HOMOGENEOUS STATIONARY SOLUTION FOR BCF MODEL DESCRIBING CRYSTAL SURFACE GROWTH

H. Fujimura and A. Yagi[1]

Received October 29, 2008; Revised November 28, 2008

Abstract. This paper continues a study on the initial-boundary value problem for a nonlinear parabolic equation of forth order which was presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [9] for describing the process of growth of a crystal surface under molecular beam epitaxy(MBE). In the previous papers [5, 6], we have constructed a dynamical system determined from the model equation and have studied asymptotic behavior of solutions. This paper is then devoted to investigating stability or instability of homogeneous stationary solution. Using the instability dimension, we will make a lower dimension estimate for the attractor of the dynamical system constructed in [5].

1 Introduction We continue a study on the initial-boundary value problem for a nonlinear parabolic equation of fourth order

$$(1.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = -a\Delta^2 u - \mu\nabla\cdot\left(\dfrac{\nabla u}{1+|\nabla u|^2}\right) & \text{in}\quad \Omega\times(0,\infty),\\ \dfrac{\partial u}{\partial n} = \dfrac{\partial}{\partial n}\Delta u = 0 & \text{on}\quad \partial\Omega\times(0,\infty),\\ u(x,0)=u_0(x) & \text{in}\quad \Omega \end{cases}$$

in a two-dimensional bounded domain $\Omega\subset\mathbb{R}^2$. This model was presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [9] for describing the process of growing of a crystal surface on the basis of the BCF theory due to Burton-Cabrera-Frank [3] (cf. also [8, 13, 14, 15, 19]). Here, $u=u(x,t)$ denotes a displacement of height of surface from the standard level ($u=0$) at a position $x\in\Omega$. We assume that u and Δu satisfy the homogeneous Neumann boundary conditions on $\partial\Omega$.

The term $-a\Delta^2 u$ in the equation of (1.1) denotes a surface diffusion of adatoms which is caused by the difference of the chemical potential, see Mullins [11], $a>0$ denotes a surface diffusion constant. In the meantime, $-\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)$ denotes the effect of surface roughening. Such roughening is caused by the Schwoebel barriers [4, 16] (cf. also [19]). The macroscopic representation of the roughening by the term $-\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right)$ is formulated in the paper [9] mentioned above, where $\mu>0$ is a coefficient of surface roughening. Some numerical simulations for one or two-dimensional model of (1.1) were performed by the papers [8, 13, 14, 15].

As in the preceding papers [5, 6], we will handle (1.1) in the underlying space $L^2(\Omega)$. In [5], we have already constructed a dynamical system $(S(t), H^1_m(\Omega), L^2_m(\Omega))$ determined

[1]This work is supported by Grant-in-Aid for Scientific Research (No. 20340035) of the Japan Society for the Promotion of Science.

2000 *Mathematics Subject Classification.* 37L15, 37L30, 74E15.

Key words and phrases. Crystal surface, Dynamical system, Stationary problem.

from (??) with the phase space $H^1_m(\Omega)$ in the universal space $L^2_m(\Omega)$, here $H^1_m(\Omega)$ (resp. $L^2_m(\Omega)$) is a closed subspace of $H^1(\Omega)$ (resp. $L^2(\Omega)$) consisting of functions $u \in H^1(\Omega)$ (resp. $f \in L^2(\Omega)$) with null mean, i.e., $m(f) \equiv |\Omega|^{-1} \int_\Omega f\,dx = 0$. And then we have constructed in [6] exponential attractors and a Lyapunov function for $(S(t), H^1_m(\Omega), L^2_m(\Omega))$. As a result, it was shown that the ω-limit set $\omega(u_0)$ of any initial value $u_0 \in H^1_m(\Omega)$ consists of equilibria of $S(t)$.

