A KANTOROVICH TYPE INEQUALITY WITH A NEGATIVE PARAMETER

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Abstract. In [5], one of the authors showed Kantorovich type inequalities with two positive parameters, which is a difference version of Furuta-Giga's one [4]. In this note, we show the following order for $p \leqq -1$: Let $A \geq B > 0$ and $MI \geq A \geq mI > 0$ for some positive numbers $M > m > 0$. Then

$$B^p + C(m, M, p)I \geq B^p + C\left(\frac{1}{M}, \frac{1}{m}, -p\right) I \geq A^p$$

holds for $p \leqq -1$ where the constant is defined as

$$C(m, M, p) \equiv (p-1)\left(\frac{M^p - m^p}{p(M-m)}\right)^{\frac{p}{p-1}} + \frac{Mm^p - M^p m}{M-m}.$$

We also show a similar inequality for the chaotic order: Let A and B positive operators on a Hilbert space with $MI \geq A \geq mI > 0$ for some positive numbers $M > m > 0$. If $\log A \geq \log B$, then

$$B^p + \frac{M}{m}(m^p - M^p)I \geq B^p + C\left(\frac{1}{M}, \frac{1}{m}, 1-p\right) MI \geq A^p$$

for $p \leqq 0$.

Throughout this paper, we consider bounded linear operators on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geqq 0$ for all $x \in H$. The positivity defines the usual order $A \geq B$ for selfadjoint operators A and B. For the sake of convenience, $T > 0$ means T is positive and invertible. The Löwner-Heinz inequality asserts that $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for all $0 \leqq \alpha \leqq 1$. However $A \geq B \geq 0$ does not ensure $A^\alpha \geq B^\alpha$ for $\alpha > 1$ in general. In 1997, M. Fujii, S. Izumino, R. Nakamoto and Y. Seo [1] showed the following reverse inequality for t^2:

$$A \geq B \geq 0,\ MI \geq A \geq mI > 0 \implies \frac{(M+m)^2}{4Mm} A^2 \geq B^2.$$

It is obtained as an application of the celebrated Kantorovich inequality, i.e.,

$$\begin{aligned} &MI \geq A \geq mI > 0,\ M > m > 0 \\ &\implies \langle A^{-1}x, x\rangle \langle Ax, x\rangle \leqq \frac{(M+m)^2}{4Mm} \quad \text{for all unit vectors } x \in H. \end{aligned}$$

The constant $\frac{(M+m)^2}{4Mm}$ is called the Kantorovich constant.

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As an extension of this, T. Furuta and M. Giga [4] gave a complementary result of Kantorovich type order preserving inequality by Mićić-Pečarić-Seo [6]. For our note, we need two constants. One is the generalized Kantorovich constant introduced by Furuta [2, 3]:

$$K(m,M,p)=\frac{mM^p-Mm^p}{(p-1)(M-m)}\left(\frac{(p-1)}{p}\frac{(M^p-m^p)}{(mM^p-Mm^p)}\right)^p$$

for $M>m>0$ and $p\in\mathbb{R}$. Another is the following by J. Mićić, Y.Seo, S-E. Takahashi and M. Tominaga [7]:

$$C(m,M,p)=(p-1)\left(\frac{M^p-m^p}{p(M-m)}\right)^{\frac{p}{p-1}}+\frac{Mm^p-M^pm}{M-m}$$

for $M>m>0$ and $p\in\mathbb{R}$. Moreovere we prepare the following theorems.

Furuta [3] showed the following inequality.

Theorem A. *If $M>m>0$ for positive numbers M and m, then*

$$K(m,M,p)\leqq\left(\frac{M}{m}\right)^{p-1}\quad \textit{for } p>1.$$

Yamazaki [8] showed Theorem B and Theorem C.

