BIPOLAR FUZZY IMPLICATIVE HYPER BCK-IDEALS IN HYPER BCK-ALGEBRAS

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Abstract. Bipolar-valued fuzzification of the notion of implicative hyper BCK-ideals is considered. Using the concepts of positive and negative lever sets, a characterization of a bipolar fuzzy implicative hyper BCK-ideal is provided. A relation between a bipolar fuzzy hyper BCK-ideal and a bipolar fuzzy implicative hyper BCK-ideal is established. A condition for a bipolar fuzzy hyper BCK-ideal to be a bipolar fuzzy implicative hyper BCK-ideal is given. By using a collection of implicative hyper BCK-ideals, a bipolar fuzzy implicative hyper BCK-ideal is established.

1 Introduction The hyper structure theory (called also multialgebras) was introduced in 1934 by Marty [12] at the 8th congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia, and Japan. Over the following decades, many important results appeared, but above all since the 70's onwards the most luxuriant flourishing of hyper structures has been seen. Hyper structures have many applications to several sectors of both pure and applied sciences. In [9], Jun et al. applied the hyper structures to BCK-algebras, and introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra. They also introduced the notion of a (weak, s-weak, strong) hyper BCK-ideal, and gave relations among them. Harizavi [2] studied prime weak hyper BCK-ideals of lower hyper BCK-semilattices. Jun et al. discussed the notion of hyperatoms and scalar elements of hyper BCK-algebras (see [5]). Jun et al. also discussed the fuzzy structures of (implicative) hyper BCK-ideals in hyper BCK-algebras (see [4, 7]).

Fuzzy set theory is established in the paper [13]. In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0, 1]$. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not belong to the fuzzy set. The membership degrees on the interval $(0, 1)$ indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [1, 14]) In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements. Consider a fuzzy set "*young*" defined on the *age* domain $[0, 100]$ (see Fig. 1 in [10]). Now consider two ages 50 and 95 with membership

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degree 0. Although both of them do not satisfy the property "*young*", we may say that age 95 is more apart from the property rather than age 50 (see [10]).

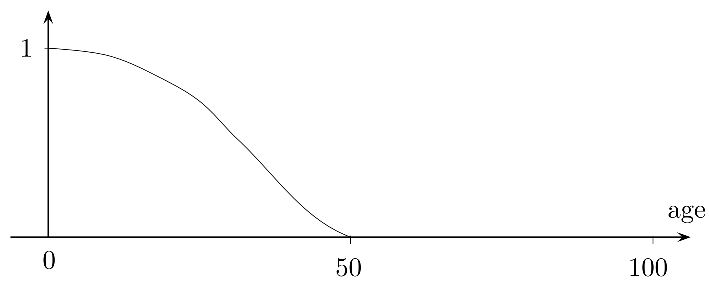

Figure 1. A fuzzy set "*young*"

Only with the membership degrees ranged on the interval [0, 1], it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, Lee [10] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. He gave two kinds of representations of the notion of bipolar-valued fuzzy sets.

In this paper, using the the notion of bipolar-valued fuzzy set, we introduce the notion of a bipolar fuzzy implication hyper BCK-ideal in hyper BCK-algebras, and investigate some of their properties. We give a relationship between bipolar fuzzy implicative hyper BCK-ideal and bipolar fuzzy hyper BCK-ideal. Using the positive (resp. negative) level set, we give a characterization of bipolar fuzzy implicative hyper BCK-ideals. We provide conditions for a bipolar fuzzy hyper BCK-ideal to be a bipolar fuzzy implicative hyper BCK-ideal. Using a collection of implicative hyper BCK-ideals, we establish a bipolar fuzzy implicative hyper BCK-ideal.

2 Preliminaries

2.1 Basic results on bipolar valued fuzzy sets As an extension of fuzzy sets, Lee [10] introduced the notion of bipolar-valued fuzzy sets. So, this subsection is based on his paper (see [10, 11]). Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets etc. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Bipolar-valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter-property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1, 0)$ indicate that elements somewhat satisfy the implicit counter-property (see [10]). Figure 2 shows a bipolar-valued fuzzy set redefined for the fuzzy set "*young*" of Figure 1. The negative membership degrees indicate the satisfaction extent of elements to an implicit counter-property (e.g., old against the property young). This kind of bipolar-valued fuzzy set representation enables the elements with membership degree 0 in traditional fuzzy sets, to be expressed into the elements with membership degree 0 (irrelevant elements) and the elements with negative membership degrees (contrary elements). The age elements 50 and 95, with membership degree 0 in the fuzzy sets of Figure 1, have 0 and a negative membership degree in the bipolar-valued fuzzy set of Figure 2, respectively. Now it is manifested that

50 is an irrelevant age to the property young and 95 is more apart from the property young than 50, i.e., 95 is a contrary age to the property young (see [10]).

