A FIXED POINT THEOREM FOR SEMIGROUPS ON METRIC SPACES WITH UNIFORM NORMAL STRUCTURE

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Abstract. In this work, we define new generalized uniformly Lipschitzian type conditions for a one-parameter family of selfmappings which is an asymptotically regular semigroup in a complete bounded metric space. Some fixed point results for this semigroup are presented.

1 Introduction In 1989, Khamsi [5] defined normal and uniform normal structure for metric spaces and proved that if (X, d) is a complete bounded metric space with uniform normal structure, then it has the fixed point property for nonexpansive mappings and a kind of intersection property which extends a results of Maluta [10] to metric spaces. In 1995, T.-C. Lim and Hong-Kun Xu [9] proved a fixed point theorem for uniformly Lipschitian mappings in metric spaces with both property (P) and uniform normal structure which extends the result of Khamsi [5]. This is the metric space version of Casini and Maluta's theorem [1]. Recently, Jen-Chih Yao and Lu-Chuan Zeng [11] established a fixed point theorem for an asymptotically regular semigroup of uniformly Lipschitian mappings with property $(*)$ in a complete bounded metric space with uniform normal structure which extends the results T.-C. Lim and Hong-Kun Xu [9].

In this paper we shall define new notions of generalized uniformly Lipschitzian mappings and establish a fixed point theorem for asymptotically regular generalized uniformly Lipschitzian semigroups in a complete bounded metric space with property $(*)$ and uniform normal structure extending the fixed point theorem of J.-C. Yao and L.-C. Zeng [11].

2 Preliminaries Throughout of this paper, X or (X, d) will denote a metric space.

Definition 2.1 [1]. A mapping $T : X \to X$ is said to be a Lipschitzian mapping if each integer $n \geq 1$ there exists a constant $k_n > 0$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y) \quad \forall \ x, y \in X. \tag{1}$$

If $k_n = k \ \forall \ n \geq 1$, then T is called uniformly Lipschitzian and if $k_n = 1 \ \forall \ n \geq 1$, then T is called nonexpansive.

Definition 2.2. A mapping $T : X \to X$ is said to be a generalized Lipschitzian mapping if each integer $n \geq 1$ there exists a constant $k_n > 0$ such that

$$d(T^n x, T^n y) \leq k_n \max\{d(x, y), \frac{1}{2}d(x, T^n x), \frac{1}{2}d(y, T^n y)\} \quad \forall \ x, y \in X. \tag{2}$$

If $k_n = k \ \forall \ n \geq 1$, then T is called generalized uniformly Lipschitzian.

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Remark 2.1. Every uniformly Lipschitzian mapping is generalized uniformly Lipschitzian but the converse is not true.

In the following we give an example to show that there exist a generalized Lipschitzian mapping T which is not Lipschitzian.

Example 2.1. Let $X = \{a, b, c\}$ and $d : X \times X \to [0, \infty)$ defined by $d(a, c) = d(c, a) = \frac{1}{2}$, $d(a, b) = d(b, a) = d(b, c) = d(c, b) = 2$ and $d(x, x) = 0 \ \forall \ x \in X$. Then (X, d) is bounded metric space. Now define $T : X \to X$ as follows $Tb = Tc = b$, $Ta = c$. If $k = 2$, then T is generalized Lipschitzian mapping but not Lipschitzian mapping because $d(Tc, Ta) \nleq 2\, d(c, a)$.

Definition 2.3 [4]. A mapping $T : X \to X$ is called asymptotically regular, if

$$\lim_{n\to\infty} d(T^{n+1}x, T^n y) = 0 \quad \forall \ x, y \in X. \tag{3}$$

Let G be a subsemigroup of $[0, \infty)$ with addition " + " such that

$$t - h \in G \ \ \forall \ t, h \in G \ \ with \ \ t \geq h.$$

This condition is satisfied if $G = [0, \infty)$ or $G = Z^+$, the set of nonnegative integers. Let $\Im = \{T(t) : t \in G\}$ be a family of selfmappings on X. Then $\Im$ is called a (one-parameter) semigroup on X if the following conditions are satisfied:

(i) $T(0)x = x \ \ \forall \ \ x \in X$;

(ii) $T(s + t)x = T(s)(T(t)x) \ \ \forall \ \ s, t \in G$ and $x \in X$;

(iii) $\forall \ \ x \in X$, a mapping $t \to T(t)x$ from G into X is continuous when G has the relative topology of $[0, \infty)$;

(iv) for each $t \in G$, $T(t) : X \to X$ is continuous.

A semigroup $\Im = \{T(t) : t \in G\}$ on X is said to be asymptotically regular at a point $x \in X$ if

$$\lim_{t\to\infty} d(T(t + h)x, T(t)x) = 0 \quad \forall \ \ h \in \ G.$$

If $\Im$ is asymptotically regular at each $x \in X$, then $\Im$ is called an asymptotically regular semigroup on X.

