ON T-FUZZY IDEALS IN HILBERT ALGEBRAS

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ABSTRACT. Using a t-norm T, we introduce the notion of (imaginable) T-fuzzy subalgebras, (imaginable) T-fuzzy deductive systems and (imaginable) T-fuzzy ideals, and obtain some related results. We give relations between an imaginable T-fuzzy subalgebra and an imaginable fuzzy deductive system.

1. INTRODUCTION

Following the introduction of Hilbert algebras by Diego [4], the algebra and related concepts were developed by Busneag [2, 3]. The concept of of fuzzy sets was established by Zadeh [7]. This concept has been applied to Hilbert algebras by W. A. Dudek [5]. In the present paper, using a t-norm T we will redefine the fuzzy subalgebras, fuzzy deductive systems and fuzzy ideals in Hilbert algebras. Furthermore, we introduce the notion of (imaginable) T-fuzzy subalgebras and (imaginable) T-fuzzy ideals, and obtain some related results. We give relations between an imaginable T-fuzzy subalgebra and an imaginable T-fuzzy deductive system.

2. PRELIMINARIES

We include some elementary aspects of Hilbert algebras that are necessary for this paper, and for more details we refer to (see [2,3,4]).

A *Hilbert algebra* is a triple $(H, *, 1)$, where H is a nonempty set, " $*$ " is a binary operation on H, $1 \in H$ is an element such that the following three axioms are satisfied for every $x, y, z \in H$:

(H1) $x * (y * x) = 1$,
(H2) $(x * (y * z)) * ((x * y) * (x * z)) = 1$,
(H3) if $x * y = y * x = 1$ then $x = y$.

If H is a Hilbert algebra, then the relation $x \leq y$ if and only if $x * y = 1$ is a partial order on H, which will be called the *natural ordering* on H. With respect to this ordering, 1 is the largest element of H. A subset S of a Hilbert algebra H is called a *subalgebra* of H if $x * y \in S$ for all $x, y \in S$.

A mapping $f : H \to H'$ of Hilbert algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in H$.

In a Hilbert algebra H, the following hold.

(H4) $x * x = 1$,
(H5) $x * 1 = 1$,
(H6) $x * (y * z) = (x * y) * (x * z)$,

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(H7) $1 * x = x$,
(H8) $x * (y * z) = y * (x * z)$. for all $x, y, z \in H$.
(H9) $x * ((x * y) * y)) = 1$
(H10) $x \leq y$ implies $z * x \leq z * y$ for all $x, y, z \in H$.

Definition 2.1. If H is a Hilbert algebra, a subset D of H is a *deductive system of* H if it satisfies:

(1) $1 \in D$,
(2) $x \in D$ and $x * y \in D$ imply $y \in D$.

Definition 2.2. If H is a Hilbert algebra, a subset I of H is a *an ideal of* H if it satisfies:

(1) $1 \in I$,
(2) $x * y \in I$ for $x \in H$ and $y \in I$,
(3) $(y_1 * (y_2 * x)) * x \in I$ for $y_1, y_2 \in I$ and $x \in H$.

Let $(H, *, 0)$ be a Hilbert algebra. A fuzzy set μ in H is a map $\mu : H \to [0, 1]$. Let μ be a fuzzy set in H. For $\alpha \in [0, 1]$, the set $R_\mu^\alpha = \{x \in H \mid \mu(x) \geq \alpha\}$ is called an *upper level set* of μ. A fuzzy set μ in a Hilbert algebra H is called a *fuzzy subalgebra* of H if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in H$.

Definition 2.3. A fuzzy set μ in a Hilbert algebra H is called *a fuzzy deductive system* if it satisfies

(1) $\mu(1) \geq \mu(x)$ for all $x \in H$,
(2) $\mu(y) \geq \min\{\mu(x * y), \mu(x)\}$ for all $x, y \in H$.