In this paper, we are concerned with stability or instability of homogeneous stationary solution. Clearly, $\overline{u} = 0$ is a unique homogeneous stationary solution of (1.1) satisfying $m(\overline{u}) = 0$, namely, 0 is a unique homogeneous equilibrium of $(S(t), H^1_m(\Omega), L^2_m(\Omega))$. We will appeal to the linearized principle invented by Bavin-Vishik [2] and Temam [18] in the theory of infinite-dimensional dynamical system. Application of the principle to the dynamical systems determined from semilinear abstract parabolic evolution equations was described in [1], for this we will make a brief review in Section 3. In fact, we shall prove that 0 is stable if the parameter μ is smaller than $a\lambda_1$, where $\lambda_1 > 0$ is the minimal eigenvalue of a realisation of $-\Delta$ in $L^2_m(\Omega)$ under the homogeneous Neumann boundary conditions on $\partial\Omega$. To the contrary, 0 becomes unstable if the μ is larger than $a\lambda_1$ and the instability dimension is given by $\#\{\lambda_k;\ \mu > a\lambda_k\}$, where $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$ denote the eigenvalues of $-\Delta$ in $L^2_m(\Omega)$ under the Neumann boundary conditions. Using the instability dimension, we can give a lower dimension estimate for exponential attractors $\mathcal{M}$ of $(S(t), H^1_m(\Omega), L^2_m(\Omega))$. By definition, $\mathcal{M}$'s are a finite-dimensional compact subset of $L^2_m(\Omega)$.

Throughout the paper, Ω is a bounded domain of $\mathcal{C}^4$ class in $\mathbb{R}^2$. According to [7], the Poisson problem $-\Delta u = f$ in Ω under the homogeneous Neumann boundary conditions $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ enjoys the shift property that $f \in H^2(\Omega)$ implies $u \in H^4(\Omega)$.

2 Reviews of known results

Let us review known results obtained in the previous papers [5, 6] concerning Problem (1.1).

Local solutions. We formulate (1.1) as the Cauchy problem for a semilinear abstract evolution equation of the form

$$(2.1) \qquad \begin{cases} \dfrac{du}{dt} + Au = F(u), & 0 < t < \infty, \\ u(0) = u_0 \end{cases}$$

in a Banach space X. For the underlying space X, we set $X = L^2(\Omega)$. The linear operator A is defined by $A = \Lambda^2$, where Λ is the realization of $-\sqrt{a}\Delta + 1$ in $L^2(\Omega)$ under the homogeneous Neumann boundary conditions. Clearly, $A \ge 1$ is also a positive definite self-adjoint operator of X. We can verify the following properties.

Proposition 2.1. ([5, Proposition 3.1]) *For $0 \le \theta \le 1$, $\theta \ne \frac{3}{8}, \frac{7}{8}$, we have*

$$(2.2) \quad \begin{cases} \mathcal{D}(A^\theta) = H^{4\theta}(\Omega), & \text{when } 0 \le \theta < \frac{3}{8}, \\ \mathcal{D}(A^\theta) = H^{4\theta}_N(\Omega) = \{u \in H^{4\theta}(\Omega);\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}, & \text{when } \frac{3}{8} < \theta < \frac{7}{8}, \\ \mathcal{D}(A^\theta) = H^{4\theta}_{N^2}(\Omega) = \{u \in H^{4\theta}(\Omega);\ \frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\Delta u = 0 \text{ on } \partial\Omega\}, & \text{when } \frac{7}{8} < \theta \le 1 \end{cases}$$

with norm equivalence.

We remark that, even when $\theta = \frac{3}{8}$ or $\frac{7}{8}$, it is true that $\mathcal{D}(A^{\frac{3}{8}}) \subset H^{\frac{3}{2}}(\Omega)$ and $\mathcal{D}(A^{\frac{7}{8}}) \subset H^{\frac{7}{2}}(\Omega)$, respectively, with continuous embedding.

