A NOTE ON THE MARTIN TOPOLOGY OF THE SPACE OF THE FORMAL BALLS

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Abstract. Let (X, d) a metric space and $\mathbf{B}X = X \times \mathbb{R}$ denote the partially ordered set of generalized formal balls in X. We investigate the relations between the Martin topology and the product topology of certain topologies of X and the Sorgenfrey line. We give a condition that the Martin topology coincides with the product topology of a metric topology and the Sorgenfrey topology, and consider on the conditions that the Martin topology is homeomorphic to the product topology of a metric topology and the Sorgenfrey topology. We also show that the space of formal balls on $\mathbb{R}$ with the Martin topology is homeomorphic to the square of the Sorgenfrey lines.

1 Introduction Basic tools of topological approaches to domain theory are "metric-like" functions such as a qausi-metric and a (weak) partial metric. The Scott topology and the Lawson topology are known as the fundamental topologies related to the order structures in posets. Several authors investigated the relations between order structures, metric-like functions above, the Scott topology and the Lawson topology (cf. [4]).

K. Martin [7] introduced a notion of a measurement in a domain to describe a quantitative statements on programs, and P. Waszkiewicz [10] discussed on the relations between the measurements and the partial metrics due to S. G. Matthews [8] on continuous posets. K. Martin also showed that every measurement induces a topology that we call the Martin topology. It is shown that the Martin topology has a clopen base, and it is stronger than the Lawson topology. However, a few facts are known about the Martin topology.

Let (X, d) be a metric space. Then, an element of $\mathbf{B}^+X = X \times [0, +\infty)$ is called a *formal ball*. We induce a partial order $\sqsubseteq$ on $\mathbf{B}^+X$ as $(x, r) \sqsubseteq (y, s)$ if $d(x, y) \leq r - s$. The notion of formal balls is introduced by Weihrauch and Schreiber to represent a metric space in a domain as a computational model [11]. Several authors sudied the poset of formal balls as an approximating structure of a metric space [1, 2, 5, 6].

Recently, Tsuiki-Hattori [9] introduced formal balls with negative radiuses and study the partially ordered set $\mathbf{B}X = X \times \mathbb{R}$ with an order relation which is similar to $\mathbf{B}^+X$. An element of $\mathbf{B}X$ is called a *generalized formal ball*. The sets $\mathbf{B}X$ obviously has the Lawson topology as a poset. It is easy to see that the relative Lawson topology on every slice $X \times \{t\} \subset \mathbf{B}X$ $(t \in \mathbb{R})$ is homeomorphic to the metric topology of X and every slice $\{x\} \times \mathbb{R} \subset \mathbf{B}X$ $(x \in X)$ is homeomorphic to the usual real line $\mathbb{R}$. In this direction, Tsuiki-Hattori considered the differences, or coincidences of the Lawson topology and the product topology of X and $\mathbb{R}$ on $\mathbf{B}X$.

The Martin topology on the space of formal balls seems to be more complicated, because the relative Martin topology on every slice $X \times \{t\} \subset \mathbf{B}X$ $(t \in \mathbb{R})$ is a discrete space, which is not homeomorphic to the metric topology of X in general, and every slice $\{x\} \times \mathbb{R} \subset \mathbf{B}X$ $(x \in X)$ is homeomorphic to the Sorgenfrey line. In the present note, we will consider

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the relations between the Martin topology in the spaces of formal balls and the product topology of certain topologies on the space X and the Sorgenfrey line.

2 Preliminaries We recall basic concepts in domain theory that we use in the present note.

Let $(L, \sqsubseteq)$ be a partially ordered set (poset). $(L\ \sqsubseteq)$ is called *directed* if every finite subset D of L has an upper bound. If every directed subset of L has a least upper bound, then L is called a *directed complete poset* (abbrev. *dcpo*). Let $x,\ y \in L$. If for every directed subset D of L for which $\sup D$ exists and $y \sqsubseteq \sup D$, there is $d \in D$ such that $x \sqsubseteq d$, then we say that x is *way below* y and write $x \ll y$. Clearly, if $x \ll y$ then $x \sqsubseteq y$. For a poset $(L, \sqsubseteq)$, $x \in L$, $A \subseteq L$, we put

$\uparrow x = \{y \in L : x \sqsubseteq y\}$, $\Uparrow x = \{y \in L : x \ll y\}$,

$\uparrow A = \{y \in L : x \sqsubseteq y \text{ for some } x \in A\}$.