Lemma 2.4. *Let μ be a fuzzy deductive system of a Hilbert algebra H and $x \leq y$. Then $\mu(y) \geq \mu(x)$.*

Theorem 2.5. *Let μ be a fuzzy set of a Hilbert algebra H. Then μ is a fuzzy deductive system of a Hilbert algebra H if and only if for all $x, y, s \in H$, the inequality $x * y \geq s$ implies $\mu(y) \geq \min\{\mu(x), \mu(s)\}$.*

Proof. Suppose that for all $x, y, s \in X, x * y \geq s$ implies $\mu(y) \geq \min\{\mu(x), \mu(s)\}$. Since $x * y \geq x * y$, it follows that $\mu(y) \geq \min\{\mu(x * y), \mu(x)\}$. Also, since $x * 1 \leq x * 1$, we have $x*(x*1) \leq x*(x*1)$. So, $\mu(1) = \mu(x*1) \geq \min\{\mu(x*(x*1)), \mu(x)\} = \min\{\mu(1), \mu(x)\} = \mu(x)$. Hence μ is a fuzzy deductive system of a Hilbert algebra H. Conversely, suppose that μ is a fuzzy deductive system of a Hilbert algebra H and $x * y \geq s$. It follows from Lemma 2.4 that $\mu(x * y) \geq \mu(s)$. So, by Definition 2.3,

$$\mu(y) \geq \min\{\mu(x * y), \mu(x)\} \geq \min\{\mu(x), \mu(s)\}.$$

This completes the proof. □

Definition 2.6 ([5])**.** A fuzzy set μ in a Hilbert algebra H is called *a fuzzy ideal* if it satisfies

(F1) $\mu(1) \geq \mu(x)$ for all $x \in H$,
(F2) $\mu(x * y) \geq \mu(y)$ for all $x, y \in H$,
(F3) $\mu((y_1 * (y_2 * x)) * x) \geq \min\{\mu(y_1), \mu(y_2)\}$ for all $x, y_1, y_2 \in H$.

3. T-FUZZY IDEALS IN HILBERT ALGEBRAS

In the sequel, we use H to denote a Hilbert algebra unless otherwise specified. We begin with the following definition.

Definition 3.1 ([1]). By a *t-norm* T, we mean a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:

(T1) $T(x, 1) = x$,
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$,
(T3) $T(x, y) = T(y, x)$,
(T4) $T(x, T(y, z)) = T(T(x, y), z)$,

for all $x, y, z \in [0, 1]$.

Every t-norm T has a useful property:

$$T(\alpha, \beta) \leq \min(\alpha, \beta) \text{ for all } \alpha, \beta \in [0, 1].$$

For a t-norm T on $[0,1]$, denote by Δ_T the set of element $\alpha \in [0,1]$ such that $T(\alpha, \alpha) = \alpha$, i.e., $\Delta_T := \{\alpha \in [0, 1] \mid T(\alpha, \alpha) = \alpha\}$.

Definition 3.2. Let T be a t-norm. A fuzzy set μ in X is said to satisfy *imaginable property* if $\mathrm{Im}(\mu) \subseteq \Delta_T$.

Lemma 3.3. *Let T be a t-norm and Λ an index set. Then T satisfy the following conditions:*

(1) $T(\alpha \wedge \beta, \gamma) = T(\alpha, \gamma) \wedge T(\beta, \gamma)$ *for all* $\alpha, \beta \in [0, 1]$.
(2) $T(\inf_{i\in\Lambda} \alpha_i, \beta) \leq \inf_{i\in\Lambda} T(\alpha_i, \beta)$ *for any* $\alpha_i, \beta \in [0, 1]$, $i \in \Lambda$.

Proof. (1) If $\alpha < \beta$, then $T(\alpha \wedge \beta, \gamma) = T(\alpha, \gamma)$ and $T(\alpha, \gamma) \wedge T(\beta, \gamma) = T(\alpha, \gamma)$, since $T(\alpha, \gamma) \leq T(\beta, \gamma)$. Hence $T(\alpha \wedge \beta, \gamma) = T(\alpha, \gamma) \wedge T(\beta, \gamma)$. In the cases of $\alpha = \beta$ and $\alpha > \beta$, we can prove similarly.