The nonlinear operator F is then defined by

$$F(u) = -\mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right) - 2\sqrt{a}\Delta u + u, \qquad u \in \mathcal{D}(A^{\frac{7}{8}}) \subset H^{\frac{7}{2}}(\Omega). \tag{2.3}$$

According to [5, Proposition 3.2], it holds that

$$\|F(u)-F(v)\| \le C[\|A^{\frac{1}{2}}(u-v)\|+(\|A^{\frac{7}{8}}u\|+\|A^{\frac{7}{8}}v\|)\|A^{\frac{1}{4}}(u-v)\|], \qquad u,\, v \in \mathcal{D}(A^{\frac{7}{8}}). \tag{2.4}$$

The general result on abstract semilinear evolution equations (cf. [5, Theorem 2.1]) then provides local existence of solutions. For any $u_0 \in \mathcal{D}(A^{\frac{1}{4}}) = H^1(\Omega)$, (2.1) and hence (1.1) has a unique local solution in the function space:

$$u \in \mathcal{C}((0, T_{u_0}]; H^4_{N^2}(\Omega)) \cap \mathcal{C}([0, T_{U_0}]; H^1(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; L^2(\Omega))$$

with the estimate

$$t^{\frac{3}{4}}\|u(t)\|_{H^4} + \|u(t)\|_{H^1} \le C_{u_0}, \qquad 0 < t \le T_{u_0},$$

$T_{u_0} > 0$ and $C_{u_0} > 0$ being determined by $\|u_0\|_{H^1}$ alone.

It is as well possible to show continuous dependence of solutions on initial values. Let $0 < R < \infty$. For any $u_0 \in \overline{B}^{H^1}(0; R)$ (closed ball of $H^1(\Omega)$ centered at the origin with radius R), let Problem (2.1) have a local solution on an interval $[0, T_R]$, $T_R > 0$ being dependent only on R. In fact, we have

$$\|u(t) - v(t)\|_{H^1} \le C_R\|u_0 - v_0\|_{H^1}, \qquad 0 \le t \le T_R;\ u_0,\, v_0 \in \overline{B}^{H^1}(0; R), \tag{2.5}$$

where u (resp. v) denotes the local solution of (3.1) with initial value u_0 (resp. v_0). For the proof, see [12, Corollary 3.2] (confer also [5, Proposition 4.3]). In addition, it is possible to verify that

$$\|A^\alpha[u(t)-v(t)]\|_{L^2} \le C_{R,\alpha}\|A^\alpha[u_0-v_0]\|_{L^2},\ \ 0 \le t \le T_R;\ u_0,\, v_0 \in \overline{B}^{H^1}(0; R)\cap\mathcal{D}(A^\alpha) \tag{2.6}$$

for any exponent $\frac{1}{4} \le \alpha < 1$.

Global solutions. As shown in [5, Proposition 4.1], we can build a priori estimates for any local solution u of (2.1) with $u_0 \in H^1(\Omega)$ which is given by

$$\|u(t)\|_{H^1} \le p(\|u_0\|_{H^1}), \tag{2.7}$$

where $p(\cdot)$ is a specific continuous increasing function. By the standard argument, this together with the local existence of solutions provides global existence, see [5, Theorem 4.1]. Indeed, for any $u_0 \in H^1(\Omega)$, (2.1) has a unique global solution in the function space:

$$u \in \mathcal{C}((0, \infty); H^4_{N^2}(\Omega)) \cap \mathcal{C}([0, \infty); H^1(\Omega)) \cap \mathcal{C}^1((0, \infty); L^2(\Omega)).$$

Dynamical system. For $u_0 \in H^1(\Omega)$, let $u(t; u_0)$ denote the global solution with the initial value u_0. For each $0 \le t < \infty$, we set $S(t)u_0 = u(t; u_0)$. Then, $S(t)$ defines a nonlinear semigroup on $H^1(\Omega)$ which is, on account of (2.5), continuous from $H^1(\Omega)$ into itself. Thus, $(S(t), H^1(\Omega), H^1(\Omega))$ is a dynamical system determined from (2.1).

We notice thanks to (2.6) that $S(t)$ is continuous from $\mathcal{D}(A^\alpha)$ into itself for any exponent $\frac{1}{4} \le \alpha < 1$. This means that $(S(t), \mathcal{D}(A^\alpha), \mathcal{D}(A^\alpha))$ is also a dynamical system for any $\frac{1}{4} \le \alpha < 1$.

