HÖLDER CONDITIONS IN THE DIVERGENCE THEOREM

WASHEK F. PFEFFER

ABSTRACT. In the framework of Lebesgue integration and bounded sets of finite perimeter, we present a straightforward proof of the divergence theorem for bounded vector fields satisfying Hölder conditions on sets of appropriate Hausdorff measures.

Given a compact BV set $A \subset \mathbb{R}^n$, the divergence theorem holds for every vector field $v : A \to \mathbb{R}^n$ that is continuous outside an $\mathcal{H}^{n-1}$ negligible set, and pointwise Lipschitz outside an $\mathcal{H}^{n-1}$ σ-finite set [9, Theorem 2.9]. Since continuous and pointwise Lipschitz are extreme points of the scale represented by Hölder conditions, it is natural to ask whether the divergence theorem remains valid under the following assumptions: for $0 < s < 1$, the vector field v is pointwise Lipschitz outside an $\mathcal{H}^{n-1+s}$ σ-finite (negligible) set $E \subset A$, and the s-Hölder constant of v is zero (finite) at each $x \in E$. We use results of W.B. Jurkat (see Remark 2.5 below) to prove the divergence theorem which takes into account Hölder conditions for all $0 \le s \le 1$ simultaneously. One of the consequences, obtained by restricting the number s to 0 and 1, is a new and simpler proof of the divergence theorem cited above.

Our starting point is the well-known divergence theorem for bounded BV sets and continuously differentiable vector fields, which is assumed without proof. Modulo a few technicalities resulting from the use of BV sets, the exposition is elementary. The crux of the proof involves only dyadic cubes. As an application, we present two theorems about integration by parts. Both are easy corollaries of the main result.

1. THE MAIN THEOREM

The sets of all reals and all positive reals are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. For each integer $m \ge 1$, we denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^m$ induced by the usual inner product $x \cdot y$. Unless specified otherwise, all functions we consider are real-valued. Given a collection $\mathcal{A}$ of sets and a set B, we let $\mathcal{A}(B) := \{A \in \mathcal{A} : A \subset B\}$.

Throughout, the ambient space is $\mathbb{R}^n$ where $n \ge 1$ is a fixed integer. The diameter, closure, interior, and boundary of a set $E \subset \mathbb{R}^n$ are denoted by $d(E)$, $\mathrm{cl}\, E$, $\mathrm{int}\, E$, and ∂E, respectively. By $B(x, r)$ we denote the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$. The distance from $x \in \mathbb{R}^n$ to $E \subset \mathbb{R}^n$ is denoted by $\mathrm{dist}(x, E)$. Equalities such as $\gamma := \gamma(n)$, $\kappa := \kappa(n), \ldots$, indicate that $\gamma, \kappa, \ldots$, are constants depending only on the dimension n.

Let $E \subset \mathbb{R}^n$ and $0 \le s \le 1$. The *s-Hölder constant* of $v : E \to \mathbb{R}^m$ at $x \in E$ is the extended real number

$$H_s v(x) := \limsup_{y \in E, \, y \to x} \frac{|v(y) - v(x)|}{|y - x|^s}.$$

Clearly, $H_0 v(x) < \infty$ if and only if v is bounded in a neighborhood of x, and $H_0 v(x) = 0$ if and only if v is continuous at x. Note that $\operatorname{Lip} v(x) := H_1 v(x)$ is the *Lipschitz constant* of v at the point x.

In $\mathbb{R}^n$ we use Lebesgue measure $\mathcal{L}^n$, as well as the Hausdorff measures $\mathcal{H}^t$ where $0 \le t \le n$ is a real number. For $E \subset \mathbb{R}^n$, we write $|E|_t$ instead of $\mathcal{H}^t(E)$; we note that

$$|E|_n = \mathcal{H}^n(E) = \mathcal{L}^n(E)$$

[4, Section 2.2], and denote this common value by $|E|$. The restricted measures $\mathcal{L}^n \llcorner E$ and $\mathcal{H}^t \llcorner E$ are defined in the usual way [4, Section 2.1]. Without additional attributes, the word "measurable" and the expressions "almost all" and "almost everywhere" refer to Lebesgue measure $\mathcal{L}^n$. We say that sets $A, B \subset \mathbb{R}^n$ *overlap* whenever $|A \cap B| > 0$.