(2) Since $T(\inf_{i\in\Lambda} \alpha_i, \beta) \leq T(\alpha_i, \beta)$ for each $i \in \Lambda$, $T(\inf_{i\in\Lambda} \alpha_i, \beta)$ is a lower bound of the set $\{T(\alpha_i, \beta) \mid i \in \Lambda\}$. Hence $T(\inf_{i\in\Lambda} \alpha_i, \beta) \leq \inf_{i\in\Lambda} T(\alpha_i, \beta)$. □

The equality of the above Lemma 3.3.(2) is not holed, in general, as following example.

Example 3.1. Let $\mathbb{N}$ be the natural number. We define a function $T : [0,1] \times [0,1] \to [0,1]$ by

$$T(\alpha, \beta) = \begin{cases} \alpha & \text{if } \beta = 1 \\ \beta & \text{if } \alpha = 1 \\ \frac{1}{2} & \text{if } \alpha, \beta \in (\frac{1}{2}, 1) \\ 0 & \text{otherwise} \end{cases}$$

for each $\alpha, \beta \in [0, 1]$. Then the function T is a t-norm. Let $a_n = \frac{1}{2} + \frac{1}{n} + 1$. Then we have

$$T(\inf_{n\in\mathbb{N}} a_n, \frac{2}{3}) = T(\frac{1}{2}, \frac{2}{3}) = 0$$

and

$$\inf_{n\in\mathbb{N}} T(a_n, \frac{2}{3}) = \inf_{n\in\mathbb{N}} \frac{1}{2} = \frac{1}{2}.$$

Hence $T(\inf_{n\in\mathbb{N}} \alpha_n, \frac{2}{3}) \lneqq \inf_{n\in\mathbb{N}} T(\alpha_n, \frac{2}{3})$

Proposition 3.4. *Let i is an arbitrary index set and T a t-norm. If for any $\beta \in [0,1]$ there is an $\alpha \in [0,1]$ such that $T(\alpha, \beta) = t$ for all $t \in [0, \beta]$, then $T(\inf_{i\in\Lambda} \alpha_i, \beta) = \inf_{i\in\Lambda} T(\alpha_i, \beta)$ for any $\alpha_i, \beta \in [0, 1]$, $i \in \Lambda$.*

Proof. Since $\inf_{i\in\Lambda} \alpha_i \le \alpha_\gamma$ for each $\gamma \in \Lambda$, $T(\inf_{i\in\Lambda} \alpha_i, \beta) \le T(\alpha_\gamma, \beta)$ for each $\gamma \in \Lambda$. This implies

$$T(\inf_{i\in\Lambda} \alpha_i, \beta) \le \inf_{i\in\Lambda} T(\alpha_i, \beta).$$

Conversely, let $\gamma = \inf_{i\in\Lambda} \alpha_i$ and $s = \inf_{i\in\Lambda} T(\alpha_i, \beta)$. If $T(\gamma, \beta) < s$, then there exists t such that $T(\gamma, \beta) < t < s$. Since $t < s \le T(\alpha_i, \beta) \le \beta$, there is a $\alpha \in [0, 1]$ such that $T(\alpha, \beta) = t$ by hypothesis. Hence $T(\alpha, \beta) < T(\alpha_i, \beta)$ for all $i \in \Lambda$. This implies $\alpha < \alpha_i$ for all $i \in \Lambda$, and $\alpha \le \gamma = \inf_{i\in\Lambda} \alpha_i$. That is, $t = T(\alpha, \beta) \le T(\gamma, \beta)$. This is a contradiction for $T(\gamma, \beta) < s$. Hence $T(\gamma, \beta) \le s$. □

Definition 3.5. A function μ in H is called a *fuzzy ideal* of H with respect to a t-norm T (briefly, a T-*fuzzy ideal* of H) if
(TF1) $\mu(1) \ge \mu(x)$ for all $x \in H$,
(TF2) $\mu(x * y) \ge \mu(y)$ for all $x, y \in H$,
(TF3) $\mu((y_1 * (y_2 * x)) * x) \ge T(\mu(y_1), \mu(y_2))$ for all $x, y_1, y_2 \in H$.