Definition 2.4. A semigroup $\Im = \{T(t) : t \in G\}$ on X is called a generalized uniformly Lipschitzian semigroup if

$$\sup\{k(t) : t \in G\} = k < \infty,$$

where

$$k(t) = \sup\{\frac{d(T(t)x, T(t)y)}{\max\{d(x, y), \frac{1}{2}d(x, T(t)x), \frac{1}{2}d(y, T(t)y)\} \neq 0} : \ x, y \in \ X\}.$$

Definition 2.5. The simplest generalized uniformly Lipschitzian semigroup is a semigroup of iterates of a mapping $T : X \to X$ with

$$\sup\{k_n : n \in N\} = k < \infty,$$

where

$$k_n = \sup\{\frac{d(T^n x, T^n y)}{\max\{d(x, y), \frac{1}{2}d(x, T^n x), \frac{1}{2}d(y, T^n y)\} \neq 0} : \ x, y \in \ X\}.$$

In a metric space (X, d) let F denote a nonempty family of subsets of X. Following Khamsi [5], we say that F defines a convexity structure on X if F is stable under intersection. We say that F has *Property (R)* if any decreasing sequence $\{C_n\}$ of closed bounded nonempty subsets of X with $C_n \in F$ has a nonvoid intersection. Recall that a subset of X is said to be admissible [2] if it is an intersection of closed balls. We denote by $A(X)$ the family of all admissible subsets of X. It is obvious that $A(X)$ defines a convexity structure on X. In this paper any other convexity structure F on X is always assumed to contain $A(X)$.

Let M be a bounded subset of X. Following Lim and Xu [9], we shall adopt the following notations:

$B(x, r)$ is the closed ball centered at x with radius r,

$$r(x, M) = \sup\{d(x, y) : y \in M\} \text{ for } x \in X,$$

$$\delta(M) = \sup\{r(x, M) : x \in M\},$$

$$R(M) = \inf\{r(x, M) : x \in M\}.$$

For a bounded subset A of X, we define the admissible hull of A, denoted by $ad(A)$, as the intersection of all those admissible subsets of X which contain A, i.e.,

$$ad(A) = \bigcap\{B : A \subseteq B \subseteq X \quad \text{with } B \text{ admissible }\}$$

Proposition 2.1 [9]. For a point $x \in X$ and a bounded subset A of X, we have

$$r(x, ad(A)) = r(x, A),$$

Definition 2.6 [5]. A metric space (X, d) is said to have normal (resp. uniform normal) structure if there exists a convexity structure F on X such that $R(A) < \delta(A)$ (resp. $R(A) \leq c.\delta(A)$ for some constant $c \in (0, 1)$) for all $A \in F$ which is bounded and consists of more than one point. In this case F is said to be normal (resp. uniformly normal) in X.

We define the normal structure coefficient $\bar{N}(X)$ of X (with respect to a given convexity structure F) as the number

$$\sup\left\{\frac{R(A)}{\delta(A)}\right\}.$$

where the supremum is taken over all bounded $A \in F$ with $\delta(A) > 0$. X then has uniform normal structure if and only if $\bar{N}(X) < 1$.

Khamsi proved the following result that will be very useful in the proof of our main theorem.

Proposition 2.2 [5]. Let X be a complete bounded metric space and F be a convexity structure of X with uniform normal structure. Then F has the property (R).

Definition 2.7 [11]. Let (X, d) be a metric space and $\Im = \{T(t) : t \in G\}$ be a semigroup on X. Let us write the set

$$w(\infty) = \{\{t_n\} : \{t_n\} \subset G \ \ and \ \ t_n \to \infty\}$$

Definition 2.8 [11]. Let (X, d) be a complete bounded metric space and $\Im = \{T(t) : t \in G\}$ be a semigroup on X. Then $\Im$ has property $(*)$ if for each $x \in X$ and each $\{t_n\} \in w(\infty)$, the following conditions are satisfied:

(a) the sequence $\{T(t_n)x\}$ is bounded;

(b) for any sequence $\{z_n\}$ in $ad\{T(t_n)x : \ n \geq 1\}$ there exists some $z \in \bigcap_{n=1}^{\infty} ad\{z_j : j \geq n\}$ such that

$$\limsup_{n\to\infty} d(z, T(t_n)x) \leq \limsup_{j\to\infty} \limsup_{n\to\infty} d(z_j, T(t_n)x).$$

Jen-Chih Yao and Lu-Chuan Zeng [11] proved the following result that will be used in the proof of our result.