We can define $\downarrow x$, $\Downarrow x$ and $\downarrow A$ similarly. If $\Downarrow x$ is directed and $x = \sup \Downarrow x$ for all $x \in L$, then L is called *continuous*. A subset U of a poset L is *Scott open* if U satisfies following two conditions.

(1) $U = \uparrow U$,

(2) If $D \subseteq L$ is directed and $\sup D \in U$, then $D \cap U \neq \emptyset$.

Then $\sigma(L) = \{U \subseteq L : U \text{ is Scott open}\}$ is a topology of L and we say it *Scott topology*. It is well known that $\{\Uparrow x : x \in L\}$ is an open base for $\sigma(L)$ if L is continuous. We notice that $\sigma(L)$ is T_0 but not T_1 in general.

The *lower topology* is a topology of a poset L generated by $\{L - \uparrow x : x \in L\}$ and we write it $\omega(L)$. The join $\sigma(L) \vee \omega(L)$ of the Scott topology $\sigma(L)$ and the lower topology $\omega(L)$ is called the *Lawson topology* and denoted by $\lambda(L)$. We notice thsat if L is continuous, then $\lambda(L)$ is generated by $\{\Uparrow x : x \in L\} \cup \{\ L - \uparrow x : x \in L\}$, and $\lambda(L)$ is Hausdorff.

Let $[0, \infty)$ be the set of non-negative real numbers. We define order on $[0, \infty)$ as the opposite relation from the usual relation, and we denote it $[0, \infty)^{op}$. Let L be a poset and $\mu : L \to [0, \infty)^{op}$ a monotone mapping. For any $x \in L$ and $\varepsilon > 0$, we put

$$\begin{aligned}\mu(x, \varepsilon) &= \{y \in L : y \sqsubseteq x,\ \mu(x) + \varepsilon \ll \mu(y)\} \\ &= \{y \in L : y \sqsubseteq x,\ \mu(x) + \varepsilon > \mu(y)\}.\end{aligned}$$

For a subset X of L, we say that μ *induces the Scott topology on* X if for all $U \in \sigma(L)$ and $x \in U \cap X$, there exists $\varepsilon > 0$ such that $\mu(x, \varepsilon) \subseteq U$. We denote it as $\mu \to \sigma(X)$. If L is continuous, $\mu \to \sigma(L)$ and μ is Scott-continuous, then we say that μ is a *measurement* on L.

Let L be a continuous poset and μ a measurement on L. The collection $\{\mu(x, \varepsilon) : x \in L, \varepsilon > 0\}$ is a base for a topology on L. We call the topology the *Martin topology* and denote $\mu(L)$. It is known that $\{\downarrow x \cap \Uparrow y : x, y \in L\}$ is a base for the Martin topology on L. In fact, for all $x,\ y \in L$, $\downarrow x \cap \Uparrow y$ is Martin open. Because for every $z \in \downarrow x \cap \Uparrow y$, there exists $\varepsilon > 0$ such that $\mu(z, \varepsilon) \subseteq \Uparrow y$, since $\Uparrow y \in \sigma(L)$ and $\mu \to \sigma(L)$. Thus, $z \in \mu(z, \varepsilon) \subseteq \downarrow x \cap \Uparrow y$. Conversely, let U be a Martin open set. For every $x \in U$, there is $\varepsilon > 0$ such that $x \in \mu(x, \varepsilon) \subseteq U$. Notice that $\mu(x, \varepsilon) = \downarrow x \cap \mu^{-1}([0, \mu(x) + \varepsilon))$. Since $[0, \mu(x) + \varepsilon)$ is a Scott open set in $[0, \infty)^{op}$ and μ is Scott-continuous, then $\mu^{-1}([0, \mu(x) + \varepsilon))$ is Scott open in L. Thus, there exists $y \in L$ such that $x \in \Uparrow y \subseteq \mu^{-1}([0, \mu(x) + \varepsilon))$ and $x \in \downarrow x \cap \Uparrow y \subseteq \downarrow x \cap \mu^{-1}([0, \mu(x) + \varepsilon)) = \mu(x, \varepsilon) \subseteq U$. Furthermore, it follows that $\downarrow x \cap \Uparrow y$ is a clopen set in the Martin topology. This implies that $(L, \mu(L))$ is 0-dimensional. We abbreviate $(L, \mu(L))$ as (L, μ).