Example 3.2. Let $H = \{a, b, c, d, 1\}$ in which $*$ is defined by

$*$	a	b	c	d	1
a	1	1	1	1	1
b	a	1	c	1	1
c	a	b	1	1	1
d	a	b	c	1	1
1	a	b	c	d	1

Then H is a Hilbert algebra. Define $\mu : H \to [0, 1]$ by $\mu(1) = 0.9, \mu(a) = \mu(c) = 0.8, \mu(b) = \mu(d) = 0.7$. Let T_m be a t-norm defined by

$$T_m(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$$

for all $\alpha, \beta \in [0, 1]$. Then, routine calculations give that μ is a T_m-fuzzy ideal of a Hilbert algebra H.

Proposition 3.6. *If $\{\mu_i : i \in \Lambda\}$ is a family of fuzzy ideals of a Hilbert algebra H, then $\bigwedge_{i\in\Lambda} \mu_i$ is a T-fuzzy ideal of a Hilbert algebra H.*

Proof. Let $x \in H$. Then, we have

$$\begin{aligned}(\bigwedge_{i\in\Lambda} \mu_i)(1) &= \inf\{\mu_i(1) : i \in \Lambda\} \\ &\ge \inf\{\mu_i(x) : i \in \Lambda\} \\ &= (\bigwedge_{i\in\Lambda} \mu_i)(x).\end{aligned}$$

Let y, y_1 and $y_2 \in H$. Then we have

$$\begin{aligned}(\bigwedge_{i\in\Lambda} \mu_i)(x * y) &= \inf\{\mu_i(x * y) : i \in \Lambda\} \\ &\ge \inf\{\mu_i(y) : i \in \Lambda\} \\ &= (\bigwedge_{i\in\Lambda} \mu_i)(y),\end{aligned}$$

and by Lemma 3.2(2),

$$\begin{aligned}(\bigwedge_{i\in\Lambda}\mu_i)((y_1*(y_2*x))*x) &= \inf\{\mu_i((y_1*(y_2*x))*x) : i\in\Lambda\}\\ &\geq \inf\{T(\mu_i(y_1),\mu_i(y_2)) : i\in\Lambda\}\\ &\geq T(\inf\{\mu_i(y_1) : i\in\Lambda\},\inf\{\mu_i(y_2) : i\in\Lambda\})\\ &= T((\bigwedge_{i\in\Lambda}\mu_i)(y_1),(\bigwedge_{i\in\Lambda}\mu_i)(y_2)).\end{aligned}$$

Hence $\bigwedge_{i\in\Lambda}\mu_i$ is a T-fuzzy ideal of a Hilbert algebra H. □

Definition 3.7. A T-fuzzy ideal of H is said to be *it imaginable* if it satisfies the imaginable property.

Theorem 3.8. *Let T be a t-norm. Then every imaginable T-fuzzy ideal of H is a fuzzy ideal of H.*

Proof. Let μ be an imaginable T-fuzzy ideal of H. Then

$$\mu((y_1*(y_2*x))*x)\geq T(\mu(y_1),\mu(y_2)) \text{ for all } x,y_1,y_2\in H.$$

Since μ satisfies the imaginable property, we have

$$\begin{aligned}\min(\mu(y_1),\mu(y_2)) &= T(\min(\mu(y_1),\mu(y_2)),\min(\mu(y_1),\mu(y_2)))\\ &\leq T(\mu(y_1),\mu(y_2))\\ &\leq \min(\mu(y_1),\mu(y_2)).\end{aligned}$$

It follows that $\mu((y_1*(y_1*x))*x)\geq T(\mu(y_1),\mu(y_2))=\min(\mu(y_1),\mu(y_1))$ so that μ is a fuzzy ideal of H. □

Proposition 3.9. *A fuzzy set μ in Hilbert algebra H is a fuzzy ideal of H if and only if each non-empty level subset R^t_μ of μ is an ideal of H* (see [5]).