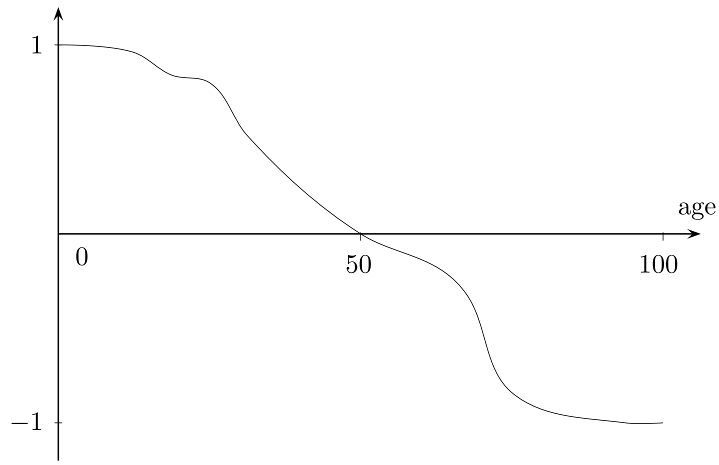

Figure 2. A bipolar fuzzy set "*young*"

In the definition of bipolar-valued fuzzy sets, there are two kinds of representations, so called canonical representation and reduced representation. In this paper, we use the canonical representation of a bipolar-valued fuzzy sets. Let X be the universe of discourse. A *bipolar-valued fuzzy set* Φ in X is an object having the form

$$\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$$

where $\mu_\Phi^P : X \to [0,1]$ and $\mu_\Phi^N : X \to [-1,0]$ are mappings. The positive membership degree $\mu_\Phi^P(x)$ denoted the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$, and the negative membership degree $\mu_\Phi^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$. If $\mu_\Phi^P(x) \neq 0$ and $\mu_\Phi^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$. If $\mu_\Phi^P(x) = 0$ and $\mu_\Phi^N(x) \neq 0$, it is the situation that x does not satisfy the property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$ but somewhat satisfies the counter-property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$. It is possible for an element x to be $\mu_\Phi^P(x) \neq 0$ and $\mu_\Phi^N(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [11]). For the sake of simplicity, we shall use the symbol $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ for the bipolar-valued fuzzy set $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

2.2 Basic results on hyper BCK-algebras We include some elementary aspects of hyper BCK-algebras that are necessary for this paper, and for more details we refer to [7], [8], and [9].

Let H be a nonempty set endowed with a hyperoperation "$\circ$". For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

By a *hyper BCK-algebra* we mean a nonempty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,

(HK3) $x \circ H \ll \{x\}$,

(HK4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in H.

Note that the condition (HK3) is equivalent to the condition:

$$(\forall x, y \in H)(x \circ y \ll \{x\}). \tag{2.1}$$

In any hyper BCK-algebra H, the following hold:

(a1) $x \circ 0 \ll \{x\},\ 0 \circ x \ll \{0\}\ 0 \circ 0 \ll \{0\}$,

(a2) $(A \circ B) \circ C = (A \circ C) \circ B,\ A \circ B \ll A,\ 0 \circ A \ll \{0\}$,

(a3) $0 \circ 0 = \{0\}$,

(a4) $0 \ll x$,

(a5) $x \ll x$,

(a6) $A \ll A$,

(a7) $A \subseteq B \ \Rightarrow\ A \ll B$,

(a8) $0 \circ x = \{0\}$,

(a9) $0 \circ A = \{0\}$,

(a10) $A \ll \{0\} \ \Rightarrow\ A = \{0\}$,

(a11) $A \circ B \ll A$,

(a12) $x \in x \circ 0$,

(a13) $x \circ 0 \ll \{y\} \ \Rightarrow\ x \ll y$,

(a14) $y \ll z \ \Rightarrow\ x \circ z \ll x \circ y$,

(a15) $x \circ y = \{0\} \ \Rightarrow\ (x \circ z) \circ (y \circ z) = \{0\},\ x \circ z \ll y \circ z$,

(a16) $A \circ \{0\} = \{0\} \ \Rightarrow\ A = \{0\}$,

(a17) $x \circ 0 = \{x\},\ A \circ 0 = A$

for all $x, y, z \in H$ and for all nonempty subsets A, B and C of H.

A nonempty subset I of a hyper BCK-algebra H is said to be a *hyper BCK-ideal* of H if it satisfies

(I1) $0 \in I$,

(I2) $(\forall x \in H)\ (\forall y \in I)\ (x \circ y \ll I \ \Rightarrow\ x \in I)$.

