-1} \| \| (C+I)^{-1} \| \cdots \| (C+(n-1)I)^{-1} \|;\ n > 0. \tag{2.2}$$

Since

$$\| (A)_n \| \leq (\| A \|)_n \text{ and } \| (B)_n \| \leq (\| B \|)_n, \tag{2.3}$$

then by (2.2) and (2.3) we get

$$\| U_n \| \leq \frac{(\| A \|)_n (\| B \|)_n \gamma(n)}{(pn)!}.$$

Hence

$$\begin{aligned}
\limsup_{n\to\infty}\left|\frac{\| U_{n+1} \| z^{n+1}}{\| U_n \| z^n}\right| &\leq \limsup_{n\to\infty}\left|\frac{(\| A \|)_{n+1}(\| B \|)_{n+1}\gamma(n+1)(pn)!z^{n+1}}{(\| A \|)_n(\| B \|)_n\gamma(n)(pn+p)!z^n}\right| \\
&= \limsup_{n\to\infty}\left|\frac{(\| A \| + n)(\| B \| + n) \| (C+nI)^{-1} \|}{n^p(p+\frac{p}{n})(p+\frac{p-1}{n})\dots(p+\frac{1}{n})}\right||z| \\
&\leq \limsup_{n\to\infty}\frac{n^2(\frac{\|A\|}{n}+1)(\frac{\|B\|}{n}+1)}{n^{p+1}(p+\frac{p}{n})(p+\frac{p-1}{n})\dots(p+\frac{1}{n})(1-\frac{\|C\|}{n})}|z| = 0 \quad \text{for all } z.
\end{aligned}$$

Then the p-hypergeometric matrix function is an entire function for all $p = 2, 3, \dots$.

Putting $p = 1$ we see that the hypergeometric matrix function $F(A, B; C; z)$ converges in $|z| < 1$ (c.f. [5]).

2.1 Integral Form of the p-Hypergeometric Matrix Function Suppose that B and C are matrices in the space $\mathbb{C}^{N\times N}$ of the square complex matrices of the same order N, such that

$$BC = CB, \tag{2.4}$$

and

$$C, B \text{ and } C - B \text{ are positive stable matrices.} \tag{2.5}$$

Since

$$B(B+I)\dots(B+(n-1)I) = \Gamma^{-1}(B)\Gamma(B+nI), \tag{2.6}$$

then

$$\begin{aligned}
(B)_n[(C)_n]^{-1} &= \Gamma^{-1}(B)\Gamma(B+nI)[\Gamma^{-1}(C)\Gamma(C+nI)]^{-1} \\
&= \Gamma^{-1}(B)\Gamma^{-1}(C-B)\Gamma(C-B)\Gamma(B+nI)\Gamma^{-1}(C+nI)\Gamma(C).
\end{aligned} \tag{2.7}$$

By Lemma 2 of [5] and (2.5), we see that

$$\begin{aligned}\int_0^1 t^{B+(n-1)I}(1-t)^{C-B-I}dt &= B(B+nI, C-B)\\ &=\Gamma(C-B)\Gamma(B+nI)\Gamma^{-1}(C+nI).\end{aligned} \tag{2.8}$$

¿From relations (2.7) and (2.8) we get

$$(B)_n[(C)_n]^{-1} = \Gamma^{-1}(B)\Gamma^{-1}(C-B)\left(\int_0^1 t^{B+(n-1)I}(1-t)^{C-B-I}dt\right)\Gamma(C). \tag{2.9}$$

Hence, formally one can write

$$\begin{aligned}{}^pF(A,B;C;z) &= \sum_{n\geq 0}\frac{(A)_n(B)_n[(C)_n]^{-1}}{(pn)!}z^n\\ &= \sum_{n\geq 0}\frac{(A)_n\Gamma^{-1}(B)\Gamma^{-1}(C-B)}{(pn)!}\left(\int_0^1 t^{B+(n-1)I}(1-t)^{C-B-I}dt\right)\Gamma(C)z^n\\ &=\left(\int_0^1 {}^pF(A,-;-;tz)\, t^{B-I}(1-t)^{C-B-I}dt\right)\Gamma^{-1}(B)\Gamma^{-1}(C-B)\Gamma(C).\end{aligned}$$

This is the integral form of p-hypergeometric matrix function.