We here introduce a subspace of $H^s(\Omega)$ in such a way that

$$H^s_m(\Omega) = \left\{ u \in H^s(\Omega); m(u) \equiv |\Omega|^{-1} \int_\Omega u\, dx = 0 \right\}, \qquad 0 \le s < \infty.$$

It is easy to see that, if $u_0 \in H^1_m(\Omega)$, then $m(S(t)u_0) = 0$ for every $0 < t < \infty$, i.e., $S(t)u_0 \in H^4_m(\Omega)$. Furthermore, (2.7) can be strengthened in the form

$$\|S(t)u_0\|^2_{H^1} \le C[e^{-\rho t}\|u_0\|^2_{H^1} + 1], \qquad 0 < t < \infty;\ u_0 \in H^1_m(\Omega)$$

with some exponent $\rho > 0$, see [5, Corollary 4.1]. As shown in [6], see Theorem 3.1 and Corollary 3.1, this then yields that $(S(t), H^1_m(\Omega), H^1_m(\Omega))$ has a finite-dimensional exponential attractor. It is the same for $(S(t), \mathcal{D}(A^\alpha) \cap H^1_m(\Omega), \mathcal{D}(A^\alpha) \cap H^1_m(\Omega))$.

3 General framework Consider the Cauchy problem for a semilinear abstract parabolic evolution equation

$$\begin{cases} \dfrac{du}{dt} + Au = F(u), & 0 < t < \infty, \\ u(0) = u_0 \end{cases} \tag{3.1}$$

in a Banach space X. Here, A is a closed linear operator of X the spectral set of which is contained in a sectorial domain $\Sigma = \{\lambda \in \mathbb{C};\ |\arg \lambda| < \omega\}$ with angle $0 < \omega < \frac{\pi}{2}$ and the resolvent of A satisfies [5, (2.2)]. We assume that the nonlinear operator $F(u)$ satisfies the Lipschitz condition [5, (2.4)] with some exponents $0 \le \alpha \le \eta < 1$. Then, as noticed by [5, Theorem 2.1], (3.1) has a unique local solution for any initial value $u_0 \in \mathcal{D}(A^\alpha)$ satisfying [5, (2.3)], i.e., $\|A^\alpha u_0\| \le R$. The local solution exists at least on an interval $[0, T_R]$, where $T_R > 0$ is determined by R alone.

For $u_0 \in \mathcal{D}(A^\alpha)$, let $u(\cdot; u_0)$ denote any local solution of (**??**). We assume that the a priori estimate

$$\|A^\alpha u(t; u_0)\| \le p(\|A^\alpha u_0\|), \qquad u_0 \in \mathcal{D}(A^\alpha) \tag{3.2}$$

holds for any local solution with some specifically fixed continuous increasing function $p(\cdot)$. By the standard arguments, we can conclude under (3.2) that (3.1) has a global solution on the whole interval $[0, \infty)$.

Let $u(\cdot; u_0)$ denote the global solution of (3.1). We then set $S(t)u_0 = u(t; u_0)$ for $u_0 \in \mathcal{D}(A^\alpha)$. Then, $S(t)$ is a continuous nonlinear semigroup acting on $\mathcal{D}(A^\alpha)$ and $(S(t), \mathcal{D}_\alpha, \mathcal{D}_\alpha)$ defines a dynamical system with phase space $\mathcal{D}_\alpha$ in the universal space $\mathcal{D}_\alpha$, $\mathcal{D}(A^\alpha)$ being abbreviated by $\mathcal{D}_\alpha$.

Let $\overline{u} \in \mathcal{D}(A)$ be a stationary solution of (3.1), i.e., $A\overline{u} = F(\overline{u})$. Clearly, $\overline{u}$ is an equilibrium of $(S(t), \mathcal{D}_\alpha, \mathcal{D}_\alpha)$. We are concerned with investigating stability or instability of $\overline{u}$.