Using Theorem 3.10 and Proposition 3.11, we have the following corollary:

Corollary 3.10. *If μ is an imaginable T-fuzzy ideal of a Hilbert algebra H, then each non-empty level subset R^t_μ of μ is an ideal of H.*

The following example shows that there exists a t-norm T such that a fuzzy ideal of H may not be an imaginable T-fuzzy ideal of H.

Example 3.3. Let $H=\{a,b,c,d,1\}$ be a Hilbert algebra with the following Cayley table:

$*$	a	b	c	1
a	1	1	1	1
b	a	1	c	1
c	a	b	1	1
1	a	b	c	1

Then a fuzzy set $\mu : H\to[0,1]$ defined by $\mu(1)=0.9$, $\mu(a)=\mu(b)=\mu(c)=0.4$ is a fuzzy ideal of H (see [3, Example 2.7]). Let $\gamma\in(0,1)$ and define the binary operation T_γ on $[0,1]$ as follows:

$$T_\gamma(\alpha,\beta)=\begin{cases}\alpha\wedge\beta & \text{if } \max(\alpha,\beta)=1\\ 0 & \text{if } \max(\alpha,\beta)<1 \text{ and } \alpha+\beta\leq 1+\gamma\\ \gamma & \text{otherwise}\end{cases}$$

$\forall\ \alpha, \beta \in [0,1]$. Then T_γ is a t-norm on $[0,1]$ (see [6, Example 1.2.1]). Thus $T_\gamma(\mu(0), \mu(0)) = T_\gamma(0.9, 0.9) = \gamma \neq \mu(0)$ whenever $\gamma < 0.8$, and so $\mu(0) \notin \Delta_{T_\gamma}$, i.e., $\mathrm{Im}(\mu) \not\subseteq \Delta_{T_\gamma}$ whenever $\gamma < 0.8$. Hence μ is not an imaginable T_γ-fuzzy ideal of H whenever $\gamma < 0.8$.

Now we consider the converse of Corollary 3.12.

Theorem 3.11. *Let T be a t-norm and let μ be an imaginable fuzzy set in H. If each non-empty level subset R^t_μ of μ is an ideal of a Hilbert algebra H, then μ is an imaginable T-fuzzy ideal of H.*

Proof. Suppose that each non-empty level subset R^t_μ of μ is an ideal of H. Then μ is a fuzzy ideal of H (see Proposition 3.11), and so

$$\mu((y_1 * (y_2 * x)) * x) \geq \min(\mu(y_1), \mu(y_2)) \geq T(\mu(y_1), \mu(y_2))$$

for all $x, y \in H$. Hence μ is an imaginable T-fuzzy ideal of H. □

Proposition 3.12. *Every imaginable T-fuzzy ideal of a Hilbert algebra H is order preserving.*

Proof. Let μ be an imaginable T-fuzzy ideal of H and let $x, y \in H$ be such that $x \leq y$. Then

$$\begin{aligned}\mu(y) = \mu((1 * (x * y) * y) &\geq \min(\mu(x), \mu(1)) \quad \text{[by Theorem 3.10]}\\ &\geq T(\mu(x), \mu(1)) \quad [T(\alpha,\beta) \leq \min(\alpha,\beta) \text{ for all } \alpha,\beta \in [0,1]]\\ &\geq T(\mu(x), \mu(x)) \quad \text{[by (T2) and (T3)]}\\ &= \mu(x), \quad \text{[since } \mu \text{ satisfies the imaginable property]}\end{aligned}$$

ending the proof. □

Definition 3.13. A function μ in H is called a *fuzzy deductive system* of H with respect to a t-norm T (briefly, a *T-fuzzy deductive system* of H) if
(FS1) $\mu(1) \geq \mu(x)$ for all $x \in H$,
(FS2) $\mu(y) \geq T(\mu(x * y), \mu(x))$ for all $x, y \in H$.