Let $E \subset \mathbb{R}^n$ be a measurable set, and for $0 \le \theta \le 1$ let

$$E(\theta) := \big\{ x \in \mathbb{R}^n : \lim_{r \to 0+} r^{-n} \big| B(x,r) \cap E \big| = \theta \big\}.$$

The sets $\operatorname{int}_* E := E(1)$, $\operatorname{cl}_* E := \mathbb{R}^n - E(0)$, and $\partial_* E := \operatorname{cl}_* E - \operatorname{int}_* E$ are called, respectively, the *essential interior, essential closure,* and *essential boundary* of E. Clearly, the inclusions $\operatorname{int} E \subset \operatorname{int}_* E \subset \operatorname{cl}_* E \subset \operatorname{cl} E$ and $\partial_* E \subset \partial E$ hold. The sets $\operatorname{int}_* E$, $\operatorname{cl}_* E$, and $\partial_* E$ are Borel, and the density theorem [13, Chapter 4, Theorem 6.1] implies

(1.1) $$|\operatorname{int}_* E| = |\operatorname{cl}_* E| = |E|.$$

The extended real number $\|E\| := |\partial_* E|_{n-1}$ is called the *perimeter* of E. A *unit exterior normal* of E at $x \in \mathbb{R}^n$ is a vector $\nu \in \mathbb{R}^n$ such that $|\nu| = 1$ and

$$\lim_{r \to 0+} r^{-n} \big| \{ y \in B(x,r) \cap E : \nu \cdot (y - x) > 0 \} \big| = 0,$$
$$\lim_{r \to 0+} r^{-n} \big| \{ y \in B(x,r) - E : \nu \cdot (y - x) < 0 \} \big| = 0.$$

If a unit exterior normal of E at x exists, it is unique and we denote it by $\nu_E(x)$.

Given a measurable set $E \subset \mathbb{R}^n$, we say that $v : \operatorname{cl}_* E \to \mathbb{R}^m$ is *relatively differentiable* at the point $x \in \operatorname{int}_* E$ if there is a linear map $Dv(x) : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{y \in \operatorname{cl}_* E, \, y \to x} \frac{\big| v(y) - v(x) - Dv(x)(y - x) \big|}{|y - x|} = 0.$$

If $m = n$, the trace of $Dv(x)$ is called the *relative divergence* of v at x, denoted by $\operatorname{div} v(x)$. Since the concepts of relative differentiability and differentiability coincide whenever x belogs to $\operatorname{int} E$, using the standard notation will cause no confusion.

Relative differentiability implies *approximate differentiability* [4, Section 3.1.2]. Although the converse is false, relative and approximate differentiability exhibit similarities. In particular, an argument analogous to proving the uniqueness of approximate derivatives shows that the linear map $Dv(x)$ defined above is unique [8, Observation 2.5.6]. The following version of Stepanoff's theorem for relative differentiability is proved in [3, Proposition 2.3].

Theorem 1.1. *Let $E \subset \mathbb{R}^n$. A map $v : \operatorname{cl}_* E \to \mathbb{R}^n$ is relatively differentiable almost everywhere in $\big\{ x \in \operatorname{int}_* E : \operatorname{Lip} v(x) < \infty \big\}$.*

A *set of finite perimeter*, or a *BV set*, is a measurable set $A \subset \mathbb{R}^m$ with $|A| < \infty$ and $\|A\| < \infty$; cf. Remark 3.3 below. The family of all bounded BV sets is denoted by $\mathcal{BV}_c$. For the basic properties of BV sets we refer to [4, Chapter 5]. The next two theorems are proved in [4, Chapter 5] and [14].

Theorem 1.2. *If $A \in \boldsymbol{\mathcal{BV}}_c$, then the map $\nu_A : x \mapsto \nu_A(x)$ is defined $\mathcal{H}^{n-1}$ almost everywhere in $\partial_* A$, it is $\mathcal{H}^{n-1}$ measurable, and for each $v \in C^1(\mathbb{R}^n; \mathbb{R}^n)$,*

$$\int_A \operatorname{div} v \, d\mathcal{L}^n = \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{n-1}.$$

Theorem 1.3. *For each set $A \in \boldsymbol{\mathcal{BV}}_c$, there is a sequence $\{A_i\}$ in $\boldsymbol{\mathcal{BV}}_c(A)$ such that $\lim \|A - A_i\| = 0$ and the essential closure of each A_i is a closed set.*

Using Theorems 1.1 and 1.2, and employing properties of dyadic cubes (Proposition 2.4 below), we prove the following divergence theorem.

Main Theorem. *Let $A \in \boldsymbol{\mathcal{BV}}_c$ be such that $A = \operatorname{cl}_* A$, let $v : A \to \mathbb{R}^n$ be bounded, and let $E := \{x \in \operatorname{int}_* A : \operatorname{Lip} v(x) < \infty\}$. Assume that for $k = 1, 2, \ldots$, there are disjoint sets E_k and numbers $0 \le s_k < 1$ such that $A - E = \bigcup_{k=1}^{\infty} E_k$ and one of the following conditions holds:*

(i) *$\mathcal{H}^{n-1+s_k} \llcorner E_k$ is σ-finite, and $H_{s_k} v(x) = 0$ for each $x \in E_k$;*
(ii) *$\mathcal{H}^{n-1+s_k} \llcorner E_k = 0$, and $H_{s_k} v(x) < \infty$ for each $x \in E_k$.*

If the relative divergence $\operatorname{div} v$ belongs to $L^1(A, \mathcal{L}^n)$, then

$$(1.2) \qquad \int_A \operatorname{div} v \, d\mathcal{L}^n = \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{n-1}.$$

Remark 1.4. A few comments concerning the Main Theorem are in order.