To this end, we assume that $F : \mathcal{D}(A^\eta) \to X$ is of class $\mathcal{C}^{1,1}$ in a neighborhood of $\overline{u}$. That is, F is Fréchet differentiable from $\mathcal{D}(A^\eta)$ to X in a neighborhood of $\overline{u}$ in the topology of $\mathcal{D}_\alpha$ and the derivative satisfies

$$\|[F'(u) - F'(v)]h\| \le C\|A^\alpha(u - v)\|\|A^\eta h\|, \qquad u, v \in \mathcal{O}(\overline{u});\ h \in \mathcal{D}(A^\eta), \tag{3.3}$$

where $\mathcal{O}(\overline{u})$ is a neighborhood of $\overline{u}$.

These assumptions in fact imply that the semigroup $S(t):\mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is Fréchet differentiable; in addition, $S(t)$ is of class $\mathcal{C}^{1,1}$ in a neighborhood $\mathcal{O}'(\overline{u})$ of $\overline{u}$ in $\mathcal{D}_\alpha$, i.e.,

$$\|S(t)'u - S(t)'v\|_{\mathcal{L}(\mathcal{D}_\alpha,\mathcal{D}_\alpha)} \le C\|A^\alpha(u-v)\|, \qquad u,\, v \in \mathcal{O}'(\overline{u});\ 0 \le t \le t^*, \tag{3.4}$$

$t^* > 0$ being arbitrarily fixed time. For detail, see the proof of [1, Theorem 5.1].

We further assume a spectral separation condition for $\sigma(A - F'(\overline{u}))$ of the form

$$\sigma(A - F'(\overline{u})) \cap \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda = 0\} = \emptyset. \tag{3.5}$$

Then, since $S(t)'\overline{u} = e^{-t\overline{A}}$, where $\overline{A} = A - F'(\overline{u})$, we have the spectral separation for $S(t)'\overline{u}$, i.e.,

$$\sigma^{\mathcal{D}_\alpha}(S(t)'\overline{u}) \cap \{\lambda \in \mathbb{C};\ |\lambda| = 1\} = \emptyset. \tag{3.6}$$

According to [18, Chapter VII, Theorem 3.1], under (3.4) and (3.6), there exists a smooth local unstable manifold $\mathcal{M}_+(\overline{u};\mathcal{O})$ in a neighborhood $\mathcal{O}$ of $\overline{u}$ in $\mathcal{D}_\alpha$.

When

$$\sigma(A - F'(\overline{u})) \subset \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda > 0\}, \tag{3.7}$$

it actually follows that $\mathcal{M}_+(\overline{u};\mathcal{O}) = \{\overline{u}\}$. Whence, under (3.7), $\overline{u}$ is a stable stationary solution. In the meantime, when

$$\sigma(A - F'(\overline{u})) \cap \{\lambda \in \mathbb{C};\ \operatorname{Re}\lambda < 0\} \ne \emptyset, \tag{3.8}$$

$\mathcal{M}_+(\overline{u};\mathcal{O})$ is not trivial and $\overline{u}$ is an unstable stationary solution.

4 Differentiability of $F(u)$ Let us apply the general results explained in the preceding section by setting $X_m = L^2_m(\Omega)$ and $A_m = (-\sqrt{a}\varDelta + 1)^2$ is considered in $L^2_m(\Omega)$. So we have

$$\mathcal{D}(A_m) = \{u \in H^4_{N^2}(\Omega);\ m(u) = 0\}.$$

The nonlinear operator $F_m : \mathcal{D}(A_m^{\frac{7}{8}}) \to X_m$ is given by (2.3) again. We take as $\overline{u}$ the zero solution which is a unique homogeneous stationary solution to (1.1) in the space X_m.