Lemma 1.1. Let (X, d) be a complete bounded metric space with uniform normal structure and $\Im = \{T(t) : t \in G\}$ be a semigroup on X with property $(*)$. Then for each $x \in X$, each $\{t_n\} \in w(\infty)$ and for any constant $\tilde{N}(X) < \bar{c}$, the normal structure coefficient with respect to the given convexity structure F, there exists some $z \in \bigcap_{n=1}^{\infty} ad\{z_j : j \geq n\}$ satisfying the properties:

(I) $\limsup_{n\to\infty} d(z, T(t_n)x) \leq \bar{c}.A(\{T(t_n)x\})$,
where $A(\{T(t_n)x\}) = \limsup_{n\to\infty} d(T(t_i)x, T(t_j)x) : i, j \geq n$;

(II) $d(z, y) \leq \limsup_{n\to\infty} d(T(t_n)x, y)$ for all $y \in X$.

3 The main theorem Theorem 3.1. Let (X, d) be a complete bounded metric space with uniform normal structure and let $\Im = \{T(t) : t \in G\}$ be an asymptotically regular and generalized uniformly Lipschitzian semigroup on X with $\widetilde{k} < \frac{1}{\sqrt{\tilde{N}(X)}}$ and satisfying property $(*)$.
Then there exist some $z \in X$ such that $T(t)z = z$ for all $t \in G$.

Proof. Choose a constant $\bar{c}$ such that $\tilde{N}(X) < \bar{c} < 1$ and $\widetilde{k} < \frac{1}{\sqrt{\bar{c}}}$. We can select a sequence $\{t_n\} \in w(\infty)$ such that $\{t_{n+1} - t_n\} \in w(\infty)$ and $\lim_{n\to\infty} k(t_n) = \widetilde{k}$, where $\widetilde{k} > 0$.

Observe that

$$\{d(T(t_j)x, T(t_i)x) : i, j \geq n\} = \{d(T(t_j)x, T(t_i)x) : j > i \geq n\} \cup \{0\}$$

for each $n \in N$ and $x \in X$,

Now fix an $x_0 \in X$. Then by Lemma 1.1, we can inductively construct a sequence $\{x_l\}_{l=1}^{\infty} \subset X$ such that

$x_{l+1} \in \bigcap_{n=1}^{\infty} \text{ad}\{T(t_i)x_l : i \geq n\}$; for each integer $l \geq 0$,

(III) $\limsup_{n\to\infty} d(T(t_n)x_l, x_{l+1}) \leq \bar{c}.A(\{T(t_n)x_l\})$,
where $A(\{T(t_n)x_l\}) = \limsup_{n\to\infty}\{d(T(t_i)x_l, T(t_j)x_l) : i, j \geq n\}$;

(IV) $d(x_{l+1}, y) \leq \limsup_{n\to\infty} d(T(t_n)x_l, y)$ for all $y \in X$.

Let

$$D_l = \limsup_{n\to\infty} d(x_{l+1}, T(t_n)x_l) \;\; \forall \;\; l \geq 0 \;\; and \quad h = \bar{c}.\widetilde{k} < 1.$$

Observe that for each $i > j \geq 1$, using (IV) and the asymptotic regularity of $\Im$ on X, we have

$$\begin{aligned} d(x_l, T(t_i - t_j)x_l) &\leq \limsup_{n\to\infty} d(x_l, T(t_n + t_i - t_j)x_{l-1}) \\ &\leq \limsup_{n\to\infty} d(x_l, T(t_n)x_{l-1}) + \limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_n + t_i - t_j)x_{l-1}) \\ &\leq D_{l-1}, \end{aligned} \tag{4}$$

$$\begin{aligned} d(x_l, T(t_j)x_l) &\leq \limsup_{n\to\infty} d(x_l, T(t_n + t_j)x_{l-1}) \\ &\leq \limsup_{n\to\infty} d(x_l, T(t_n)x_{l-1}) + \limsup_{n\to\infty} d(T(t_n)x_{l-1}, T(t_n + t_j)x_{l-1}) \leq D_{l-1}, \end{aligned} \tag{5}$$

$$\begin{aligned}
d(T(t_i-t_j)x_l,T(t_i)x_l) &\le \limsup_{n\to\infty} d(T(t_n+t_i-t_j)x_{l-1},T(t_i)x_l)\\
&\le \limsup_{n\to\infty} d(T(t_n+t_i-t_j)x_{l-1},T(t_n)x_{l-1}) + \limsup_{n\to\infty} d(T(t_n)x_{l-1},x_l)\\
&+d(x_l,T(t_i)x_l) \le 2D_{l-1}, \qquad (6)
\end{aligned}$$

$$\begin{aligned}
d(T(t_i)x_l,T(t_j)x_l) &= d(T(t_j)x_l,T(t_j)T(t_i-t_j)x_l)\\
&\le k(t_j)\max\{d(x_l,T(t_i-t_j)x_l),\frac{1}{2}d(x_l,T(t_j)x_l),\frac{1}{2}d(T(t_i-t_j)x_l,T(t_i)x_l)\} \qquad (7)
\end{aligned}$$