The reader refers [4] for the theory of the domain theory and [3] for topology.

3 Results The set of formal balls is the important object as a continuous poset and a computational model for a metric space. The notion of a formal ball is a generalization of a usual closed ball in a metric space whose center is x and the radius is r. The partially ordered sets $(\mathbf{B}^+X, \sqsubseteq)$ and $(\mathbf{B}X, \sqsubseteq)$ are continuous posets. It is well known that $(x, r) \ll (y, s)$ if and only if $d(x, y) < r - s$ for each (x, r), $(y, s) \in \mathbf{B}^+X$ $(\mathbf{B}X)$.

Recently, Tsuiki and Hattori [9] investigated the relations between the Lawson topology and product topology of a metric topology and the usual topology of $\mathbb{R}$. In this note, we consider about the relations between the Martin topology on and product topology of certain topologies and the Sorgenfrey line.

Let $\mathbb{S}$ be the Sorgenfrey line (i.e. the underline set of $\mathbb{S}$ is the set $\mathbb{R}$ of real numbers and the topology of $\mathbb{S}$ is generated by $\{[a, b) : a < b,\ a, b \in \mathbb{R}\}$) and (X, d) a metric space. We denote the product topology of $X \times \mathbb{S}$ by $\Im_{X\times\mathbb{S}}$. We notice that $\Im_{X\times\mathbb{S}} \subseteq \mu$. Indeed, it is clear that for each $(x, r) \in \mathbf{B}X$ and $\varepsilon > 0$, $\downarrow(x, r) \cap \Uparrow(x, r + \varepsilon) \subseteq U(x; \varepsilon) \times [r, r + \varepsilon)$. We also notice that for each $(x, r) \in \mathbf{B}X$, the family of the sets of the form $\downarrow(x, r) \cap \Uparrow(x, r + \varepsilon)$, $\varepsilon > 0$, is a neighborhood base at (x, r) in $(\mathbf{B}X, \mu)$. Since $(\mathbf{B}X, \mu)$ is 0-dimensional, it follows that X is 0-dimensional if $\Im_{X\times\mathbb{S}} = \mu$. Now, we have a simple characterization.

Proposition 3.1 *Let (X, d) be a metric space. Then, $\Im_{X\times\mathbb{S}} = \mu$ if and only if d induces the discrete topology.*

Proof. Suppose that $\Im_{X\times\mathbb{S}} = \mu$ and d dose not induce the discrete topology on X. There is a point $x \in X$ such that $\{x\}$ is not open in X. Then $\downarrow(x, 1) \cap \Uparrow(x, 2)$ is not open in the product topology. In fact, since $\{x\}$ is not open in X, for each $\varepsilon > 0$ there is $x' \in U(x; \varepsilon)$ such that $x \neq x'$. Then, $(x', 1) \in U(x; \varepsilon) \times [1, 1 + \varepsilon)$, but $(x', 1) \notin \downarrow(x, 1)$. This is a contradiction. Conversely, suppose that d induces the discrete topology. It suffices to show $\Im_{X\times\mathbb{S}} \supseteq \mu$. Let (x, r), $(y, s) \in \mathbf{B}X$ and $(z, t) \in \downarrow(x, r) \cap \Uparrow(y, s)$. We put $\varepsilon = s - t - d(z, y)$. Since $d(z, y) < s - t$, it follows that $\varepsilon > 0$. Then, we obtain $\{z\} \times [t, t + \varepsilon) \subseteq \downarrow(z, t) \cap \Uparrow(y, s) \subseteq \downarrow(x, r) \cap \Uparrow(y, s)$. This implies that $\downarrow(x, r) \cap \Uparrow(y, s)$ is open in the product topology. □