Definition 3.14. A T-fuzzy deductive system μ of a Hilbert algebra H is said to be *imaginable* if it satisfies the imaginable property.

We give a relation between an imaginable T-fuzzy deductive system of H and an imaginable T-fuzzy ideal of H.

Theorem 3.15. *Let T be a t-norm. Then every imaginable T-fuzzy ideal of H is an imaginable T-fuzzy deductive system of H.*

Proof. Since $y = 1 * y$ for all $y \in X$, we have

$$\begin{aligned}\mu(y) = \mu(1 * y) &= \mu(((x * y) * (x * y)) * y)\\ &\geq T(\mu(x * y), \mu(x)).\end{aligned}$$

Hence μ is an imaginable T-fuzzy deductive system of H. □

Example 3.4. Let H be a Hilbert algebra in Example 3.7, and let T_m be a t-norm in Example 3.4. Define a fuzzy set $\mu : H \to [0,1]$ by $\mu(1) = \mu(b) = 1$ and $\mu(a) = \mu(c) = 0.6$. Then μ is an imaginable T_m-fuzzy deductive system of H.

Definition 3.16. A function $\mu : H \to [0,1]$ is called a *fuzzy subalgebra* of H with respect to a t-norm T (briefly, a *T-fuzzy subalgebra* of H) if

$$\mu(x * y) \geq T(\mu(x), \mu(y))$$

for all $x, y \in H$.

Definition 3.17. A T-fuzzy subalgebra μ of H is said to be *imaginable* if it satisfies the imaginable property.

We give a relation between an imaginable T-fuzzy subalgebra of H and an imaginable T-fuzzy deductive system of H.

Theorem 3.18. *Let T be a t-norm. Then every imaginable T-fuzzy deductive system of H is an imaginable T-fuzzy subalgebra of H.*

Proof. Since $y*x \geq x$ for all $x, y \in H$, it follows from Proposition 3.15 that $\mu(y*x) \geq \mu(x)$, so by (FS2) and (T2)

$$\mu(y * x) \geq \mu(x) \geq T(\mu(y * x), \mu(y)) \geq T(\mu(x), \mu(y)) = T(\mu(y), \mu(x)).$$

Hence μ is an imaginable T-fuzzy subalgebra of H. □

Proposition 3.19. *Let μ be an imaginable T-fuzzy deductive system of H. If the inequality $x * y \geq z$ holds in H, then $\mu(y) \geq T(\mu(x), \mu(z))$ for all $x, y, z \in H$.*

Proof. Let $x, y, z \in H$ be such that $x * y \geq z$. Then

$$\mu(x * y) \geq T(\mu(z * (x * y)), \mu(z)) = T(\mu(0), \mu(z)).$$

It follows that

$$\begin{aligned}
\mu(y) &\geq T(\mu(x * y), \mu(x)) \\
&\geq T(T(\mu(1), \mu(z)), \mu(x)) \\
&= T(\mu(1), T(\mu(z), \mu(x))) \\
&\geq T(\mu(z), T(\mu(z), \mu(x))) \\
&= T(T(\mu(z), \mu(z)), \mu(x)) \\
&= T(\mu(x), \mu(z)),
\end{aligned}$$

completing the proof. □

The following example shows that, for a t-norm T, an imaginable T-fuzzy subalgebra of H may not be an imaginable T-fuzzy deductive system of H.