Substituting from (4),(5) and (6) in (7) we get

$$d(T(t_i)x_l,T(t_j)x_l) \le k(t_j)\max\{D_{l-1},\frac{1}{2}D_{l-1},D_{l-1}\} = k(t_j)D_{l-1},$$

which implies for each $n \ge 1$,

$$\begin{aligned}
\sup\{d(T(t_i)x_l,T(t_j)x_l): i,j\ge n\} &= \sup\{d(T(t_i)x_l,T(t_j)x_l): i>j\ge n\}\\
&\le \sup\{k(t_j)D_{l-1}: i>j\ge n\}\\
&\le D_{l-1}.\sup\{k(t_j): j\ge n\}. \qquad (8)
\end{aligned}$$

Hence by using (III) and (8), we have

$$\begin{aligned}
D_l = \limsup_{n\to\infty} d(x_{l+1},T(t_n)x_l) &\le \bar{c}A(\{T(t_n)x_l\}) \le \bar{c}\limsup_{n\to\infty}\{d(T(t_i)x_l,T(t_j)x_l): i,j\ge n\}\\
&\le \bar{c}.D_{l-1}.\limsup_{n\to\infty} k(t_n)\\
&\le \bar{c}.\lim_{n\to\infty} k(t_n).D_{l-1} = \bar{c}.\widetilde{k}.D_{l-1} = hD_{l-1} \le h^2D_{l-2} \le \ldots\\
&\le h^l D_0. \qquad (9)
\end{aligned}$$

Hence by the asymptotic regularity of $\Im$ on X, we have for each integer $n \ge 1$,

$$\begin{aligned}
d(x_{l+1},x_l) &\le \limsup_{n\to\infty} d(T(t_n)x_l,x_l)\\
&\le \limsup_{n\to\infty}\limsup_{m\to\infty} d(x_l,T(t_m+t_n)x_{l-1})\\
&\le \limsup_{m\to\infty} d(x_l,T(t_m)x_{l-1}) + \limsup_{n\to\infty}\limsup_{m\to\infty} d(T(t_m)x_{l-1},T(t_m+t_n)x_{l-1})\\
&\le D_{l-1}.
\end{aligned}$$

It follows from (9) that

$$d(x_{l+1},x_l) \le D_{l-1} \le h^{l-1}D_0.$$

Thus, we have $\sum_{l=0}^{\infty} d(x_{l+1},x_l) \le D_0\sum_{l=0}^{\infty} h^{l-1} < \infty$. Consequently $\{x_l\}$ is Cauchy and, hence, convergent as X is complete. Let $z = \lim_{l\to\infty} x_l$. Then we have

$$\begin{aligned}
\limsup_{n\to\infty} d(z,T(t_n)z) &= \lim_{l\to\infty}\limsup_{n\to\infty} d(x_l,T(t_n)x_l)\\
&\le \lim_{l\to\infty} D_{l-1} \le \lim_{l\to\infty} h^{l-1}D_0 = 0,
\end{aligned}$$

i.e., $\lim_{n\to\infty} d(z,T(t_n)z) = 0$. Hence for each $s \in G$, we deduce

$$\begin{aligned}
d(z,T(s)z) &= \lim_{l\to\infty} d(x_l,T(s)x_l) \le \lim_{l\to\infty}\limsup_{n\to\infty} d(x_l,T(t_n+s)x_{l-l})\\
&\le \lim_{l\to\infty}\limsup_{n\to\infty} d(x_l,T(t_n)x_{l-l}) + \lim_{l\to\infty}\limsup_{n\to\infty} d(T(t_n)x_{l-1},T(t_n+s)x_{l-l})\\
&\le \lim_{l\to\infty} D_{l-1} \le \lim_{l\to\infty} h^{l-1}D_0 = 0.
\end{aligned}$$

Then we have $d(z, T(s)z) = 0$, i.e., $T(s)z = z$ for each $s \in G$.

From Remark 2.1 and Theorem 3.1, we immediately obtain the following results

Corollary 3.2 [11]. Let (X, d) be a complete bounded metric space with uniform normal structure and let $\Im = \{T(t) : t \in G\}$ be an asymptotically regular semigroup on X with property $(*)$ and satisfying

$$(\liminf_{t\to\infty} k(t)).(\limsup_{t\to\infty} k(t)) < \bar{N}(X)^{-\frac{1}{2}}.$$

Then there exist some $z \in X$ such that $T(t)z = z$ for all $t \in G$.

Remark 3.2. It will be interesting to establish Theorem 3.1 for representative $\Im = \{T(s) : s \in S\}$ of a left amenable S as a complete bounded metric space with uniform normal structure as in Holmes and A. Lau [3], Lau and Takahashi [8] and Lau [7].

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