Proposition 2.1. *Let I be a hyper BCK-ideal of a hyper BCK-algebra H. For any $x, y \in H$, if $x \in I$ and $y \ll x$, then $y \in I$.*

A nonempty subset I of a hyper BCK-algebra H is said to be an *implicative hyper BCK-ideal* of H (see [6]) if it satisfies (I1) and

$$(\forall x, y \in H)(\forall z \in I)((x \circ z) \circ (y \circ x) \ll I \;\Rightarrow\; x \in I). \tag{2.2}$$

Note that every implicative hyper BCK-ideal is a hyper BCK-ideal, but the converse is not true (see [6]).

Lemma 2.2. [6] *Let I be a hyper BCK-ideal of a hyper BCK-algebra H. Then I is implicative if and only if the following assertion is valid:*

$$(\forall x, y \in H)(x \circ (y \circ x) \ll I \;\Rightarrow\; x \in I). \tag{2.3}$$

Definition 2.3. [3] A bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in a hyper BCK-algebra H is called a *bipolar fuzzy hyper BCK-ideal* of H if it satisfies

$$(\forall x, y \in H)(x \ll y \;\Rightarrow\; \mu_\Phi^P(x) \geq \mu_\Phi^P(y),\ \mu_\Phi^N(x) \leq \mu_\Phi^N(y)), \tag{2.4}$$

and

$$\begin{aligned} \mu_\Phi^P(x) &\geq \min\Big\{\inf_{a \in x \circ y} \mu_\Phi^P(a),\ \mu_\Phi^P(y)\Big\}, \\ \mu_\Phi^N(x) &\leq \max\Big\{\sup_{b \in x \circ y} \mu_\Phi^N(b),\ \mu_\Phi^N(y)\Big\} \end{aligned} \tag{2.5}$$

for all $x, y \in H$.

3 Bipolar fuzzy implicative hyper BCK-ideals

In what follows let H denote a hyper BCK-algebra unless otherwise specified.

Definition 3.1. A bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H is called a *bipolar fuzzy implicative hyper BCK-ideal* of H if it satisfies (2.4) and

$$\begin{aligned} \mu_\Phi^P(x) &\geq \min\Big\{\inf_{w \in (x \circ z) \circ (y \circ x)} \mu_\Phi^P(w),\ \mu_\Phi^P(z)\Big\}, \\ \mu_\Phi^N(x) &\leq \max\Big\{\sup_{w \in (x \circ z) \circ (y \circ x)} \mu_\Phi^N(w),\ \mu_\Phi^N(z)\Big\} \end{aligned} \tag{3.1}$$

for all $x, y, z \in H$.

Example 3.2. Let $H = \mathbb{N} \cup \{0\}$ and define a hyperoperation "$\circ$" on H as follows:

$$x \circ y := \begin{cases} \{0, x\} & \text{if } x \leq y, \\ \{x\} & \text{if } x > y. \end{cases}$$

Then $(H, \circ, 0)$ is a hyper BCK-algebra (see [3]). Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in H defined by

	0	1	2	3	4	$\cdots$
μ_Φ^P	$2-1$	$2-1.4$	$2-1.41$	$2-1.414$	$2-1.4142$	$\cdots$
μ_Φ^N	$-2+1$	$-2+1.4$	$-2+1.41$	$-2+1.414$	$-2+1.4142$	$\cdots$

Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy implicative hyper BCK-ideal of H.

Theorem 3.3. *Every bipolar fuzzy implicative hyper BCK-ideal is a bipolar fuzzy hyper BCK-ideal.*

Proof. Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy implicative hyper BCK-ideal of H. Then

$$\begin{aligned}
\mu_\Phi^P(x) &\geq \min\Big\{\inf_{w\in(x\circ y)\circ(0\circ x)} \mu_\Phi^P(w),\, \mu_\Phi^P(y)\Big\}\\
&= \min\Big\{\inf_{w\in(x\circ y)\circ 0} \mu_\Phi^P(w),\, \mu_\Phi^P(y)\Big\}\\
&= \min\Big\{\inf_{w\in x\circ y} \mu_\Phi^P(w),\, \mu_\Phi^P(y)\Big\},
\end{aligned}$$

$$\begin{aligned}
\mu_\Phi^N(x) &\leq \max\Big\{\sup_{w\in(x\circ y)\circ(0\circ x)} \mu_\Phi^N(w),\, \mu_\Phi^N(y)\Big\}\\
&= \max\Big\{\sup_{w\in x\circ y} \mu_\Phi^N(w),\, \mu_\Phi^N(y)\Big\}
\end{aligned}$$

for all $x, y \in H$. Therefore $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H. $\square$

The converse of Theorem 3.3 may not be true as seen in the following example.