It may happen that $(\mathbf{B}X, \mu)$ is homeomorphic to $(\mathbf{B}X, \Im_{X\times\mathbb{S}})$ even if $\mu \neq \Im_{X\times\mathbb{S}}$. For example, let $A = \{0\} \cup \{\ 1/n : n = 1, 2, \dots\}$, then $\Im_{A\times\mathbb{S}} \neq \mu$ by Proposition 3.1, but it is clear that $(\mathbf{B}A, \Im_{A\times\mathbb{S}}) \approx (\mathbf{B}A, \mu)$. We consider about homeomorphic relations between the Martin topology and the product topology. We have the following result which is a generalization of the example above.

Proposition 3.2 *Let A be a subspace of $\mathbb{R}$. If for each $a \in A$, there exists $\varepsilon > 0$ such that $(a - \varepsilon,\ a) \cap A = \emptyset$, then $(\mathbf{B}A, \Im_{A\times\mathbb{S}}) \approx (\mathbf{B}A, \mu)$.*

Proof. Let A be a subspace of $\mathbb{R}$ satisfying the condition above, and we define a mapping $f : (\mathbf{B}A, \Im_{A\times\mathbb{S}}) \to (\mathbf{B}A, \mu)$ by $f(x, r) = (x, r + x)$. Clearly, f is a bijection. For each $(x, r) \in A \times \mathbb{S}$, there is $\varepsilon_x > 0$ such that $(x - \varepsilon_x, x) \cap A = \emptyset$. And for each $\delta > 0$, we put $\delta' = \min\{\varepsilon_x, \delta/4\}$. Since $U(x; \delta') \cap A = [x, x + \delta') \cap A$, $U = ([x, x + \delta') \cap A) \times [r, r + \delta/2)$ is a neighborhood of (x, r) in $A \times \mathbb{S}$. Then, $f(U) \subseteq \downarrow(x, x + r) \cap \Uparrow(x, x + r + \delta)$ and thus f is continuous. Next, we show that f^{-1} is continuous. Let (x, r) be a point of $\mathbf{B}A = A \times \mathbb{R}$. Notice that $f^{-1}(x, r) = (x, r - x)$. There is $\varepsilon_x > 0$ such that $(x - \varepsilon_x, x) \cap A = \emptyset$. For each $\delta > 0$, we put $\delta' = \min\{\varepsilon_x, \delta\}$. Then $V = \downarrow(x, r) \cap \Uparrow(x, r + \delta')$ is a neighborhood of (x, r) in $(\mathbf{B}A, \mu)$ and $f^{-1}(V) \subseteq (U(x; \delta) \cap A) \times [r - x, r - x + \delta)$. Thus, f is a homeomorphism. □

On the other hand, we have the following by a standard argument of diagonals of the square of the Sorgenfrey lines.

Proposition 3.3 *If X is a separable metric space and $\mid X \mid \geq \mathfrak{c}$, then $X \times \mathbb{S} \not\approx (\mathbf{B}X, \mu)$.*

Proof. Suppose that there exists a homeomorphism $h : (\mathbf{B}X, \mu) \to X \times \mathbb{S}$. Since $X \times \{0\}$ is closed discrete in $(\mathbf{B}X, \mu)$, $h(X \times \{0\})$ is closed discrete in $X \times \mathbb{S}$. Let $C(h(X \times \{0\}))$ be the set of all continuous mappings from $h(X \times \{0\})$ to $\mathbb{R}$, then $\mid C(h(X \times \{0\})) \mid = \mathfrak{c}^{|X|} \geq \mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}$. Let $C(X \times \mathbb{S})$ be the set of all continuous mappings from $(X \times \mathbb{S})$ to $\mathbb{R}$, and let $\mathfrak{m} = \mid C(X \times \mathbb{S}) \mid$. Since $X \times \mathbb{S}$ is normal, $\mid C(h(X \times \{0\})) \mid \leq \mathfrak{m}$ by Tietze's extension theorem. Thus, $2^{\mathfrak{c}} \leq \mathfrak{m}$. On the other hand, since $X \times \mathbb{S}$ is separable, there is a countable dense subset D of $X \times \mathbb{S}$. Then, $\mathfrak{m} \leq \mathfrak{c}^{|D|} = \mathfrak{c}^{\aleph_0} = \mathfrak{c}$ and thus $2^{\mathfrak{c}} \leq \mathfrak{m} \leq \mathfrak{c}$. This is a contradiction. $\square$