(1) In view of equality (1.1), the assumption $A = \operatorname{cl}_* A$ is merely a matter of convenience and presents no factual restriction.

(2) Since $v : A \to \mathbb{R}^n$ is bounded and continuous outside the $\mathcal{H}^{n-1}$ negligible set

$$\{x \in A : H_0 v(x) > 0\},$$

the integral $\int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{n-1}$ is defined.

(3) As $|A - E| = 0$, Theorem 1.1 implies that the relative divergence $\operatorname{div} v(x)$ is defined for almost all $x \in A$. To see that $\operatorname{div} v \in L^1(A, \mathcal{L}^n)$ is a necessary assumption, consider $A := \operatorname{cl} B(0, 1)$ and

$$v(x) := \begin{cases} x |x| \cos |x|^{-n-1} & \text{if } x \in A - \{0\}, \\ 0 & \text{if } x = 0. \end{cases}$$

It is easy to verify that $\operatorname{div} v$ does not belong to $L^1(A, \mathcal{L}^n)$, although v satisfies all remaining assumptions of the Main Theorem.

(4) Assume $n = 1$, and let $v : \mathbb{R} \to \mathbb{R}$ be such that $\int_a^b v' \, d\mathcal{L}^1 = v(b) - v(a)$ for every compact interval $[a, b] \subset \mathbb{R}$. Since [4, Section 2.4.3] implies

$$\left| \{x \in \mathbb{R} : H_s v(x) > 0\} \right|_s = 0$$

for each $0 \le s < 1$, condition (ii) cannot be omitted. The Cantor-Vitali function [11, Section 7.16] and its multidimensional analogue [10] provide another rationale for our assumptions.

(5) If we replace bounded BV sets by finite unions of dyadic cubes, the proof of the Main Theorem becomes completely elementary. Still, this specialized result is sufficient for useful applications (Theorems 3.1 and 3.2 below).

2. The proof

Throughout $\rho = \frac{1}{8} n^{-3/2}$, and a set $A \in \boldsymbol{\mathcal{BV}}_c$ is called *regular* if

$$|A| > 0 \quad \text{and} \quad \frac{|A|}{d(A)\|A\|} > \rho.$$

Note that $|C| = 4\rho\, d(C)\|C\|$ for each cube $C \subset \mathbb{R}^n$.

Lemma 2.1. *Let $E \subset \mathbb{R}^n$ be a closed set, and let $v : E \to \mathbb{R}^n$ be bounded and $\mathcal{H}^{m-1}$ measurable. Assume $\{B_i\}$ is a sequence in $\boldsymbol{\mathcal{BV}}_c(E)$ such that $\lim d(B_i) = 0$, and each B_i is regular and contains a fixed $x \in E$. If $0 \le s \le 1$, then*

$$\limsup \frac{1}{d(B_i)^{n-1+s}} \int_{\partial_* B_i} v \cdot \nu_{B_i} \, d\mathcal{H}^{m-1} \le \gamma H_s v(x)$$

where $\gamma := \gamma(n) > 0$. If $x \in \text{int}_* E$ and v is relatively differentiable at x, then

$$\lim \frac{1}{|B_i|} \int_{\partial_* B_i} v \cdot \nu_{B_i} \, d\mathcal{H}^{m-1} = \operatorname{div} v(x).$$

Proof. Choose $\varepsilon > 0$, and let $F(B) := \int_{\partial_* B} v \cdot \nu\, d\mathcal{H}^{n-1}$ for each $B \in \boldsymbol{\mathcal{BV}}_c(E)$. There is $\delta > 0$ such that

$$\frac{\big|v(y) - v(x)\big|}{|y - x|^s} \le H_s v(x) + \varepsilon.$$

for every $y \in E \cap B(x, \delta)$. By Theorem 1.2 and the isodiametric inequality [4, Section 2.2], there is $\beta := \beta(n) > 0$ such that for all sufficiently large i,

$$
\begin{aligned}
\big|F(B_i)\big| &= \left| \int_{\partial_* B_i} \big[v(y) - v(x)\big] \cdot \nu_{B_i}(y) \, d\mathcal{H}^{n-1}(y) \right| \\
&\le \big[H_s v(x) + \varepsilon\big] \int_{\partial_* B_i} |y - x|^s \, d\mathcal{H}^{n-1}(y) \\
&\le \big[H_s v(x) + \varepsilon\big] d(B_i)^s \|B_i\| \le \frac{\beta}{\rho} \big[H_s v(x) + \varepsilon\big] d(B_i)^{n-1+s}.
\end{aligned}
$$