We can entirely follow the arguments reviewed in Section 2 in order to construct a dynamical system $(S(t), H^1_m(\Omega), H^1_m(\Omega))$ as well as $(S(t), \mathcal{D}(A_m^\alpha), \mathcal{D}(A_m^\alpha))$ for any exponent $\frac{1}{4} \le \alpha < 1$. Proposition 4.2 which will be shown below suggests that it is natural to take $\alpha = \frac{1}{2}$. In view of (2.2), we have

$$\mathcal{D}(A_m^{\frac{1}{2}}) = H^2_{N,m}(\Omega) \equiv \{u \in H^2_N(\Omega);\ m(u) = 0\}.$$

In this section, we intend to verify Fréchet differentiability of F_m and the conditions (3.3) with $\alpha = \frac{1}{2}$.

Proposition 4.1. *$F_m : \mathcal{D}(A_m^{\frac{7}{8}}) \to X_m$ is Fréchet differentiable and the derivative is given by*

$$F'_m(u)h = -\mu\nabla\cdot\left(\frac{\nabla h}{1+|\nabla u|^2}\right) + 2\mu\nabla\cdot\left(\frac{(\nabla u\cdot\nabla h)\nabla u}{(1+|\nabla u|^2)^2}\right) - 2\sqrt{a}\varDelta h + h,\ u,\ h \in \mathcal{D}(A_m^{\frac{7}{8}}). \tag{4.1}$$

Proof. Let u, $h \in \mathcal{D}(A_m^{\frac{7}{8}})$. By (2.3) we have

$$\begin{aligned} F_m(u+h) - F_m(u) &= -\mu\nabla\cdot\left[\left(\frac{1}{1+|\nabla(u+h)|^2} - \frac{1}{1+|\nabla u|^2}\right)\nabla(u+h)\right] \\ &\quad - \mu\nabla\cdot\left(\frac{\nabla(u+h)-\nabla u}{1+|\nabla u|^2}\right) - 2\sqrt{a}\Delta h + h \\ &= -\mu\nabla\cdot\left[\frac{(-2\nabla u\cdot\nabla h - |\nabla h|^2)\nabla(u+h)}{(1+|\nabla(u+h)|^2)(1+|\nabla u|^2)}\right] - \mu\nabla\cdot\left(\frac{\nabla h}{1+|\nabla u|^2}\right) - 2\sqrt{a}\Delta h + h. \end{aligned}$$

By the similar calculations as for the proof of [5, Proposition 3.2],

$$\|F_m(u+h) - F_m(u) - F_m'(u)h\|_{L^2} \le C\|A_m^{\frac{7}{8}}h\|_{L^2}^2(\|A_m^{\frac{7}{8}}u\|_{L^2} + \|A_m^{\frac{7}{8}}h\|_{L^2}).$$

Hence, $F_m : \mathcal{D}(A_m^{\frac{7}{8}}) \to X_m$ is Fréchet differentiable at u, and the derivative is given by (4.1). □

Proposition 4.2. *Let $u \in \mathcal{D}(A_m^{\eta})$ varies in the ball $B^{\mathcal{D}(A_m^{\frac{1}{2}})}(0;1)$. Then, $F_m'(u)$ satisfies the Lipschitz condition*

$$\|[F_m'(u) - F_m'(v)]h\|_{L^2} \le C\|A_m^{\frac{1}{2}}(u-v)\|_{L^2}\|A_m^{\frac{7}{8}}h\|_{L^2},$$
$$u,\ v \in \mathcal{D}(A_m^{\frac{7}{8}}) \cap B^{\mathcal{D}(A_m^{\frac{1}{2}})}(0;1);\ h \in \mathcal{D}(A_m^{\frac{7}{8}}).$$

Proof. We know that $F_m'(u)$ is given by (4.1). Then, the desired estimate can be seen directly by the similar calculations as for the proof of [5, Proposition 3.2]. □

5 Spectral separation condition Under the same situation as in Section 4, let us now verify the condition (3.5).

Let Λ denote the realization of $-\Delta$ in $L_m^2(\Omega)$ under the Neumann boundary conditions. The operator Λ possesses denumerable positive eigenvalues and the corresponding real eigenfunctions can constitute an orthonormal basis of $L_m^2(\Omega)$. So, let