Example 3.4. Let $H = \{0, a, b\}$ be a hyper BCK-algebra with the following Cayley table:

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

	0	a	b
μ_Φ^P	0.8	0.6	0.4
μ_Φ^N	−0.9	−0.7	−0.5

It is routine to check that $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H. But it is not a bipolar fuzzy implicative hyper BCK-ideal of H since

$$\mu_\Phi^P(a) = 0.6 < 0.8 = \min\Big\{\inf_{w\in(a\circ 0)\circ(b\circ a)} \mu_\Phi^P(w),\, \mu_\Phi^P(0)\Big\}$$

and/or

$$\mu_\Phi^N(a) = -0.7 > -0.9 = \max\Big\{\sup_{w\in(a\circ 0)\circ(b\circ a)} \mu_\Phi^N(w),\, \mu_\Phi^N(0)\Big\}.$$

Proposition 3.5. *Every bipolar fuzzy implicative hyper BCK-ideal $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ of H satisfies the following assertion:*

$$(\forall x, y \in H)\big(\mu_\Phi^P(x) \geq \inf_{w\in x\circ(y\circ x)} \mu_\Phi^P(w),\ \mu_\Phi^N(x) \leq \sup_{w\in x\circ(y\circ x)} \mu_\Phi^N(w)\big). \tag{3.2}$$

Proof. Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy implicative hyper BCK-ideal of H. Then

$$\mu_\Phi^P(x) \geq \min\Big\{ \inf_{w \in (x \circ 0) \circ (y \circ x)} \mu_\Phi^P(w),\ \mu_\Phi^P(0) \Big\} = \inf_{w \in x \circ (y \circ x)} \mu_\Phi^P(w),$$

$$\mu_\Phi^N(x) \leq \max\Big\{ \sup_{w \in (x \circ 0) \circ (y \circ x)} \mu_\Phi^N(w),\ \mu_\Phi^N(0) \Big\} = \sup_{w \in x \circ (y \circ x)} \mu_\Phi^N(w)$$

for all $x, y \in H$. □

For a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in a set H, the *positive level set* and *negative level set* are denoted by $P(\mu_\Phi^P; \alpha)$ and $N(\mu_\Phi^N; \beta)$, and are defined as follows:

$$P(\mu_\Phi^P; \alpha) := \{x \in H \mid \mu_\Phi^P(x) \geq \alpha\},\ \alpha \in [0, 1],$$

$$N(\mu_\Phi^N; \beta) := \{x \in H \mid \mu_\Phi^N(x) \leq \beta\},\ \beta \in [-1, 0],$$

respectively.

Lemma 3.6. [3] *Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar-valued fuzzy set in H. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H if and only if for every $(\alpha, \beta) \in [0, 1] \times [-1, 0]$, the nonempty positive level set $P(\mu_\Phi^P; \alpha)$ and the nonempty negative level set $N(\mu_\Phi^N; \beta)$ are hyper BCK-ideals of H.*

Theorem 3.7. *Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in H. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy implicative hyper BCK-ideal of H if and only if for every $(\alpha, \beta) \in [0, 1] \times [-1, 0]$, the nonempty positive level set $P(\mu_\Phi^P; \alpha)$ and the nonempty negative level set $N(\mu_\Phi^N; \beta)$ are implicative hyper BCK-ideals of H.*

Proof. Assume that $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy implicative hyper BCK-ideal of H and $P(\mu_\Phi^P; \alpha) \neq \emptyset \neq N(\mu_\Phi^N; \beta)$ for any $(\alpha, \beta) \in [0, 1] \times [-1, 0]$. Using Theorem 3.3 and Lemma 3.6, we know that $P(\mu_\Phi^P; \alpha)$ and $N(\mu_\Phi^N; \beta)$ are hyper BCK-ideals of H. Let $x, y \in H$ be such that $x \circ (y \circ x) \ll P(\mu_\Phi^P; \alpha)$. For any $w \in x \circ (y \circ x)$, there exists $z \in P(\mu_\Phi^P; \alpha)$ such that $w \ll z$. It follows from (2.4) that $\mu_\Phi^P(w) \geq \mu_\Phi^P(z) \geq \alpha$ so from Proposition 3.5 that $\mu_\Phi^P(x) \geq \inf_{w \in x \circ (y \circ x)} \mu_\Phi^P(w) \geq \alpha$. Hence $x \in P(\mu_\Phi^P; \alpha)$. Now let $u, v \in H$ be such that $u \circ (v \circ u) \ll N(\mu_\Phi^N; \beta)$. If $x \in u \circ (v \circ u)$, then $x \ll a$ for some $a \in N(\mu_\Phi^N; \beta)$. Using (2.4), we have $\mu_\Phi^N(x) \leq \mu_\Phi^N(a) \leq \beta$, which imply from Proposition 3.5 that $\mu_\Phi^N(u) \leq \sup_{x \in u \circ (v \circ u)} \mu_\Phi^N(x) \leq \beta$. Therefore $u \in N(\mu_\Phi^N; \beta)$. Using Lemma 2.2, we conclude that $P(\mu_\Phi^P; \alpha)$ and $N(\mu_\Phi^N; \beta)$ are implicative hyper BCK-ideals of H.