If v is relatively differentiable at $x \in \text{int}_* E$, let

$$w : y \mapsto v(x) + \big[Dv(x)\big](y - x) : \mathbb{R}^n \to \mathbb{R}^n,$$

and observe that $\operatorname{div} w(y) = \operatorname{div} v(x)$ for each $y \in \mathbb{R}^n$. There is $\eta > 0$ such that

$$\big|v(y) - w(y)\big| < \varepsilon |y - x|$$

for every $y \in E \cap B(x, \eta)$. By Theorem 1.2, for all sufficiently large i,

$$
\begin{aligned}
\Big|F(B_i) - \operatorname{div} v(x)|B_i|\Big| &= \left| \int_{\partial_* B_i} \big[v(x) - w(y)\big] \cdot \nu_{B_i}(y) \, d\mathcal{H}^{n-1}(y) \right| \\
&\le \varepsilon \int_{\partial_* B_i} |y - x| \, d\mathcal{H}^{n-1}(y) \le \varepsilon d(B_i)\|B_i\| = \frac{\varepsilon}{\rho}|B_i|.
\end{aligned}
$$

Letting $\gamma := \beta/\rho$ and $i \to \infty$, the lemma follows from the arbitrariness of ε. $\square$

Remark 2.2. Without the assumption of regularity in Lemma 2.1, we obtain

$$\limsup \frac{1}{\|B_i\|} \int_{\partial_* B_i} v \cdot \nu_{B_i} \, d\mathcal{H}^{n-1} \le H_0 v(x).$$

For integers $k \geq 1$ and j_i, $i = 1, \ldots, n$, the compact interval

$$C := \prod_{i=1}^{n} \left[2^{-k} j_i, 2^{-k}(j_i + 1)\right]$$

is called a *k-cube*. The collection of all k-cubes is denoted by $\mathbf{D}_k$, and we let $\mathbf{D}_{\geq k} := \bigcup_{j=k}^{\infty} \mathbf{D}_j$. The elements of $\mathbf{D} := \mathbf{D}_{\geq 1}$ are called *dyadic cubes*. If A and B are overlapping dyadic cubes, then either $A \subset B$ or $B \subset A$. It follows that each family $\mathbf{C}$ of dyadic cubes has a nonoverlapping subfamily $\mathbf{K}$ with $\bigcup \mathbf{K} = \bigcup \mathbf{C}$. Dyadic cubes A and B are called *adjacent* if $d(A) = d(B)$ and $A \cap B \neq \emptyset$. Every dyadic cube is adjacent to 3^n dyadic cubes, including itself.

For $x \in \mathbb{R}^n$ and an integer $k \geq 1$, let $\mathrm{st}(x, k) := \{C \in \mathbf{D}_k : x \in C\}$. Note $\mathrm{st}(x, k)$ consists of 2^n k-cubes, and x is the interior point of $\bigcup \mathrm{st}(x, k)$. A family $\mathbf{C}$ of dyadic cubes is called a *complete cover* of a set $E \subset \mathbb{R}^n$ if for each $x \in E$ there is an integer $k_x \geq 1$ such that $\mathrm{st}(x, k_x) \subset \mathbf{C}$; in this case $E \subset \bigcup_{x \in E} \mathrm{int} \bigcup \mathrm{st}(x, k_x)$. Thus a complete cover of a compact set has a finite nonoverlapping subcover.

Lemma 2.3. *Let δ be a positive function on a set $E \subset \mathbb{R}^n$, and let $0 \leq t \leq n$. Given $\varepsilon > 0$, the set E is covered completely by a family $\mathbf{C}_E$ of dyadic cubes such that each $C \in \mathbf{C}_E$ contains $x_C \in E$ with $\delta(x_C) > d(C)$, and for $\kappa := \kappa(n) > 0$,*

$$\sum_{C \in \mathbf{C}_E} d(C)^t \leq \kappa\left(|E|_t + \varepsilon\right).$$

Proof. Avoiding trivialities, assume $E \neq \emptyset$. Let $\mathbf{B}$ consist of all dyadic cubes C containing $x_C \in E$ with $\delta(x_C) > d(C)$. Denote by B_k the set of all $x \in \mathbb{R}^n$ such that each $C \in \mathbf{D}_{\geq k}$ containing x belongs to $\mathbf{B}$. From $\mathbb{R}^n - B_k = \bigcup(\mathbf{D}_{\geq k} - \mathbf{B})$ infer B_k is a Borel set. Since $\{x \in E : \delta(x) > 2^{-k}\sqrt{n}\} \subset B_k \subset B_{k+1}$, letting $E_1 := E \cap B_1$ and $E_k := E \cap (B_k - B_{k-1})$ for $k \geq 2$, we obtain $E = \bigcup_{k=1}^{\infty} E_k$ and $|E|_t = \sum_{k=1}^{\infty} |E_k|_t$. By elimination, we may assume each E_k is a nonempty set. There is a cover $\mathbf{K}_k \subset \mathbf{D}_{\geq k}$ of E_k such that every $C \in \mathbf{K}_k$ meets E_k, and

$$\sum_{C \in \mathbf{K}_k} d(C)^t \leq \vartheta\left(|E_k|_t + \varepsilon 2^{-k}\right)$$

where $\vartheta := \vartheta(n) > 0$; see [5, Theorem 5.1]. If $\mathbf{C}_k$ consists of all dyadic cubes that meet E_k and are adjacent to some $C \in \mathbf{K}_k$, then $\mathbf{C}_k$ is a complete cover of E_k and

$$\sum_{C \in \mathbf{C}_k} d(C)^t \leq 3^n \sum_{C \in \mathbf{K}_k} d(C)^t \leq 3^n \vartheta\left(|E_k|_t + \varepsilon 2^{-k}\right).$$