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \to \infty$$

be eigenvalues of Λ and let ϕ_1, ϕ_2, $\phi_3, \ldots$ be corresponding real eigenfunctions which constitute an orthonormal basis. For each $k = 1, 2, 3, \ldots$, let X_k be the eigenspace of λ_k which is a one-dimensional subspace of $L_m^2(\Omega)$. Any two subspaces X_k and X_ℓ are orthogonal if $k \ne \ell$, and $X_m = L_m^2(\Omega)$ is given by an infinite sum $X_m = \sum_{k=1}^{\infty} X_k$.

According to (**??**), we have

$$F_m'(0)h = -(\mu + 2\sqrt{a})\Delta h + h, \qquad h \in \mathcal{D}(A_m^{\frac{7}{8}}).$$

Therefore, the operator $\overline{A}_m = A_m - F_m'(0) = a\Delta^2 + \mu\Delta$ maps the subspace X_k into itself, namely, X_k is an invariant set of $\overline{A}_m$ for every k. Consequently, the operator $\overline{A}_m$ can also be decomposed as $\overline{A}_m = \sum_{k=1}^{\infty} \overline{A}_k$, where $\overline{A}_k$ is the part of $\overline{A}_m$ in X_k, i.e.,

$$\overline{A}_k\phi_k = (a\lambda_k^2 - \mu\lambda_k)\phi_k.$$

Hence, $\sigma(\overline{A}_k) = \{a\lambda_k^2 - \mu\lambda_k\}$.

Let $\lambda \in i\mathbb{R}$. Let k be sufficiently large so that $a\lambda_k > \mu$ holds. Then, $\lambda \in \rho(\overline{A}_k)$ and

$$\|(\lambda - \overline{A}_k)^{-1}\|_{\mathcal{L}(X_k)} \leq \frac{1}{(a\lambda_k - \mu)\lambda_k}.$$

This means that $\lambda \in i\mathbb{R}$ belongs to $\rho(\overline{A})$ if and only if $\lambda \in \rho(\overline{A}_k)$ for every $k = 1, 2, 3, \ldots$. In other words, $\lambda \notin \sigma(\overline{A})$ if and only $\lambda \notin \sigma(\overline{A}_k) = \{a\lambda_k^2 - \mu\lambda_k\}$ for every k. In view of this fact, we will make the following assumption

$$\lambda_k \neq \frac{\mu}{a} \qquad \text{for every } k = 1, 2, 3, \ldots. \tag{5.1}$$

Under (5.1), it is true that $\sigma(\overline{A}) \cap i\mathbb{R} = \emptyset$, namely, the spectral separation condition (3.5) is fulfilled.

6 Stability or instability conditions Let $\lambda \in \mathbb{C}$ satisfy $\operatorname{Re}\lambda \leq 0$. By the same reason as before, we see that $\lambda \notin \sigma(\overline{A}_m)$ if and only if $\lambda \notin \sigma(\overline{A}_k)$ for every k. Therefore, if the condition

$$\mu < a\lambda_1 \tag{6.1}$$

is valid, then, as $\bigcup_{k=1}^{\infty} \sigma(\overline{A}_k) \subset \{\lambda; \operatorname{Re}\lambda > 0\}$, λ such that $\operatorname{Re}\lambda \leq 0$ cannot belong to $\sigma(\overline{A}_m)$, namely, $\sigma(\overline{A}_m) \subset \{\lambda; \operatorname{Re}\lambda > 0\}$. Thus, under (6.1), (3.7) is fulfilled and 0 is a stable stationary solution of $(S(t), H^2_{N,m}(\Omega), H^2_{N,m}(\Omega))$.

On the other hand, if the condition

$$N = \#\{\lambda_k;\, \mu > a\lambda_k\} \neq 0 \tag{6.2}$$

is satisfied, then $\sigma(\overline{A}) \cap \{\lambda;\ \operatorname{Re}\lambda < 0\} \neq \emptyset$, namely, (3.8) is fulfilled. Thus, under (5.1) and (6.2), 0 has a nontrivial unstable manifold $\mathcal{M}_+(0)$ and is an unstable equilibrium of $(S(t), H^2_{N,m}(\Omega), H^2_{N,m}(\Omega))$.