Theorem B. *If $M>m>0$ for positive numbers M and m, then*

$$C(m,M,p)=\frac{mM^p-Mm^p}{M-m}(K(m,M,p)^{\frac{1}{p-1}}-1)\quad \textit{for } p\in\mathbb{R}.$$

Theorem C. *Let A and B be positive operators on H such that $A\geq B\geq 0$ and $MI\geq B\geq mI>0$ for some positive numbers $M>m>0$. Then*

$$A^p+C(m,M,p)I\geq B^p\quad \textit{for } p>1.$$

Theorem (The chaotic Furuta inequality). *Let A and B be positive invertible operators on H. Then the chaotic order $\log A\geq\log B$ is equivalent to the inequality*

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}}\geq B^r\quad \textit{for all } p,r\geqq 0.$$

In the below, we observe the case $p<0$. If $-1\leqq p<0$, then $B^p\geq A^p$ holds by the Löwner-Heinz inequality. The other case $p\leqq -1$, we have the following inequality:

Theorem 1. *Let A and B be positive and invertible operators on H such that $A\geq B>0$ and $MI\geq A\geq mI>0$ for some positive numbers $M>m>0$. Then*

$$B^p+C(m,M,p)I\geq A^p\quad \textit{for}\ \ p<-1.$$

Proof. It is shown in [7] that

$$0\leqq\langle A^px,x\rangle-\langle Ax,x\rangle^p\leqq C(m,M,p)$$

holds for unit vectors x. Then Hölder-McCarthy inequality shows

$$\langle A^px,x\rangle\leqq\langle Ax,x\rangle^p+C(m,M,p)\leqq\langle Bx,x\rangle^p+C(m,M,p)\leqq\langle B^px,x\rangle+C(m,M,p),$$

which implies the required inequality. ∎

To discuss more precise estimation than Theorem 1, we prepare the following Lemma 2 by Yamazaki[8]. For the reader's convenience, we give a proof:

Lemma 2. *If $q \geqq 1$ and $M > m > 0$, then*

$$0 \leqq C(m, M, q) \leqq M(M^{q-1} - m^{q-1}).$$

Incidentally, $C(m, M, 1) = 0$.

Proof. By Theorems B and A, we have

$$\begin{aligned} C(m, M, q) &= \frac{mM^q - Mm^q}{M - m}\left(K(m, M, q)^{\frac{1}{q-1}} - 1\right) \\ &\leqq \frac{mM^q - Mm^q}{M - m}\left(\left(\left(\frac{M}{m}\right)^{q-1}\right)^{\frac{1}{q-1}} - 1\right) \\ &= M(M^{q-1} - m^{q-1}) \quad \text{for } \ q > 1. \end{aligned}$$

The case $q = 1$ is clear. □

Now we show the following reverse inequality. It characterizes the usual order $A \geq B$ by Lemma 2:

Theorem 3. *If A and B be positive and invertible operators on H such that $MI \geq A \geq mI > 0$ for some positive numbers $M > m > 0$. If $A \geq B$, then*

$$B^p + \frac{m^{p+1} - M^{p+1}}{m} I \geq B^p + C\left(\frac{1}{M}, \frac{1}{m}, -p\right) I \geq A^p$$

holds for $p \leqq -1$.

Proof. By Lemma 2, we have the left inequality since

$$C\left(\frac{1}{M}, \frac{1}{m}, -p\right) \leq \frac{1}{m}\left(\left(\frac{1}{m}\right)^{-p-1} - \left(\frac{1}{M}\right)^{-p-1}\right) = \frac{m^{p+1} - M^{p+1}}{m}.$$

For the proof of the right inequality, we use the Theorem C. Let $p_1 := -p$, $A_1 := B^{-1}$, $B_1 := A^{-1}$, $L := m^{-1}$, and $l := M^{-1}$. By the inequalities

$$B^{-1} \geq A^{-1} > 0, \quad m^{-1}I \geq A^{-1} \geq M^{-1}I,$$

we have that

$$(B^{-1})^{-p} + C\left(\frac{1}{M}, \frac{1}{m}, -p\right) I \geq (A^{-1})^{-p}.$$

Then we obtain the second inequality.

$$B^p + C\left(\frac{1}{M}, \frac{1}{m}, -p\right) I \geq A^p \quad \text{for } p < -1.$$

Taking limit $p \longrightarrow -1$, we have that all the inequalities hold also for $p \leqq -1$. □

Theorem 3 is better than Theorem 1 by the following comparison:

Theorem 4. *If $q > 1$ and $M > m > 0$, then*

$$C(m, M, -q) \geqq C(1/M, 1/m, q).$$

Proof. Let $G_1(x) = a_1x + b_1$ be an affine function on $[m, M]$ with

$$G_1(m) = m^{-q} \quad \text{and} \quad G_1(M) = M^{-q}. \tag{0}$$

Since x^{-q} is a convex function, we have

$$x^{-q} \leqq G_1(x) \tag{1}$$

on $[m, M]$. Moreover, since G_1 is a tangent line for the curve $x^{-q} + C(m, M, -q)$, we have

$$G_1(x) \leqq x^{-q} + C(m, M, -q) \tag{2}$$

on $[m, M]$.