Since $\mathbf{C}_k \subset \mathbf{D}_{\geq k}$ and $E_k \subset B_k$, and since each $C \in \mathbf{C}_k$ meets E_k, the definition of B_k implies $\mathbf{C}_k \subset \mathbf{B}$. Thus the family $\mathbf{C}_E := \bigcup_{k=1}^{\infty} \mathbf{C}_k$ is a complete cover of E and each $C \in \mathbf{C}_E$ contains $x_C \in E$ with $\delta(x_C) > d(C)$. For $\kappa := 3^n \vartheta$,

$$\sum_{C \in \mathbf{C}_E} d(C)^t \leq \sum_{k=1}^{\infty} \sum_{C \in \mathbf{C}_k} d(C)^t \leq \kappa \sum_{k=1}^{\infty} \left(|E|_t + \varepsilon 2^{-k}\right) = \kappa\left(|E|_t + \varepsilon\right). \quad \square$$

A *dyadic figure* is a finite union of dyadic cubes. A *partition* is a collection

$$P := \left\{(B_1, x_1), \ldots, (B_p, x_p)\right\}$$

where $B_1, \ldots, B_p$ are nonoverlapping bounded BV sets, and $x_i \in B_i$ for $i = 1, \ldots, p$. The *body* of P is the set $[P] := \bigcup_{i=1}^{p} B_i$. If each B_i is a dyadic cube, P is called a *dyadic partition*. Given $E \subset \mathbb{R}^n$ and $\delta : E \to \mathbb{R}_+$, we say that P is *δ-fine* if $x_i \in E$ and $d(B_i) < \delta(x_i)$ for $i = 1, \ldots, p$.

Proposition 2.4. *Let $\mathcal{E}$ be a family of disjoint subsets of a dyadic figure A, and for each $E \in \mathcal{E}$ select real numbers $0 \leq t_E \leq n$ and $\varepsilon_E > 0$. Given $\delta : A \to \mathbb{R}_+$, there is a δ-fine dyadic partition $P := \{(C_1, x_1), \ldots, (C_p, x_p)\}$ such that $[P] = A$, and for $\kappa := \kappa(n) > 0$ and every $E \in \mathcal{E}$,*

$$\sum_{x_i \in E} d(C_i)^{t_E} \leq \kappa\big(|E|_{t_E} + \varepsilon_E\big).$$

Proof. Enlarging $\mathcal{E}$, we may assume $\bigcup \mathcal{E} = A$. Find an integer $k \geq 0$ so that A is the union of k-cubes and, making δ smaller, assume $\delta < 2^{-k}\sqrt{n}$. Let $\mathcal{C}_E$ be a complete cover of $E \in \mathcal{E}$ associated with $\delta_E := \delta \restriction E$, t_E, and ε_E according to Lemma 2.3. For every $C \in \mathcal{C}_E$, select $x_C \in E \cap C$ with $d(C) < \delta_E(x_C)$. Since $\mathcal{C} := \bigcup_{E \in \mathcal{E}} \mathcal{C}_E$ covers the compact set A completely, there are nonoverlapping dyadic cubes $C_1, \ldots, C_p$ in $\mathcal{C}$ whose union contains A. Our restriction of δ implies $P := \{(C_i, x_{C_i}) : C_i \subset A\}$ is a δ-fine dyadic partition with $[P] = A$. As $\mathcal{E}$ is a disjoint family, $\{C_i : x_{C_i} \in E\} \subset \mathcal{C}_E$. Hence for the same κ as in Lemma 2.3, we obtain the desired inequality

$$\sum_{x_{C_i} \in E} d(C_i)^{t_E} \leq \sum_{C \in \mathcal{C}_E} d(C)^{t_E} \leq \kappa\big(|E|_{t_E} + \varepsilon_E\big). \quad \square$$

Remark 2.5. Lemma 2.3 and Proposition 2.4 are due to W.B. Jurkat [7, Section 4]. The classical Cousin's lemma [8, Lemma 2.6.1], as well as its generalization obtained by E.J. Howard [6, Lemma 5], are immediate consequences of Proposition 2.4.

The *critical interior* of a measurable set $E \subset \mathbb{R}^n$, denoted by $\mathrm{int}_c E$, is the set of all $x \in \mathrm{int}_* E$ for which

$$\lim_{r \to 0+} r^{1-n}\big|B(x,r) \cap \partial_* E\big|_{n-1} = 0.$$

If E is a BV set, then $|\mathrm{int}_* E - \mathrm{int}_c E|_{n-1} = 0$ by [4, Theorem 1, Section 2.3].