3 Composite Hypergeometric Matrix Function Let us introduce the following notation

$$\begin{aligned}\underline{n} &= (n_1, n_2, \ldots, n_k),\\ (\underline{n}) &= n_1 + n_2 + \cdots + n_k,\\ \underline{z^n} &= z_1^{n_1} z_2^{n_2}\ldots z_k^{n_k},\\ \underline{A} &= (A_1, A_2, \ldots, A_k),\\ \underline{B} &= (B_1, B_2, \ldots, B_k),\\ \underline{C} &= (C_1, C_2, \ldots, C_k),\\ (\underline{A})_{\underline{n}} &= (A_1)_{n_1}(A_2)_{n_2}\ldots(A_k)_{n_k},\\ (\underline{B})_{\underline{n}} &= (B_1)_{n_1}(B_2)_{n_2}\ldots(B_k)_{n_k},\\ (\underline{C})_{\underline{n}} &= (C_1)_{n_1}(C_2)_{n_2}\ldots(C_k)_{n_k},\end{aligned}$$

and

$$\underline{F} = (F_1, F_2, \ldots, F_k).$$

Suppose that

$$F_i(A_i, B_i; C_i; z_i) = \sum_{n_i\geq 0}\frac{(A_i)_{n_i}(B_i)_{n_i}[(C_i)_{n_i}]^{-1}}{(n_i)!}z_i^{n_i},\ i = 1, 2, \ldots, k, \tag{3.1}$$

are k hypergeometric functions with square complex matrices $A_1, A_2, \ldots, A_k$, B_1, B_2 ,$\ldots$, B_k and $C_1, C_2, \ldots, C_k$ of the same order N.

Construct the function

$$F(\underline{A},\underline{B};\underline{C};\underline{z}) = \sum_{\underline{n}\geq 0} \frac{(\underline{A})_{\underline{n}}(\underline{B})_{\underline{n}}[(\underline{C})_{\underline{n}}]^{-1}}{(\underline{n})!}\underline{z}^{\underline{n}} = \sum_{\underline{n}\geq 0} U_{\underline{n}}\underline{z}^{\underline{n}}. \tag{3.2}$$

This function, will be called the composite hypergeometric matrix function of several complex variables $z_1, z_2, \ldots, z_k$.

We begin the study of this function by calculating its radius of convergence R. For this purpose, we recall relation (1.3.10) of [8] and keeping in mind that $\sigma_{\underline{n}} \geq 1$. Hence

$$\begin{aligned}
\frac{1}{R} &= \limsup_{(\underline{n})\to\infty} \left(\frac{\| U_{\underline{n}} \|}{\sigma_{\underline{n}}}\right)^{\frac{1}{(\underline{n})}} = \limsup_{(\underline{n})\to\infty} \left(\frac{\| (\underline{A})_{\underline{n}}(\underline{B})_{\underline{n}}[(\underline{C})_{\underline{n}}]^{-1} \|}{(\underline{n})!}\right)^{\frac{1}{(\underline{n})}} \left(\frac{1}{\sigma_{\underline{n}}}\right)^{\frac{1}{(\underline{n})}} \\
&\leq \limsup_{(\underline{n})\to\infty} \left(\frac{\| (A_1)_{n_1}\ldots(A_k)_{n_k}(B_1)_{n_1}\ldots(B_k)_{n_k}[(C_k)_{n_k}]^{-1}\ldots[(C_1)_{n_1}]^{-1} \|}{n_1!n_2!\ldots n_k!}\right)^{\frac{1}{(\underline{n})}} \\
&\leq \limsup_{(\underline{n})\to\infty} \Bigg(\Bigg\|\left(\frac{n_1^{-A_1}(A_1)_{n_1}}{(n_1-1)!}\right)(n_1-1)!n_1^{A_1}\ldots\left(\frac{n_k^{-A_k}(A_k)_{n_k}}{(n_k-1)!}\right)(n_k-1)!n_k^{A_k} \\
&\quad \left(\frac{n_1^{-B_1}(B_1)_{n_1}}{(n_1-1)!}\right)(n_1-1)!n_1^{B_1}\ldots\left(\frac{n_k^{-B_k}(B_k)_{n_k}}{(n_k-1)!}\right)(n_k-1)!n_k^{B_k} \\
&\quad \frac{n_k^{C_k}}{(n_k-1)!}(n_k-1)![(C_k)_{n_k}]^{-1}n_k^{-C_k}\ldots\frac{n_1^{C_1}}{(n_1-1)!}(n_1-1)![(C_1)_{n_1}]^{-1}n_1^{-C_1}\Bigg\| \\
&\quad \frac{1}{n_1!n_2!\ldots n_k!}\Bigg)^{\frac{1}{(\underline{n})}} \\
&\leq \limsup_{(\underline{n})\to\infty} \left(\| \Gamma^{-1}(A_1) \| \cdots \| \Gamma^{-1}(A_k) \|\| \Gamma^{-1}(B_1) \| \cdots \| \Gamma^{-1}(B_k) \|\right) \\
&\quad \left(\| \Gamma(C_k) \| \cdots \| \Gamma(C_1) \|\right)^{\frac{1}{(\underline{n})}} \\
&\quad \limsup_{(\underline{n})\to\infty} \left(\frac{\| n_1^{A_1} \| \cdots \| n_k^{A_k} \|\| n_1^{B_1} \| \cdots \| n_k^{B_k} \|\| n_k^{-C_k} \| \cdots \| n_1^{-C_1} \|}{n_1 n_2 \ldots n_k}\right)^{\frac{1}{(\underline{n})}},
\end{aligned} \tag{3.3}$$