Example 3.5. Let $H = \{1, x, y, z, 0\}$ in which $*$ is defined by

$*$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then H is a Hilbert algebra. Define $\mu : H \to [0, 1]$ by $\mu(1) = 1, \mu(x) = \mu(y) = 0.8, \mu(z) = 0.7, \mu(0) = 0.4$. Let T_m be a t-norm defined by

$$T_m(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$$

for all $\alpha, \beta \in [0, 1]$. Then, routine calculations give that μ is a T_m-fuzzy subalgebra of a Hilbert algebra H. But μ is not an imaginable T-fuzzy deductive system, because

$$\mu(0) = 0.4 \leq T(\mu(z * 0), \mu(z)) = 0.5.$$

Now we give conditions in order that an imaginable T-fuzzy subalgebra of H would be an imaginable T-fuzzy deductive system of H.

Theorem 3.20. *Let T be a t-norm. An imaginable T-fuzzy subalgebra μ of H is an imaginable T-fuzzy deductive system of H if and only if for all $x, y, z \in H$ the inequality $x*y \geq z$ implies that $\mu(y) \geq T(\mu(x), \mu(z))$.*

Proof. ($\Rightarrow$) It follows from Proposition 3.23.

($\Leftarrow$) Let $x, y, z \in H$ and let μ be an imaginable T-fuzzy subalgebra of H satisfying $\mu(y) \geq T(\mu(x), \mu(z))$ whenever $x*y \geq z$. Since $x*y \geq x*y$, it follows that $\mu(y) \geq T(\mu(x*y), \mu(x))$. Since μ satisfies the imaginable property, we have

$$\mu(1) = \mu(x * x) \geq T(\mu(x), \mu(x)) = \mu(x).$$

Hence μ is an imaginable T-fuzzy deductive system of H. □

Theorem 3.21. *Let T be a t-norm and let H be a Hilbert algebra in which the equality $y = (x * y) * x$ holds for all distinct elements x and y of H. Then every imaginable T-fuzzy subalgebra of H is an imaginable T-fuzzy deductive system of H.*

Proof. Let μ be an imaginable T-fuzzy subalgebra of H. It is sufficient to show that μ satisfies the conditions (FS1) and (FS2). Since $x * x = 1$ for all $x \in H$ and since μ satisfies the imaginable property, we have

$$\mu(1) = \mu(x * x) \geq T(\mu(x), \mu(x)) = \mu(x)$$

for all $x \in H$. Let $x, y \in H$. Then

$$\mu(y) = \mu((x * y) * x) \geq T(\mu(x * y), \mu(x))$$

when $x \neq y$. If $x = y$ then

$$\mu(x) = \mu(1 * x) \geq T(\mu(1), \mu(x)) = T(\mu(x * x), \mu(x)).$$

Hence μ is an imaginable T-fuzzy deductive system of H. □

Let f be a mapping defined on X. If ν is a fuzzy set in $f(X)$ then the fuzzy set $\mu = \nu \circ f$ in X (i.e., the fuzzy set defined by $\mu(x) = \nu(f(x))$ for all $x \in X$) is called the *preimage* of ν under f.

Theorem 3.22. *Let T be a t-norm and let $f : H \to H'$ be an epimorphism of Hilbert algebras, ν an imaginable T-fuzzy ideal of H' and μ the preimage of ν under f. Then μ is an imaginable T-fuzzy ideal of H.*

Proof. For any $x, y \in H$ we have

$$\mu(x) = \nu(f(x)) \leq \nu(1') = \nu(f(1)) = \mu(1),$$

and

$$\mu(x * y) = \nu(f(x * y)) = \nu(f(x) *' f(y)) \geq \nu(f(y)).$$

Furthermore,

$$\begin{aligned} \mu((y_1 * (y_2 * x)) * x) &= \nu(f(y_1 * (y_2 * x)) * x)) \\ &= \nu((f(y_1) *' (f(y_2) *' f(x))) *' f(x)) \\ &\geq T(\nu(f(y_1), \nu(f(y_2))) \\ &= T(\mu(y_1), \mu(y_2)) \end{aligned}$$

for any $x, y_1, y_2 \in H$. Clearly μ satisfies the imaginable property. □

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