ON t-LEVEL SUBALGEBRAS OF BCK-ALGEBRAS

KUNG HO KIM

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Abstract. Using t-norm T, we introduce the notion of t-level subalgebra, and some related properties are investigated.

1. **Introduction.** Y. Imai and Iseki [2, 3] introduced two classes of abstract: BCK-algebras and BCI-algebras and was extensively investigated by several researchers. L. A. Zander [8] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches, such as group, functional analysis, probability theory, topology, and so on. In 1991, O. G. Xi [5] applied this concept to BCK-algebras, and he introduced the notion of fuzzy subalgebras(ideals) of the BCK-algebras with respect to minimum. In this paper, using t-norm T, we introduce the notion of t-level subalgebra, and some related properties are investigated.

2. **Preliminaries.** In what follows we use X to denote a BCK-algebra unless otherwise specified.

A BCK-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms

(1) $\{(x*y)*(x*z)\}*(z*y) = 0$,
(2) $\{x*(x*y)\}*y = 0$,
(3) $x*x = 0$,
(4) $x*y = 0$ and $y*x = 0$ imply $x = y$,
(5) $x*0 = 0$ imply $x = 0$,

for all $x,\ y \in X$. A partial ordering $\leq$ on X can be defined by $x \leq y$ if and only if $x*y = 0$.

Let I be a nonempty subset of a BCK-algebra X. Then I is called a *subalgebra* of X if $x*y \in I$, for all $x,\ y \in I$.

Definition 2.1 ([7])**.** By a *t-norm* T, we mean a function $T : [0,1]\times[0,1] \to [0,1]$ satisfying the following conditions:

(T1) $T(x,1) = x$,
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$,
(T3) $T(x,y) = T(y,x)$,
(T4) $T(x,T(y,z)) = T(T(x,y),z)$,

for all $x,\ y,\ z \in [0,1]$.

For a t-norm T on [0, 1], denote by Δ_T the set of element $\alpha \in [0,1]$ such that $T(\alpha,\alpha) = \alpha$, i.e., $\Delta_T := \{\alpha \in [0,1] \mid T(\alpha,\alpha) = \alpha\}$.

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Proposition 2.2. *Every t-norm T has a useful property:*

$$T(\alpha, \beta) \leq \min(\alpha, \beta)$$

for all α, $\beta \in [0, 1]$.

A *fuzzy set* in X is a function $A : X \to [0, 1]$. A fuzzy set A in X is called a *fuzzy subalgebra* of X if $A(x * y) \geq \min\{A(x), A(y)\}$ for all x, $y \in X$.

3. t-LEVEL SUBALGEBRAS OF BCK-ALGEBRAS.

Definition 3.1. A function $A : X \to [0, 1]$ is called a T*-fuzzy subalgebra* of X with respect to a t-norm T (briefly, a T*-fuzzy subalgebra* of X) if $A(x*y) \geq T(A(x), A(y))$ for all x, $y \in X$.

It is easy to show that every fuzzy subalgebra is a T-fuzzy subalgebra of X with $T(\alpha, \beta) = \alpha \wedge \beta$ for each α, $\beta \in [0, 1]$.

Definition 3.2. Let T be a t-norm. A fuzzy set A in X is said to satisfy *idempotent property* if $\mathrm{Im}(A) \subseteq \Delta_T$.

Proposition 3.3. *Let T be a t-norm on $[0, 1]$. If A is an idempotent T-fuzzy subalgebra of X, then we have $A(0) \geq A(x)$ for all $x \in X$.*

Proof. For every $x \in X$, we have

$$A(0) = A(x * x) \geq T(A(x), A(x)) = A(x).$$

This completes the proof. □

Proposition 3.4. *Let T be a t-norm on $[0, 1]$. If A is an idempotent T-fuzzy subalgebra of X, then the set*

$$A^\omega = \{x \in X \mid A(x) \geq A(\omega)\}$$

is a subalgebra of X.

Proof. Let x, $y \in A^\omega$. Then $A(x) \geq A(\omega)$ and $A(y) \geq A(\omega)$. Since A is an idempotent T-fuzzy subalgebra of X, it follows that

$$A(x * y) \geq T(A(x), A(y)) \geq T(A(x), A(\omega)) \geq T(A(\omega), A(\omega)) = A(\omega).$$

Thus, we have $A(x * y) \geq A(\omega)$, that is., $x * y \in A^\omega$. This completes the proof. □

Corollary 3.5. *Let T be a t-norm. If A is an idempotent T-fuzzy subalgebra of X, then the set*

$$A_X = \{x \in X \mid A(x) = A(0)\}$$

is a subalgebra of a BCK-algebra X.