Corollary 3.1 *Let $\mathbb{P}$ be the space of the irrational numbers. Then $(\mathbf{B}\mathbb{P}, \mu)$ is not homeomorphic to $\mathbb{P} \times \mathbb{S}$.*

We may ask the following.

Question 3.1 Dose Proposition 3.3 hold for every uncountable separable metric space X?

Question 3.2 Let X be a countable metric space. Is $(\mathbf{B}X, \mu)$ homeomorphic to $X \times \mathbb{S}$? In particular, how about the case $X = \mathbb{Q}$?

We notice that $(\mathbf{B}\mathbb{R}, \mu)$ is not homeomorphic to $\mathbb{R} \times \mathbb{S}$ by Proposition 3.3. However, we have the following.

Proposition 3.4 $(\mathbf{B}\mathbb{R}, \mu)$ *is homeomorphic to* $\mathbb{S} \times \mathbb{S}$.

Proof. We define a mapping $f : \mathbb{S} \times \mathbb{S} \to (\mathbf{B}\mathbb{R}, \mu)$ by $f(x, r) = (\frac{1}{\sqrt{2}}(x - r), \frac{1}{\sqrt{2}}(x + r))$. It is clear that f is a bijection. For each $(x, r) \in \mathbb{S} \times \mathbb{S}$, let V be an arbitrary neighborhood of $f(x, r)$ in $(\mathbf{B}\mathbb{R}, \mu)$. We may assume that $V = \downarrow(\frac{1}{\sqrt{2}}(x - r), \frac{1}{\sqrt{2}}(x + r)) \cap \Uparrow(\frac{1}{\sqrt{2}}(x - r), \frac{1}{\sqrt{2}}(x + r) + \varepsilon)$ for some $\varepsilon > 0$. We put $U = [x, x + \frac{\varepsilon}{\sqrt{2}}) \times [r, r + \frac{\varepsilon}{\sqrt{2}})$. Then U is neighborhood of (x, r) in $\mathbb{S} \times \mathbb{S}$, and one can easily show that $f(U) \subseteq V$. Hence, f is continuous.

To show that f^{-1} is continuous, let $(x, r) \in \mathbf{B}\mathbb{R}$ and U be an arbitrary neighborhood of $f^{-1}(x, r)$ in $\mathbb{S} \times \mathbb{S}$. We notice that $f^{-1}(x, r) = (\frac{1}{\sqrt{2}}(x + r), -\frac{1}{\sqrt{2}}(x - r))$. We may assume that $U = [\frac{1}{\sqrt{2}}(x + r), \frac{1}{\sqrt{2}}(x + r) + \varepsilon) \times [-\frac{1}{\sqrt{2}}(x - r), -\frac{1}{\sqrt{2}}(x - r) + \varepsilon)$ for some $\varepsilon > 0$. We put $V = \downarrow(x, r) \cap \Uparrow(x, r + \sqrt{2}\varepsilon)$. Then V is a neighborhood of (x, r) in $(\mathbf{B}\mathbb{R}, \mu)$, and one can easily show that $f^{-1}(V) \subseteq U$. Hence, f^{-1} is continuous and hence f is a homeomorphism. $\square$

Connects with Proposition 3.4, we may ask the following question.

Question 3.3 For each subset X of $\mathbb{R}$, let $X_{\mathbb{S}}$ denote the subspace X of $\mathbb{S}$. Then, is $X_{\mathbb{S}} \times \mathbb{S}$ homeomorphic to $(\mathbf{B}X, \mu)$? In particular, how about the case $X = \mathbb{Q}$ or $\mathbb{P}$?

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