Lemma 2.6. *Let $A \in \mathcal{BV}_c$ and $x \in \mathrm{int}_c A$. If $C \subset \mathbb{R}^n$ is a sufficiently small cube containing the point x, then $A \cap C$ is a regular BV set.*

Proof. If $r := d(C)$ then $\|A \cap C\| \leq \|C\| + \big|B(x,r) \cap \partial_* A\big|_{n-1}$, and hence

$$\frac{r\|A \cap C\|}{|C|} \leq \frac{r\|C\|}{|C|} + \frac{r^n}{|C|} \cdot \frac{\big|B(x,r) \cap \partial_* A\big|_{n-1}}{r^{n-1}}$$

$$= \frac{1}{4\rho} + n^{n/2}\frac{\big|B(x,r) \cap \partial_* A\big|_{n-1}}{r^{n-1}}.$$

By our assumptions and the density theorem, for all sufficiently small r,

$$n^{n/2}\frac{\big|B(x,r) \cap \partial_* A\big|_{n-1}}{r^{n-1}} < \frac{1}{4\rho} \quad \text{and} \quad \frac{|A \cap C|}{|C|} > \frac{1}{2}.$$

As $d(A \cap C) \leq r$, the regularity of $A \cap C$ follows. $\square$

Corollary 2.7. *Let $A \in \mathcal{BV}_c$. There is $\delta : A \to \mathbb{R}_+$ such that for each δ-fine dyadic partition $\{(C_1, x_1), \ldots, (C_p, x_p)\}$, the collection*

$$\big\{(A \cap C_1, x_1), \ldots, (A \cap C_p, x_p)\big\}$$

is a δ-fine partition, and $A \cap C_i$ is regular whenever $x_i \in \mathrm{int}_c A$.

Lemma 2.8. *Let $A \in \mathcal{BV}_c$ be a closed set, let $v : A \to \mathbb{R}^n$, and for $x \in A$, let*

$$f(x) := \begin{cases} \operatorname{div} v(x) & \text{if } x \in \operatorname{int}_* A \text{ and } v \text{ is relatively differentiable at } x, \\ 0 & \text{otherwise.} \end{cases}$$

Let $E := \big\{ x \in \operatorname{int}_* A : \operatorname{Lip} v(x) < \infty \big\}$, and assume that for $k = 1, 2, \dots$, there are disjoint sets E_k and numbers $0 \le s_k < 1$ such that $A - E = \bigcup_{k=1}^\infty E_k$, and one of the following conditions holds:

(i) $\mathcal{H}^{n-1+s_k} \, \llcorner \, E_k$ *is σ-finite, and $H_{s_k} v(x) = 0$ for each $x \in E_k$;*

(ii) $\mathcal{H}^{n-1+s_k} \, \llcorner \, E_k = 0$, *and $H_{s_k} v(x) < \infty$ for each $x \in E_k$.*

Then given $\varepsilon > 0$ and $\delta : A \to \mathbb{R}_+$, there is a δ-fine partition $P := \big\{ (B_1, x_1), \dots, (B_p, x_p) \big\}$ such that $[P] = A$ and

(2.1) $$\left| \sum_{i=1}^p f(x)|B_i| - \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{n-1} \right| < \varepsilon.$$

Proof. Enlarging the family $\{E_k\}$, condition (i) can be replaced by

(i*) $|E_k|_{n-1+s_k} < \infty$, and $H_{s_k} v(x) = 0$ for each $x \in E_k$.

Since $|A - \operatorname{int}_c A|_{n-1} < \infty$, we may assume that E and $\bigcup_{\{k : s_k > 0\}} E_k$ are subsets of $\operatorname{int}_c A$. Denote by D the set of all $x \in E$ at which v is relatively differentiable, and let $E_0 := E - D$ and $s_0 := 1$. For integers $k, j \ge 0$, let $t_k := n - 1 + s_k$ and

$$E_{k,j} := \big\{ x \in E_k : j - 1 < H_{s_k} v(x) \le j \big\}.$$

As $|E_{k,j}|_{s_k} = 0$ for $j \ge 1$, the set $S := \bigcup_{\{k : s_k = 0\}} \bigcup_{j=1}^\infty E_{k,j}$ is $\mathcal{H}^{n-1}$ negligible. Since A is compact, $|v|$ is bounded by $c > 0$. Choose $\varepsilon > 0$, and find $\eta > 0$ so that $\|A \cap B\| \le \varepsilon/c$ whenever $B \in \mathcal{BV}_c$ and $\|B\| < \eta$; see [8, Proposition 1.9.2]. Let $c_k := 1 + |E_{k,0}|_{s_k}$, and let γ and κ be the same constants as in Lemma 2.1 and Proposition 2.4, respectively.