where

$$\sigma_{\underline{n}} = \begin{cases} \left(\frac{n_1+\ldots+n_k}{n_1}\right)^{\frac{n_1}{2}}\left(\frac{n_1+\ldots+n_k}{n_2}\right)^{\frac{n_2}{2}}\ldots\left(\frac{n_1+\ldots+n_k}{n_k}\right)^{\frac{n_k}{2}} & ,\underline{n} \neq 0 \\ 1 & ,\underline{n} = 0. \end{cases}$$

For positive numbers μ_i and positive integer n, we can write

$$n_i = \mu_i n, \quad i = 1, 2, \ldots, k.$$

Using relation (8) of [5] for any square complex matrix A of size N

$$\| e^{tA} \| \leq e^{tM(A)} \sum_{j=0}^{N-1} \frac{(\| A \| N^{\frac{1}{2}} t)^j}{j!}; \ t \geq 0,$$

and considering that $n^A = e^{A\ln n}$ one gets

$$\| n^A \| \leq n^{M(A)} \sum_{j=0}^{N-1} \frac{(\| A \| N^{\frac{1}{2}} \ln n)^j}{j!}. \tag{3.4}$$

Substitute from (3.4) into (3.3) one gets

$$
\begin{aligned}
\frac{1}{R} \leq \limsup_{n(\mu_1+\cdots+\mu_k)\to\infty} \Bigg\{ &(\mu_1 n)^{M(A_1)} \sum_{j=0}^{N-1} \frac{(\| A_1 \| N^{\frac{1}{2}} \ln \mu_1 n)^j}{j!} \ldots \\
&(\mu_k n)^{M(A_k)} \sum_{j=0}^{N-1} \frac{(\| A_k \| N^{\frac{1}{2}} \ln \mu_k n)^j}{j!} (\mu_1 n)^{M(B_1)} \sum_{j=0}^{N-1} \frac{(\| B_1 \| N^{\frac{1}{2}} \ln \mu_1 n)^j}{j!} \ldots \\
&(\mu_k n)^{M(B_k)} \sum_{j=0}^{N-1} \frac{(\| B_k \| N^{\frac{1}{2}} \ln \mu_k n)^j}{j!} (\mu_1 n)^{-m(C_1)} \sum_{j=0}^{N-1} \frac{(\| C_1 \| N^{\frac{1}{2}} \ln \mu_1 n)^j}{j!} \\
&\ldots (\mu_k n)^{-m(C_k)} \sum_{j=0}^{N-1} \frac{(\| C_k \| N^{\frac{1}{2}} \ln \mu_k n)^j}{j!} \frac{1}{(\mu_1 n)! \ldots (\mu_k n)!} \Bigg\}^{\frac{1}{n(\mu_1+\cdots+\mu_k)}}.
\end{aligned} \tag{3.5}
$$

Since

$$
\sum_{j=0}^{N-1} \frac{(\| A \| N^{\frac{1}{2}} \ln \mu n)^j}{j!} \leq (N \ln \mu n)^{N-1} \sum_{j=0}^{N-1} \frac{(\| A \|)^j}{j!} = (N \ln \mu n)^{N-1} e^{\|A\|},
$$