Proof. ¿From the Proposition 3.3, $A_X = \{x \in X \mid A(x) = A(0)\} = \{x \in X \mid A(x) \geq A(0)\}$, hence A_X is a subalgebra of X from Proposition 3.4. □

Let χ_I denote the characteristic function of a non-empty subset I of X.

Theorem 3.6. *Let $I \subseteq X$. Then I is a subalgebra of a BCK-algebra X if and only if χ_I is a T-fuzzy subalgebra of X.*

Proof. Let I be a subalgebra of X. Then it is easy to show that χ_I is an T-fuzzy subalgebra of X. In fact, let $x,\, y \in I$. Then $x * y \in I$. Hence

$$\chi_I(x * y) = 1 = T(\chi_I(x), \chi_I(y))$$

If $x \in I$, $y \notin I$ (or $x \notin I$ and $y \in I$), then we have $\chi_I(x) = 1$ or $\chi_I(y) = 0$. This means that

$$\chi_I(x * y) \geq T(\chi_I(x), \chi_I(y)) = 0.$$

Conversely, suppose that χ_I is a T-fuzzy subalgebra of X. Now let $x,\, y \in I$. Then $\chi_I(x*y) \geq T(\chi_I(x), \chi_I(y)) = 1$, and so $\chi_I(x * y) = 1$, that is, $x * y \in I$. This proves the theorem. $\square$

Lemma 3.7 ([2]). *Let T be a t-norm. Then*

$$T(T(\alpha, \beta), T(\gamma, \delta)) = T(T(\alpha, \gamma), T(\beta, \delta))$$

for all $\alpha,\, \beta,\, \gamma,\, \delta \in [0, 1]$.

Proposition 3.8. *If A and B are T-fuzzy subalgebras of X, then $A \wedge B : X \to [0, 1]$ defined by*

$$(A \wedge B)(x) = T(A(x), B(x))$$

for all $x \in X$ is a T-fuzzy subalgebra of X.

Proof. Let $x,\, y \in X$. Then we have

$$\begin{aligned}(A \wedge B)(x * y) &= T(A(x * y), B(x * y)) \\ &\geq T(T(A(x), A(y)), T(B(x), B(y))) \\ &= T(T(A(x), B(x)), T(A(y), B(y))) \\ &= T((A \wedge B)(x), (A \wedge B)(y)).\end{aligned}$$

This completes the proof. $\square$

Definition 3.9. A fuzzy subalgebra A of a BCK-algebra X is said to be *normal* if $A(0) = 1$.

Theorem 3.10. *Let A be a T-fuzzy subalgebra of a BCK-algebra X and let A° be a fuzzy set in X defined by $A^\circ(x) = A(x) + 1 - A(0)$ for all $x \in X$. Then A° is a normal T-fuzzy subalgebra of a BCK-algebra X containing A.*

Proof. For $x,\, y \in X$, we have

$$\begin{aligned}A^\circ(x * y) &= A(x * y) + 1 - A(0) \\ &\geq T(A(x), A(y)) + 1 - A(0) \\ &= T(A(x) + 1 - A(0), A(y) + 1 - A(0)) \\ &= T(A^\circ(x), A^\circ * (y))\end{aligned}$$

Hence A° is a T-fuzzy subalgebra of X. Clearly, $A^\circ(0) = 1$ and $A \subset A^\circ$. $\square$

Definition 3.11. Let A be a fuzzy subset of a set X, T a t-norm and $\alpha \in [0, 1]$. Then we define a *t-level subset* of a fuzzy subset A as

$$A^T_\alpha = \{x \in X \mid T(A(x), \alpha) \geq \alpha\}.$$

Theorem 3.12. *Let X be a BCK-algebra and A a T-fuzzy subalgebra of X. Then t-level subset A^T_α is an subalgebra of X where $T(A(0), \alpha) \geq \alpha$ for $\alpha \in [0, 1]$.*

Proof. $A_\alpha^T = \{x \in M \mid T(A(x), \alpha) \geq \alpha\}$ is clearly nonempty. Let $x, y \in A_\alpha^T$. Then we have $T(A(x), \alpha) \geq \alpha$ and $T(A(y), \alpha) \geq \alpha$. Since A is a T-fuzzy subalgebra of X, $A(x * y) \geq T(A(x), A(y))$ is satisfied. This means that

$$T(A(x * y), \alpha) \geq T(T(A(x), A(y)), \alpha) = T(A(x), T(A(y), \alpha)) \geq T(A(x), \alpha) \geq \alpha.$$