Let $F(B) := \int_{\partial_* B} v \cdot \nu_B \, d\mathcal{H}^{n-1}$ for each $B \in \mathcal{BV}_c(A)$; see Remark 1.4, (2). By Remark 2.2 and Lemma 2.1, there is $\beta : \operatorname{cl}_* A \to \mathbb{R}_+$ such that for each $B \in \mathcal{BV}_c(A)$,

(1) $\big| F(B) \big| \le \varepsilon c_k^{-1} 2^{-k} \|B\|$ if $s_k = 0$ and $d(B) < \beta(x)$ for $x \in E_{k,0} \cap B$,

and for each regular $B \in \mathcal{BV}_c(A)$,

(2) $\big| f(x)|B| - F(B) \big| \le \varepsilon |B|$ if $d(B) < \beta(x)$ for $x \in D \cap B$,

(3) $\big| F(B) \big| \le \varepsilon c_k^{-1} 2^{-k} d(B)^{t_k}$ if $d(B) < \beta(x)$ for $x \in E_{k,0} \cap B$,

(4) $\big| F(B) \big| \le \gamma j d(B)^{t_k}$ if $d(B) < \beta(x)$ for $x \in E_{k,j} \cap B$ and $j \ge 1$.

Choose $\delta : A \to \mathbb{R}_+$ and, making it smaller, assume $\delta \le \beta$ and Corollary 2.7 holds for δ. Select a dyadic cube C containing A, and define $\theta : C \to \mathbb{R}_+$ by the formula

$$\theta(x) := \begin{cases} \delta(x) & \text{if } x \in A, \\ \operatorname{dist}(x, \operatorname{cl}_* A) & \text{if } x \in C - A. \end{cases}$$

By Proposition 2.4, there is a θ-fine dyadic partition $P := \big\{(C_1, x_1), \ldots, (C_p, x_p)\big\}$ such that $[P] = C$, and

$$(2.2) \qquad \begin{aligned} \sum_{x_i \in S} d(C_i)^{n-1} < \eta/(2n), \qquad \sum_{x_i \in E_{k,0}} d(C_i)^{t_k} \le \kappa c_k \text{ for all } k \ge 0, \\ \sum_{x_i \in E_{k,j}} d(C_i)^{t_k} \le \varepsilon j^{-1} 2^{-k-j} \text{ for all } k \text{ with } s_k > 0 \text{ and all } j \ge 1. \end{aligned}$$

If $B_i := A \cap C_i$, then Corollary 2.7 implies that $Q := \big\{(B_i, x_i) : x_i \in A\big\}$ is a δ-fine partition, $[Q] = A$, and B_i is regular whenever $x_i \in \text{int}_c A$. Let

$$K := \bigcup_{x_i \in S} C_i \quad \text{and} \quad B := \bigcup_{x_i \in S} B_i = A \cap K.$$

Then $\|K\| \le 2n \sum_{x_i \in S} d(C_i)^{n-1} < \eta$, and hence $\big|F(B)\big| \le c\|B\| \le \varepsilon$. By property (1), and the second of the inequalities (2.2),

$$\begin{aligned} \sum_{\{k:s_k=0\}} \sum_{x_i \in E_{k,0}} \big|F(B_i)\big| &\le \varepsilon \sum_{\{k:s_k=0\}} c_k^{-1} 2^{-k} \sum_{x_i \in E_{k,0}} \|A \cap C_i\| \\ &\le \varepsilon \sum_{\{k:s_k=0\}} c_k^{-1} 2^{-k} \sum_{x_i \in E_{k,0}} \big(|\text{int}\, C_i \cap \partial_* A|_{n-1} + \|C_i\|\big) \\ &\le \varepsilon\|A\| + 2n\varepsilon \sum_{\{k:s_k=0\}} c_k^{-1} 2^{-k} \sum_{x_i \in E_{k,0}} d(C_i)^{n-1} \\ &\le \varepsilon\big(\|A\| + 2n\kappa\big). \end{aligned}$$

The lemma follows from the previous inequalities and properties (2)–(4). Indeed, denoting by I the left side of inequality (2.1), we obtain

$$\begin{aligned} I \le \sum_{x_i \in D} \Big|f(x_i)|B_i| - F(B_i)\Big| &+ \big|F(B)\big| + \sum_{\{k:s_k=0\}} \sum_{x_i \in E_{k,0}} \big|F(B_i)\big| + \\ \sum_{\{k:s_k>0\}} \sum_{x_i \in E_{k,0}} \big|F(B_i)\big| &+ \sum_{\{k:s_k>0\}} \sum_{j=1}^{\infty} \sum_{x_i \in E_{k,j}} \big|F(B_i)\big| \le \\ \varepsilon \sum_{x_i \in D} |B_i| + \varepsilon &+ \varepsilon\big(\|A\| + 2n\kappa\big) + \\ \varepsilon \sum_{\{k:s_k>0\}} c_k^{-1} 2^{-k} \sum_{x_i \in E_{k,0}} d(B_i)^{t_k} &+ \gamma \sum_{\{k:s_k>0\}} \sum_{j=1}^{\infty} j \sum_{x_i \in E_{k,j}} d(B_i)^{t_k} \le \\ \varepsilon\big(|A| + \|A\| &+ 1 + 2\kappa(n+1) + 2\gamma\big). \quad \square \end{aligned}$$