then

$$
\begin{aligned}
\frac{1}{R} \leq \limsup_{n(\mu_1+\cdots+\mu_k)\to\infty} &\{ n^{M(A_1)+\cdots+M(A_k)+M(B_1)+\cdots+M(B_k)} \\
&n^{-m(C_1)-\cdots-m(C_k)-k} \}^{\frac{1}{n(\mu_1+\cdots+\mu_k)}} \\
&\limsup_{n(\mu_1+\cdots+\mu_k)\to\infty} \left((N \ln \mu_1 n)^{N-1} e^{\|A_1\|} \ldots (N \ln \mu_k n)^{N-1} e^{\|A_k\|} \right)^{\frac{1}{n(\mu_1+\cdots+\mu_k)}} \\
&\limsup_{n(\mu_1+\cdots+\mu_k)\to\infty} \left((N \ln \mu_1 n)^{N-1} e^{\|B_1\|} \ldots (N \ln \mu_k n)^{N-1} e^{\|B_k\|} \right)^{\frac{1}{n(\mu_1+\cdots+\mu_k)}} \\
&\limsup_{n(\mu_1+\cdots+\mu_k)\to\infty} \left((N \ln \mu_1 n)^{N-1} e^{\|C_1\|} \ldots (N \ln \mu_k n)^{N-1} e^{\|C_k\|} \right)^{\frac{1}{n(\mu_1+\cdots+\mu_k)}} = 1,
\end{aligned}
$$

i.e. the radius of convergence of the composite hypergeometric matrix function is one and it is regular in the sphere $\bar{S}_R$; $R = 1$ (c.f. [8]).

4 Composite p-Hypergeometric Matrix Function

Let

$$
{}^{p_i}F(A_i, B_i; C_i; z_i) = \sum_{\underline{n}\geq 0} \frac{(A_i)_n (B_i)_n [(C_i)_n]^{-1}}{(p_i n_i)!} z_i^{n_i}, i = 1, 2, \ldots, k,
$$

are k, p-hypergeometric matrices functions of the square complex matrices A_i, B_i and C_i of the same order N.

Construct the composite p-hypergeometric matrix function of these functions for any mode of arrangement we put

$$
{}^{p}\underline{F}(\underline{A}, \underline{B}; \underline{C}; \underline{z}) = \sum_{\underline{n}\geq 0} \frac{(\underline{A})_n (\underline{B})_n [(\underline{C})_n]^{-1}}{(p\underline{n}_i)!} \underline{z}^{\underline{n}} = \sum_{\underline{n}\geq 0} U_{\underline{n}} \underline{z}^{\underline{n}}, \tag{4.1}
$$

where

$$
{}^{\underline{p}}\underline{F} = {}^{p_1}F_1(A_1, B_1; C_1; z_1) \ldots {}^{p_k}F_k(A_k, B_k; C_k; z_k).
$$

Then ${}^{\underline{p}}\underline{F}(\underline{A},\underline{B};\underline{C};\underline{z})$ is the p-hypergeometric matrix function.

Now we calculate the radius of convergence of this function as follows

$$\begin{aligned}
\frac{1}{R} &= \limsup_{(\underline{n})\to\infty}\left(\frac{\| U_{\underline{n}} \|}{\sigma_{\underline{n}}}\right)^{\frac{1}{(\underline{n})}} = \limsup_{(\underline{n})\to\infty}\left(\frac{\| (\underline{A})_{\underline{n}}(\underline{B})_{\underline{n}}[(\underline{C})_{\underline{n}}]^{-1} \|}{(p\underline{n})!}\right)^{\frac{1}{(\underline{n})}}\left(\frac{1}{\sigma_{\underline{n}}}\right)^{\frac{1}{(\underline{n})}} \\
&\le \limsup_{(\underline{n})\to\infty}\left(\frac{\| (A_1)_{n_1}\dots(A_k)_{n_k}(B_1)_{n_1}\dots(B_k)_{n_k}[(C_k)_{n_k}]^{-1}\dots[(C_1)_{n_1}]^{-1} \|}{(p_1n_1)!\dots(p_kn_k)!}\right)^{\frac{1}{(\underline{n})}} \\
&\le \limsup_{(\underline{n})\to\infty}\Bigg\|\Bigg(\left(\frac{n_1^{-A_1}(A_1)_{n_1}}{(n_1-1)!}\right)(n_1-1)!n_1^{A_1}\dots\left(\frac{n_k^{-A_k}(A_k)_{n_k}}{(n_k-1)!}\right)(n_k-1)!n_k^{A_k} \\
&\quad \left(\frac{n_1^{-B_1}(B_1)_{n_1}}{(n_1-1)!}\right)(n_1-1)!n_1^{B_1}\dots\left(\frac{n_k^{-B_k}(B_k)_{n_k}}{(n_k-1)!}\right)(n_k-1)!n_k^{B_k} \\
&\quad \frac{n_k^{C_k}}{(n_k-1)!}(n_k-1)![(C_k)_{n_k}]^{-1}n_k^{-C_k}\dots\frac{n_1^{C_1}}{(n_1-1)!}(n_1-1)![(C_1)_{n_1}]^{-1}n_1^{-C_1}\Bigg\| \\
&\quad \frac{1}{(p_1n_1)!(p_2n_2)!\dots(p_kn_k)!}\Bigg)^{\frac{1}{(\underline{n})}} \qquad (4.2)\\
&= \limsup_{(\underline{n})\to\infty}\Big\{\frac{\| n_1^{A_1} \| \cdots \| n_k^{A_k} \|\| \| n_1^{B_1} \| \cdots \| n_k^{B_k} \|\| \| n_k^{-C_k} \| \cdots \| n_1^{-C_1} \|}{p_1^{p_1+1}n_1(n_1-\frac{1}{p_1})\dots(n_1-\frac{p_1-1}{p_1})p_2^{p_2+1}n_2(n_2-\frac{1}{p_2})\dots(n_2-\frac{p_2-1}{p_2})} \\
&\quad \frac{1}{p_k^{p_k+1}n_k(n_k-\frac{1}{p_k})\dots(n_k-\frac{p_k-1}{p_k})}\Big\}^{\frac{1}{(\underline{n})}},
\end{aligned}$$