On the other hand, a function x^q on $[1/M, 1/m]$ is also convex. So, putting $G_2(x) = a_2x+b_2 (a_2 > 0)$ be an affine function on $[1/M, 1/m]$ with $G_2(1/M) = M^{-q}$ and $G_1(1/m) = m^{-q}$, we have

$$x^q \leqq G_2(x) \tag{3}$$

and G_2 is a tangent line for $x^q + C(1/M, 1/m, q)$ with a contact point $x_2 \in [1/M, 1/m]$, that is,

$$G_2(x) \leqq x^q + C(1/M, 1/m, q) \quad \text{and} \quad G_2(x_2) = x_2^q + C(1/M, 1/m, q) \tag{4}$$

on $[1/M, 1/m]$. Consider a function $F(x) = G_2(1/x)$ on $[m, M]$. Then

$$F(m) = m^{-q}, \quad \text{and} \quad F(M) = M^{-q}.$$

Since F is convex, (0) implies

$$F(x) \leqq G_1(x) \tag{5}$$

and

$$F(x) \leqq x^{-q} + C(1/M, 1/m, q) \quad \text{and} \quad F(1/x_2) = x_2^q + C(1/M, 1/m, q) \tag{4'}$$

on $[m, M]$. Therefore, (5) and (2) show

$$\begin{aligned} x_2^q + C(1/M, 1/m, q) &= F(1/x_2) \leqq G_1(1/x_2) \\ &\leqq (1/x_2)^{-q} + C(m, M, -q) = x_2^q + C(m, M, -q), \end{aligned}$$

so that we have the required inequality. □

We also have a similar inequality for the chaotic order, which does not characterize the chaotic order $\log A \geq \log B$ unfortunately since $C\left(\frac{1}{M}, \frac{1}{m}, 1-p\right)/p$ does not converge to 0 as p tends to 0:

Theorem 5. *Let A and B be positive and invertible operators on H with $MI \geq A \geq mI > 0$ for some positive numbers $M > m > 0$. If $\log A \geq \log B$, then*

$$B^p + \frac{M(m^p - M^p)}{m} I \geq B^p + C\left(\frac{1}{M}, \frac{1}{m}, 1-p\right) MI \geq A^p$$

for $p \leqq 0$.

Proof. By Lemma 2, we have the following inequality.

$$C\left(\frac{1}{M},\frac{1}{m},-p+1\right)M \leqq \left(\frac{1}{m}\right)\left(\left(\frac{1}{m}\right)^{-p}-\left(\frac{1}{M}\right)^{-p}\right)M$$
$$=\frac{M}{m}(m^p-M^p)$$

holds for $p<0$. Hence we get the first inequality. By using the chaotic Furuta inequality, we have the following for $r=1$ and $q\geqq 0$,

$$\log A \geq \log B \implies (B^{\frac{1}{2}}A^qB^{\frac{1}{2}})^{\frac{1}{q+1}} \geq B.$$

Then we have

$$\log B^{-1} \geq \log A^{-1} \implies (A^{-\frac{1}{2}}B^{-q}A^{-\frac{1}{2}})^{\frac{1}{q+1}} \geq A^{-1} \quad \text{for } q\geqq 0.$$

Hence it follows from Theorem C that

$$(A^{-\frac{1}{2}}B^{-q}A^{-\frac{1}{2}})+C\left(\frac{1}{M},\frac{1}{m},q+1\right)I \geq A^{-(q+1)},$$

or equivalently,

$$B^{-q}+C\left(\frac{1}{M},\frac{1}{m},q+1\right)A \geq A^{-q}.$$

Finally, since $MI\geq A\geq mI$, we have

$$B^p+C\left(\frac{1}{M},\frac{1}{m},-p+1\right)MI \geq A^p \quad \text{by} \quad p:=-q\leqq 0.$$

Then we have the second inequality. Whence the proof is complete. □

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