Conversely, suppose that the nonempty positive level set $P(\mu_\Phi^P; \alpha)$ and the nonempty negative level set $N(\mu_\Phi^N; \beta)$ are implicative hyper BCK-ideals of H for all $(\alpha, \beta) \in [0, 1] \times [-1, 0]$. For any $x, y \in H$ with $x \ll y$, let $\mu_\Phi^P(y) = \alpha$ and $\mu_\Phi^N(y) = \beta$. Then $y \in P(\mu_\Phi^P; \alpha) \cap N(\mu_\Phi^N; \beta)$, and so $x \in P(\mu_\Phi^P; \alpha) \cap N(\mu_\Phi^N; \beta)$. Hence $\mu_\Phi^P(x) \geq \alpha = \mu_\Phi^P(y)$ and $\mu_\Phi^N(x) \leq \beta = \mu_\Phi^N(y)$. This shows that $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ satisfies (2.4). For any $x, y, z \in H$, let

$$\alpha = \min\Big\{ \inf_{c \in (x \circ z) \circ (y \circ x)} \mu_\Phi^P(c),\ \mu_\Phi^P(z) \Big\},$$

$$\beta = \max\Big\{ \sup_{d \in (x \circ z) \circ (y \circ x)} \mu_\Phi^N(d),\ \mu_\Phi^N(z) \Big\}.$$

Then $\mu_\Phi^P(z) \geq \alpha$ and $\mu_\Phi^N(z) \leq \beta$, i.e., $z \in P(\mu_\Phi^P;\alpha) \cap N(\mu_\Phi^N;\beta)$. Let $a, b \in (x \circ z) \circ (y \circ x)$. Then

$$\mu_\Phi^P(a) \geq \inf_{c \in (x\circ z)\circ(y\circ x)} \mu_\Phi^P(c) \geq \min\Big\{\inf_{c \in (x\circ z)\circ(y\circ x)} \mu_\Phi^P(c),\, \mu_\Phi^P(z)\Big\} = \alpha,$$

$$\mu_\Phi^N(b) \leq \sup_{d \in (x\circ z)\circ(y\circ x)} \mu_\Phi^N(d) \leq \max\Big\{\sup_{d \in (x\circ z)\circ(y\circ x)} \mu_\Phi^N(d),\, \mu_\Phi^N(z)\Big\} = \beta.$$

It follows that $a \in P(\mu_\Phi^P;\alpha)$ and $b \in N(\mu_\Phi^N;\beta)$ so that $(x \circ z) \circ (y \circ x) \subseteq P(\mu_\Phi^P;\alpha)$ and $(x\circ z)\circ(y\circ x) \subseteq N(\mu_\Phi^N;\beta)$. Using (a7), we have $(x\circ z)\circ(y\circ x) \ll P(\mu_\Phi^P;\alpha)$ and $(x\circ z)\circ(y\circ x) \ll N(\mu_\Phi^N;\beta)$. Since $P(\mu_\Phi^P;\alpha)$ and $N(\mu_\Phi^N;\beta)$ are implicative hyper BCK-ideal, it follows from (2.2) that $x \in P(\mu_\Phi^P;\alpha) \cap N(\mu_\Phi^N;\beta)$ so that

$$\mu_\Phi^P(x) \geq \alpha = \min\Big\{\inf_{c \in (x\circ z)\circ(y\circ x)} \mu_\Phi^P(c),\, \mu_\Phi^P(z)\Big\},$$

$$\mu_\Phi^N(x) \leq \beta = \max\Big\{\sup_{c \in (x\circ z)\circ(y\circ x)} \mu_\Phi^N(c),\, \mu_\Phi^N(z)\Big\}.$$

Hence $\Phi = (H;\mu_\Phi^P,\mu_\Phi^N)$ is a bipolar fuzzy implicative hyper BCK-ideal of H. □

Theorem 3.8. *For any subset I of H, let $\Phi_I = (H;\mu_{\Phi_I}^P,\mu_{\Phi_I}^N)$ be a bipolar fuzzy set in H defined by*

$$\mu_{\Phi_I}^P(x) := \begin{cases} \alpha_1 & \text{if} \quad x \in I, \\ \alpha_2 & \text{otherwise} \end{cases}$$

$$\mu_{\Phi_I}^N(x) := \begin{cases} \beta_1 & \text{if} \quad x \in I, \\ \beta_2 & \text{otherwise} \end{cases}$$

for all $x \in H$, where $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 > \alpha_2$ and $\beta_1, \beta_2 \in [-1,0]$ with $\beta_1 < \beta_2$. Then I is an implicative hyper BCK-ideal of H if and only if Φ_I is a bipolar fuzzy implicative hyper BCK-ideal of H.