as above putting $n_i = \mu_i n$ and using relation (3.4) we get

$$\begin{aligned}
\frac{1}{R} \le & \limsup_{n(\mu_1+\dots+\mu_k)\to\infty}\left(n^{M(A_1)+\dots+M(A_k)+M(B_1)+\dots+M(B_k)}n^{-m(C_k)-\dots-m(C_1)-k}\right)^{\frac{1}{n(\mu_1+\dots+\mu_k)}} \\
& \limsup_{n(\mu_1+\dots+\mu_k)\to\infty}\left((N\ln\mu_1 n)^{N-1}e^{\|A_1\|}\dots(N\ln\mu_k n)^{N-1}e^{\|A_k\|}\right)^{\frac{1}{n(\mu_1+\dots+\mu_k)}} \\
& \limsup_{n(\mu_1+\dots+\mu_k)\to\infty}\left((N\ln\mu_1 n)^{N-1}e^{\|B_1\|}\dots(N\ln\mu_k n)^{N-1}e^{\|B_k\|}\right)^{\frac{1}{n(\mu_1+\dots+\mu_k)}} \\
& \limsup_{n(\mu_1+\dots+\mu_k)\to\infty}\left((N\ln\mu_1 n)^{N-1}e^{\|C_k\|}\dots(N\ln\mu_k n)^{N-1}e^{\|C_1\|}\right)^{\frac{1}{n(\mu_1+\dots+\mu_k)}} \\
& \limsup_{n(\mu_1+\dots+\mu_k)\to\infty}\Big(\frac{1}{p_1^{p_1+1}(\mu_1 n-\frac{1}{p_1})\dots(\mu_k n-\frac{p_1-1}{p_1})p_2^{p_2+1}(\mu_2 n-\frac{1}{p_2})\dots(\mu_k n-\frac{p_2-1}{p_2})} \\
& \frac{1}{\dots p_k^{p_k+1}(\mu_k n-\frac{1}{p_k})\dots(\mu_k n-\frac{p_k-1}{p_k})}\Big)^{\frac{1}{n(\mu_1+\dots+\mu_k)}} = 0.
\end{aligned}$$

Then the composite p-hypergeometric matrix function is an entire function.

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References

[1] G. Bennett, Schur multipliers, *Duke Math. J.*, **44**, 603-639, (1977).

[2] A. G. Constantine & R. J. Mairhead, Partial differential equations for hypergeometric function of two argument matrix, *J. Mutivariate Anal.*, **3**, 332-338, (1972).

[3] A. T. James, *Special Functions of Matrix and Single Argument in Statistics in Theory and Application of Special Functions*, R. A. Askey (Ed) Academic Press, New York, 497-520, (1975).

[4] L. Jódar & J. Sastre, On the Laguerre matrix polynomials, *Utilities Math.*, **53**, 37-48, (1998).

[5] L. Jódar & J. C. Cortés, On the hypergeometric matrix function, *J. Computational Appl. Math.*, **99**, 205-217, (1998).

[6] L. Jódar & J. C. Cortés, Closed form general solution of the hypergeometric matrix differential equation, *Math. Comput. Modelling*, **32**, 1017-1028, (2000).

[7] J. R. Prather, Geometric properties of the Hadamard product, *Complex Variables*, **41**, 17-26, (2000).

[8] K. A. M. Sayyed, *Basic Sets of Polynomials of Two Complex Variables*, M. Sc. Thesis Assiut Univ. (1972).

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