Hence $x * y \in A_\alpha^T$. Therefore A_α^T is a subalgebra of X. □

Theorem 3.13. *Let X be a BCK-algebra and A a fuzzy subalgebra of X. Then t-level subset A_α^T is a subalgebra of X where $T(A(0), \alpha) \geq \alpha$ for $\alpha \in [0, 1]$.*

Proof. $A_\alpha^T = \{x \in X \mid T(A(x), \alpha) \geq \alpha\}$ is clearly nonempty. Let $x, y \in A_\alpha^T$. Then we have $T(A(x), \alpha) \geq \alpha$ and $T(A(y), \alpha) \geq \alpha$. Since A is a fuzzy subalgebra of X, $A(x * y) \geq \min\{A(x), A(y)\}$ is satisfied. This means that $T(A(x * y), \alpha) \geq T(\min(A(x), A(y)), \alpha)$. If $\min\{A(x), A(y)\} = A(x)$ or $\min\{A(x), A(y)\} = A(y)$, in two cases, we have

$$T(\min\{A(x), A(y)\}, \alpha) \geq \alpha$$

since $x, y \in A_\alpha^T$. Therefore, $T(A(x * y), \alpha) \geq \alpha$. Thus we get $x * y \in A_\alpha^T$. Hence A_α^T is a subalgebra of X. □

Theorem 3.14. *Let X be a BCK-algebra and A a fuzzy set of X such that A_α^T is a subalgebra of X where $T(A(x), \alpha) \geq \alpha$ for all $\alpha \in [0, 1]$. Then A is a T-fuzzy subalgebra of X.*

Proof. Let $x, y \in X, T(A(x), \alpha_1) = \alpha_1$ and $T(A(y), \alpha_2) = \alpha_2$. Then $x \in A_{\alpha_1}^T$ and $y \in A_{\alpha_2}^T$. Let $\alpha_1 < \alpha_2$. Then it follows that $T(A(x), \alpha_1) < T(A(y), \alpha_2)$ and $A_{\alpha_2}^T \subseteq A_{\alpha_1}^T$. So, $y \in A_{\alpha_1}^T$. Thus $x, y \in A_{\alpha_1}^T$ and since $A_{\alpha_1}^T$ is a subalgebra of X, by hypothesis, $x * y \in A_{\alpha_1}^T$. Therefore we have

$$T(A(x * y), \alpha_1) \geq \alpha_1 = T(A(x), \alpha_1) \geq T(A(x), T(A(y), \alpha_1)) = T(T(A(x), A(y)), \alpha_1).$$

Thus we get $T(A(x * y), \alpha_1) \geq T(T(A(x), A(y)), \alpha_1)$. As a t-norm is monotone with respect to each variable and symmetric, we have $A(x * y) \geq T(A(x), A(y))$. Thus A is a T-fuzzy subalgebra of X. □

Definition 3.15. For each $i = 1, 2, 3, \ldots, n$, let A_i be a T-fuzzy subalgebra in a BCK-algebra X_i. Let T be a t-norm. Then the *T-product* of A_i $(i = 1, 2, \ldots, n)$ is the function $A_1 \times A_2 \times A_3 \times \cdots \times A_n : X_1 \times X_2 \times X_3 \times \cdots \times X_n \to [0, 1]$ defined

$$(A_1 \times A_2 \times A_3 \times \cdots \times A_n)(x_1, x_2, x_3, \cdots, x_n) = T(A_1(x_1), A_2(x_2), A_3(x_3), \ldots, A_n(x_n))$$

for $x_i \in X_i$ $(i = 1, 2, \ldots, n)$.

Theorem 3.16 ([1])**.** *Let A and B be t-level subsets of the sets G and H, respectively, and let $\alpha \in [0, 1]$. Then $A \times B$ is also t-level subset of $G \times H$.*

Definition 3.17. Let X be a BCK-algebra and A a T-fuzzy subalgebra of X. The subalgebra A_α^T is called *t-level subalgebra* of X where $T(A(0), \alpha) \geq \alpha$ for $\alpha \in [0, 1]$.