Proof. Assume that I is an implicative hyper BCK-ideal of H. Note that

$$P(\mu_{\Phi_I}^P;\alpha) = \begin{cases} \emptyset & \text{if} \quad \alpha_1 < \alpha \leq 1, \\ I & \text{if} \quad \alpha_2 < \alpha \leq \alpha_1, \\ H & \text{if} \quad 0 \leq \alpha \leq \alpha_2, \end{cases}$$

$$N(\mu_{\Phi_I}^N;\beta) = \begin{cases} \emptyset & \text{if} \quad -1 \leq \beta < \beta_1, \\ I & \text{if} \quad \beta_1 \leq \beta < \beta_2, \\ H & \text{if} \quad \beta_2 \leq \beta \leq 0 \end{cases}$$

for all $(\alpha,\beta) \in [0,1] \times [-1,0]$. As I and H are implicative hyper BCK-ideals of H, we infer from Theorem 3.7 that $\Phi_I = (H;\mu_{\Phi_I}^P,\mu_{\Phi_I}^N)$ is a bipolar fuzzy implicative hyper BCK-ideal of H.

Conversely, if $\Phi_I = (H;\mu_{\Phi_I}^P,\mu_{\Phi_I}^N)$ is a bipolar fuzzy implicative hyper BCK-ideal of H, then $P(\mu_{\Phi_I}^P;\alpha_1) = I = N(\mu_{\Phi_I}^N;\beta_1)$ which is an implicative hyper BCK-ideal of H. □

We give a condition for a bipolar fuzzy hyper BCK-ideal to be a bipolar fuzzy implicative hyper BCK-ideal.

Lemma 3.9. [3] *Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in H. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H if and only if for every $(\alpha, \beta) \in [0,1] \times [-1,0]$, the nonempty positive level set $P(\mu_\Phi^P; \alpha)$ and the nonempty negative level set $N(\mu_\Phi^N; \beta)$ are hyper BCK-ideals of H.*

Theorem 3.10. *If a bipolar fuzzy hyper BCK-ideal $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ of H satisfies* (3.2), *then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy implicative hyper BCK-ideal of H.*

Proof. Let $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy hyper BCK-ideal of H satisfying the condition (3.2). Let $x, y, z \in H$ be such that $z \in P(\mu_\Phi^P; \alpha)$ and $(x \circ z) \circ (y \circ x) \ll P(\mu_\Phi^P; \alpha)$ where $\alpha \in [0, 1]$. Then $\mu_\Phi^P(z) \geq \alpha$ and $(x \circ (y \circ x)) \circ z \ll P(\mu_\Phi^P; \alpha)$, and so $w \circ z \ll P(\mu_\Phi^P; \alpha)$ for all $w \in x \circ (y \circ x)$. Since $P(\mu_\Phi^P; \alpha)$ is a hyper BCK-ideal of H by Lemma 3.9, it follows that $w \in P(\mu_\Phi^P; \alpha)$, that is, $x \circ (y \circ x) \subseteq P(\mu_\Phi^P; \alpha)$. Hence

$$\mu_\Phi^P(x) \geq \inf_{w \in x \circ (y \circ x)} \mu_\Phi^P(w) \geq \inf_{w \in P(\mu_\Phi^P; \alpha)} \mu_\Phi^P(w) \geq \alpha$$

by (3.2), which implies that $x \in P(\mu_\Phi^P; \alpha)$. Therefore $P(\mu_\Phi^P; \alpha)$ is an implicative hyper BCK-ideal of H. Now let $a, b, c \in H$ be such that $(a \circ c) \circ (b \circ a) \ll N(\mu_\Phi^N; \beta)$ and $c \in N(\mu_\Phi^N; \beta)$ where $\beta \in [-1, 0]$. Then $\mu_\Phi^N(c) \leq \beta$ and $(a \circ (b \circ a)) \circ c \ll N(\mu_\Phi^N; \beta)$. It follows that $d \circ c \ll N(\mu_\Phi^N; \beta)$ for all $d \in a \circ (b \circ a)$ so from Lemma 3.9 that $d \in N(\mu_\Phi^N; \beta)$, i.e., $a \circ (b \circ a) \subseteq N(\mu_\Phi^N; \beta)$. Using (3.2), we have

$$\mu_\Phi^N(a) \leq \sup_{d \in a \circ (b \circ a)} \mu_\Phi^N(d) \leq \sup_{d \in N(\mu_\Phi^N; \beta)} \mu_\Phi^N(d) \leq \beta$$

and so $a \in N(\mu_\Phi^N; \beta)$. Thus $N(\mu_\Phi^N; \beta)$ is an implicative hyper BCK-ideal of H. Therefore $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy implicative hyper BCK-ideal of H by Theorem 3.7. $\square$