We remark $\overline{A}_m$ has a real eigenfunction ϕ_k for each λ_k. This means that the unstable manifold $\mathcal{M}_+(0)$ is tangential to an N-dimensional subspace of $H^2_{N,m}(\Omega)$ whose basis is composed by real functions. In particular, it is deduced that

$$\dim \mathcal{M} \geq \dim \mathcal{M}_+(0) \geq N.$$

References

[1] M. Aida, T. Tsujikawa, M. Efendiev, A. Yagi and M. Mimura, *Lower estimate of attractor dimension for chemotaxis growth system*, J. London Math. Soc. **74** (2006), 453–474.

[2] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992.

[3] W. K. Burton, N. Cabrera and F. C. Frank, *The growth of crystals and the equilibrium structure of their surfaces*, Phil. Trans. Royal Soc. London **243** (1951), 299–358.

[4] G. Ehrlich and F. G. Hudda, *Atomic view of surface self-diffusion: tungsten on tungsten*, J. Chem. Phys. **44** (1966), 1039–1049.

[5] H. Fujimura and A. Yagi, *Dynamical system for BCF model describing crystal surface growth*, Bull. Chelyabinsk Univ., Math. Mech. Info. **10**(2008), 75–88.

[6] H. Fujimura and A. Yagi, *Dynamical system for BCF model describing crystal surface growth*, Internat. Math. Forum **3**(2008), 1803–1812.

[7] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, London, 1985.

[8] A. W. Hunt, C. Orme, D. R. M, Williams, B. G. Orr and L. M. Sander, *Instabilities in MBE growth*, Europhys. Lett. **27** (1994), 611–616.

[9] M. D. Johnson, C. Orme, A. W. Hunt,D. Graff,J. Sudijono, L. M. Sauder, and B. G. Orr, *Stable and unstable growth in molecular beam epitaxy*, Phys. Rev. Lett. **72** (1994), 116–119.

[10] S. G. Krein, *Linear Differential Equations in Banach Space*, AMS, Providence, 1971.

[11] W. W. Mullins, *Theory of thermal grooving*, J. Applied Phys. **28** (1957), 333–339.

[12] K. Osaki and A. Yagi, *Global existence for a chemotaxis-growth system in* $\mathbb{R}^2$, Adv. Math. Sci. Appl. **12** (2002), 587–606.

[13] O. Pierre-Louis, C. Misbah, and Y. Saito, J. Krug and P. Politi, *New nonlinear evolution equation for steps during molecular beam epitaxy on vicinal surfaces*, Phys. Rev. Lett. **80** (1998), 4221–4224.

[14] M. Rost and J. Krug, *Coarsening of surface structures in unstable epitaxial growth*, Phys. Rev. E **55** (1997), 3952–3957.

[15] M. Rost, P. Smilauer and J. Krug, *Unstable epitaxy on vicinal surfaces*, Surface Science **369** (1996), 393–402.

[16] R. L. Schwoebel and E. J. Shipsey, *Step motion on crystal surfaces*, J. Appl. Phys. **37** (1966), 3682–3686.

[17] H. Tanabe, *Evolution Equations*, Pitman, London, 1979.

[18] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd ed., Springer, Berlin, 1997.

[19] M. Uwaha, *Study on Mechanism of Crystal Growth*, Kyoritsu Publisher, Tokyo, 2002 (in Japanese).

[20] A. Yagi, H_∞ *functional calculus and characterization of domains of fractional powers*, Proc. 17th Conference on "Operator Theory and Applications", Birkhäuser, Basel, 2008, 217–235.

Department of Applied Physics, Osaka University, Suita, Osaka 565-0871, Japan
E-mail ; yagi@ap.eng.osaka-u.ac.jp