Theorem 3.18. *Let X_1 and X_2 be two BCK-algebras, and A, B T-fuzzy subalgebras of X_1 and X_2, respectively. Then the t-level subset $(A \times B)_\alpha^T$, for $\alpha \in [0, 1]$, is a subalgebra of $X_1 \times X_2$.*

Proof. We know that $(A \times B)_\alpha^T = \{(x, y) \mid T((A \times B)(x, y), \alpha) \geq \alpha\}$. Since

$$\begin{aligned} T((A \times B)(0_{X_1}, 0_{X_2}), \alpha) &= T(T(A(0_{X_1}), B(0_{X_2})), \alpha)) \\ &= T(A(0_{X_1}), T(B(0_{X_2}), \alpha)) \\ &\geq T(A(0_{X_1}), \alpha) \geq \alpha, \end{aligned}$$

$(A\times B)^T_\alpha$ is nonempty. Let $(x_1,y_1),(x_2,y_2)\in(A\times B)^T_\alpha$. Then $T((A\times B)(x_1,y_1),\alpha)\geq\alpha$ and $T((A\times B)(x_2,y_2),\alpha)\geq\alpha$. Since $A\times B$ is an T-fuzzy subalgebra of $X_1\times X_2$, we have

$$(A\times B)((x_1,y_1)*(x_2,y_2))=(A\times B)(x_1*x_2,y_1*y_2)\geq T(A(x_1*x_2),B(y_1*y_2)).$$

Since A and B are T-fuzzy subalgebras, we get

$$\begin{aligned}T((A\times B)(x_1*x_2,y_1*y_2),\alpha)&\geq T(T(A(x_1*x_2),B(y_1*y_2)),\alpha)\\&=T((A(x_1*x_2),T(B(y_1*y_2),\alpha))\\&\geq T(A(x_1*x_2),\alpha)\\&\geq\alpha.\end{aligned}$$

Hence $(x_1,y_1)*(x_2,y_2)\in(A\times B)^T_\alpha$. Therefore $(A\times B)^T_\alpha$ is a subalgebra of $X_1\times X_2$. □

Theorem 3.19 ([1]). *Let A and B be fuzzy sets of the sets G and H, respectively and T a t-norm and $\alpha\in[0,1]$. Then $A^T_\alpha\times B^T_\alpha=(A\times B)^T_\alpha$.*

Theorem 3.20. *Let $A_1,A_2,A_3,\ldots,A_n$ be fuzzy subalgebras under a minimum operation in BCK-algebras $X_1,X_2,X_3,\ldots,X_n$, respectively and $\alpha\in[0,1]$. Then*

$$(A_1\times A_2\times\cdots\times A_n)^T_\alpha=A^T_{1\alpha}\times A^T_{2\alpha}\times\cdots\times A^T_{n\alpha}.$$

Proof. Let $(a_1,a_2,a_3,\ldots,a_n)\in(A_1\times A_2\times\cdots\times A_n)^T_\alpha$. Then we have

$$T(\min((A_1\times A_2\times\cdots\times A_n)(a_1,a_2,a_3,\ldots,a_n),\alpha)=T(\min(A_1(a_1),A_2(a_2),\ldots,A_n(a_n)),\alpha).$$

For all $i=1,2,\ldots,n$, $\min(A_1(a_1),A_2(a_2),\ldots,A_n(a_n))=A_i(a_i)$. This gives us

$$T(\min(A_1(a_1),A_2(a_2),\ldots,A_n(a_n)),\alpha)=T(A_i(a_i),\alpha)\geq\alpha.$$

Thus we have $a_i\in A^T_{i\alpha}$. That is, $(a_1,a_2,a_3,\ldots,a_n)\in A^T_{1\alpha}\times A^T_{2\alpha}\times\cdots\times A^T_{n\alpha}$. Similarly, $(a_1,a_2,a_3,\ldots,a_n)\in A^T_{1\alpha}\times A^T_{2\alpha}\times\cdots\times A^T_{n\alpha}$. Then, for all $i=1,2,\ldots,n$, we have $a_i\in A^T_{i\alpha}$. That is, $T(A_i(a_i),\alpha)\geq\alpha$. Since $\min(A_1(a_1),A_2(a_2),\ldots,A_n(a_n))=A_i(a_i)$ and $T(A_i(a_i),\alpha)\geq\alpha$, we have

$$\begin{aligned}T((A_1\times A_2\times\cdots\times A_n)(a_1,a_2,a_3,\ldots,a_n),\alpha)&=T(\min(A_1(a_1),A_2(a_2),\ldots,A_n(a_n)),\alpha)\\&=T(A_i(a_i),\alpha)\\&\geq\alpha.\end{aligned}$$

Thus we have $(a_1,a_2,a_3,\ldots,a_n)\in(A_1\times A_2\times\cdots\times A_n)^T_\alpha$. □

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Department of Mathematics,
Chungk National University, Chungk,
Chungking 380-702, Korea
E-mail: ghkim@cjnu.ac.kr