Lemma 3.11. [6] *Let I be a hyper BCK-ideal of H. Then I is implicative if and only if it satisfies:*

$$(\forall x, y \in H)(x \circ (y \circ x) \ll I \;\Rightarrow\; x \in I).$$

Theorem 3.12. *If $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy implicative hyper BCK-ideal of a hyper BCK-algebra H, then the set*

$$I := \{x \in H \mid \mu_\Phi^P(x) = \mu_\Phi^P(0),\ \mu_\Phi^N(x) = \mu_\Phi^N(0)\}$$

is an implicative hyper BCK-ideal of H.

Proof. Obviously $0 \in I$. Let $x, y \in H$ be such that $x \circ y \ll I$ and $y \in I$. Then $\mu_\Phi^P(y) = \mu_\Phi^P(0)$ and $\mu_\Phi^N(y) = \mu_\Phi^N(0)$, and for each $a \in x \circ y$ there exists $z \in I$ such that $a \ll z$. Hence $\mu_\Phi^P(a) \geq \mu_\Phi^P(z) = \mu_\Phi^P(0)$ and $\mu_\Phi^N(a) \leq \mu_\Phi^N(z) = \mu_\Phi^N(0)$ by (2.4). It follows from [3, Proposition 3.15(1)] that $\mu_\Phi^P(a) = \mu_\Phi^P(0)$ and $\mu_\Phi^N(a) = \mu_\Phi^N(0)$. As $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal (see Theorem 3.3), we infer that

$$\mu_\Phi^P(x) \geq \min\{\inf_{a \in x \circ y} \mu_\Phi^P(a), \mu_\Phi^P(y)\} = \mu_\Phi^P(0),$$

$$\mu_\Phi^N(x) \leq \max\{\sup_{b \in x \circ y} \mu_\Phi^N(b), \mu_\Phi^N(y)\} = \mu_\Phi^N(0)$$

by (2.5). Hence $\mu_\Phi^P(x) = \mu_\Phi^P(0)$ and $\mu_\Phi^N(x) = \mu_\Phi^N(0)$, and so $x \in I$. This shows that I is a hyper BCK-ideal of H. Now let $x, y \in H$ be such that $x \circ (y \circ x) \ll I$. For every $z \in x \circ (y \circ x)$ there exists $b \in I$ such that $z \ll b$. It follows from (2.4) that

$$\mu_\Phi^P(z) \geq \mu_\Phi^P(b) = \mu_\Phi^P(0),\ \mu_\Phi^N(z) \leq \mu_\Phi^N(b) = \mu_\Phi^N(0)$$

so from [3, Proposition 3.15(1)] that $\mu_\Phi^P(z) = \mu_\Phi^P(0)$ and $\mu_\Phi^N(z) = \mu_\Phi^N(0)$. Using Proposition 3.5, we have

$$\mu_\Phi^P(x) \geq \inf_{z \in x \circ (y \circ x)} \mu_\Phi^P(z) = \mu_\Phi^P(0),$$

$$\mu_\Phi^N(x) \leq \sup_{z \in x \circ (y \circ x)} \mu_\Phi^N(z) = \mu_\Phi^N(0).$$

Thus $\mu_\Phi^P(x) = \mu_\Phi^P(0)$ and $\mu_\Phi^N(x) = \mu_\Phi^N(0)$, that is, $x \in I$. Therefore, by means of Lemma 3.11, I is an implicative hyper BCK-ideal of H. □

The converse of Theorem 3.12 is not true as seen in the following example.

Example 3.13. Consider a hyper BCK-algebra $H = \{0, 1, 2\}$ with the following Cayley table:

$\circ$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{2\}$	$\{0, 2\}$

Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

	0	1	2
μ_Φ^P	0.8	0.8	0.9
μ_Φ^N	-0.3	-0.3	-0.5

Then $I = \{x \in H \mid \mu_\Phi^P(x) = \mu_\Phi^P(0),\ \mu_\Phi^N(x) = \mu_\Phi^N(0)\} = \{0, 1\}$ which is an implicative hyper BCK-ideal of H. Note that $1 \ll 2$, but $\mu_\Phi^P(1) = 0.8 \ngeq 0.9 = \mu_\Phi^P(2)$ and/or $\mu_\Phi^N(1) = -0.3 \nleq -0.5 = \mu_\Phi^N(2)$. Hence $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is not a bipolar fuzzy implicative hyper BCK-ideal of H.

Let Λ_P (resp. Λ_N) be a subset of $[0, 1]$ (resp. $[-1, 0]$) such that every subset of Λ_P (resp. Λ_N) contains both its supremum and its infimum.

Theorem 3.14. *Let $\{(I_\alpha, J_\beta) \mid (\alpha, \beta) \in \Lambda_P \times \Lambda_N\}$ be a collection of ordered pairs of implicative hyper BCK-ideals of H such that*

1. $(\forall \alpha_1, \alpha_2 \in \Lambda_P)\ (\alpha_1 > \alpha_2\ \Rightarrow\ I_{\alpha_1} \subseteq I_{\alpha_2})$,
2. $(\forall \beta_1, \beta_2 \in \Lambda_N)\ (\beta_1 > \beta_2\ \Rightarrow\ J_{\beta_2} \subseteq J_{\beta_1})$.

Define a bipolar fuzzy set $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ in H by

$$\mu_\Phi^P(x) := \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\},\ \mu_\Phi^N(x) := \inf\{\beta \in \Lambda_N \mid x \in J_\beta\}$$

for all $x \in H$. Then $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy implicative hyper BCK-ideal of H.

Proof. Let $x, y \in H$ be such that $x \ll y$. Then

$$\mu_\Phi^P(y) = \sup\{\alpha \in \Lambda_P \mid y \in I_\alpha\} \leq \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\} = \mu_\Phi^P(x),$$

$$\mu_\Phi^N(y) = \inf\{\beta \in \Lambda_N \mid y \in J_\beta\} \geq \inf\{\beta \in \Lambda_N \mid x \in J_\beta\} = \mu_\Phi^N(x).$$

For every $x, y, z \in H$, let $\alpha_1 = \inf\limits_{a \in (x \circ z) \circ (y \circ x)} \mu_\Phi^P(a)$ and $\alpha_2 = \mu_\Phi^P(z)$. Then $a \in I_{\alpha_1}$ for all $a \in (x \circ z) \circ (y \circ x)$, i.e., $(x \circ z) \circ (y \circ x) \subseteq I_{\alpha_1}$, and $z \in I_{\alpha_2}$. Since every subset of Λ_P has both its supremum and its infimum, we know that α_1 and α_2 belong to Λ_P. We may assume that $\alpha_1 > \alpha_2$ without loss of generality. Then $I_{\alpha_1} \subseteq I_{\alpha_2}$ by (1). It follows from (a7) that $(x \circ z) \circ (y \circ x) \ll I_{\alpha_1} \subseteq I_{\alpha_2}$ so from (2.2) that $x \in I_{\alpha_2}$. Hence

$$\begin{aligned} \mu_\Phi^P(x) &= \sup\{\alpha \in \Lambda_P \mid x \in I_\alpha\} \geq \alpha_2 = \min\{\alpha_1, \alpha_2\} \\ &= \min\Big\{\inf_{a \in (x \circ z) \circ (y \circ x)} \mu_\Phi^P(a), \mu_\Phi^P(z)\Big\}. \end{aligned}$$

Now, for every $x, y, z \in H$, let $\beta_1 = \sup\limits_{b \in (x \circ z) \circ (y \circ x)} \mu_\Phi^N(b)$ and $\beta_2 = \mu_\Phi^N(z)$. Then $b \in J_{\beta_1}$ for all $b \in (x \circ z) \circ (y \circ x)$, that is, $(x \circ z) \circ (y \circ x) \subseteq J_{\beta_1}$, and $z \in J_{\beta_2}$. Since every subset of Λ_N has both its supremum and its infimum, β_1 and β_2 belong to Λ_N. We may assume that $\beta_1 < \beta_2$ without loss of generality. Using (a7) and (1), we get $(x \circ z) \circ (y \circ x) \ll J_{\beta_1} \subseteq J_{\beta_2}$ and $z \in J_{\beta_2}$. Since J_{β_2} is an implicative hyper BCK-ideal, it follows from (2.2) that $x \in J_{\beta_2}$ so that

$$\begin{aligned} \mu_\Phi^N(x) &= \inf\{\beta \in \Lambda_N \mid x \in J_\beta\} \leq \beta_2 = \max\{\beta_1, \beta_2\} \\ &= \max\Big\{\sup_{b \in (x \circ z) \circ (y \circ x)} \mu_\Phi^N(b), \mu_\Phi^N(z)\Big\}. \end{aligned}$$

Consequently, $\Phi = (H; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy hyper BCK-ideal of H. □

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