Lemma 2.9. *Let $A \in \mathcal{BV}_c$ and $f \in L^1(A, \mathcal{L}^n)$. Given $\varepsilon > 0$, there is $\delta : A \to \mathbb{R}_+$ such that for each δ-fine partition $P := \big\{(B_1, x_1), \ldots, (B_p, x_p)\big\}$ such that $[P] = A$ and*

$$(2.3) \qquad \left|\int_A f \, d\mathcal{L}^n - \sum_{i=1}^{p} f(x_i)|B_i|\right| < \varepsilon.$$

Proof. Choose $\varepsilon > 0$, and using Vitali-Carathéodory theorem [13, Chapter 3, Theorem 7.6], find extended real-valued functions g and h defined on A so that g is upper semicontinuous, h is lower semicontinuous, $g \le f \le h$, and $\int_A (h-g) \, d\mathcal{L}^n \le \varepsilon$. There is a $\delta : A \to \mathbb{R}_+$ such that

$g(y) \leq f(x) + \varepsilon$ and $h(y) \geq f(x) - \varepsilon$ for all $x, y \in A$ with $|x - y| < \delta(x)$. Let $P := \big\{(B_1, x_1),$
$\ldots, (B_p, x_p)\big\}$ be a δ-fine partition with $[P] = A$. Then

$$\int_{B_i} g \, d\mathcal{L}^n \leq \int_{B_i} f \, d\mathcal{L}^n \leq \int_{B_i} h \, d\mathcal{L}^n,$$

$$\int_{B_i} g \, d\mathcal{L}^n - \varepsilon|B_i| \leq f(x_i)|B_i| \leq \int_{B_i} h \, d\mathcal{L}^n + \varepsilon|B_i|$$

for $i = 1, \ldots, p$, and consequently

$$\left| \int_A f \, d\mathcal{L}^n - \sum_{i=1}^p f(x_i)|B_i| \right| \leq \sum_{i=1}^p \left| \int_{B_i} f \, d\mathcal{L}^n - f(x_i)|B_i| \right|$$

$$\leq \sum_{i=1}^p \left[\int_{B_i} (h - g) \, d\mathcal{L}^n + \varepsilon|B_i| \right]$$

$$\leq \int_A (h - g) \, d\mathcal{L}^n + \varepsilon|A| \leq \varepsilon\big(1 + |A|\big). \quad \square$$

Proof of the Main Theorem. Assume first that A is a closed set. Choose $\varepsilon > 0$, and define f as in Lemma 2.8. Then $f \in L^1(A, \mathcal{L}^n)$ and $\int_A f \, d\mathcal{L}^n = \int_A \operatorname{div} v \, d\mathcal{L}^n$. Select $\delta : A \to \mathbb{R}_+$ associated with f according to Lemma 2.9. By Lemmas 2.8, there is a δ-fine partition $P := \big\{(B_1, x_1), \ldots, (B_p, x_p)\big\}$ such that $[P] = A$ and (2.1) holds. The choice of δ implies that (2.3) holds as well. Thus

$$\left| \int_A \operatorname{div} v \, d\mathcal{L}^n - \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{m-1} \right| \leq$$

$$\left| \int_A f \, d\mathcal{L}^n - \sum_{i=1}^p f(x_i)|B_i| \right| + \left| \sum_{i=1}^p f(x)|B_i| - \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{n-1} \right| < 2\varepsilon,$$

and the desired equality (1.2) follows from the arbitrariness of ε.

For an arbitrary A with $A = \operatorname{cl}_* A$, Theorem 1.3 yields a sequence $\{A_i\}$ in $\boldsymbol{\mathcal{BV}}_c(A)$ such that $\lim \|A - A_i\| = 0$ and each A_i is a closed set. Observe $\operatorname{div} v(x) = \operatorname{div}(v \upharpoonright A_i)(x)$ for each $x \in \operatorname{int}_* A_i$. Consequently

$$\int_{A_i} \operatorname{div} v \, d\mathcal{L}^n = \int_{\partial_* A_i} v \cdot \nu_{A_i} \, d\mathcal{H}^{m-1}$$

for $i = 1, 2, \ldots$, by the first part of the proof. Since v is bounded,

$$\lim \left| \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{m-1} - \int_{\partial_* A_i} v \cdot \nu_{A_i} \, d\mathcal{H}^{m-1} \right| \leq$$

$$\lim \int_{\partial_*(A - A_i)} |v| \, d\mathcal{H}^{m-1} \leq \sup_{x \in A} \big|v(x)\big| \lim \|A - A_i\| = 0.$$

As the isoperimetric inequality [4, Section 5.6.2] implies $\lim |A - A_i| = 0$,

$$\int_A \operatorname{div} v \, d\mathcal{L}^n = \lim \int_{A_i} \operatorname{div} v \, d\mathcal{L}^n$$

$$= \lim \int_{\partial_* A_i} v \cdot \nu_{A_i} \, d\mathcal{H}^{m-1} = \int_{\partial_* A} v \cdot \nu_A \, d\mathcal{H}^{m-1}. \quad \square$$

3. Integration by parts

Theorem 3.1. *Let $U \subset \mathbb{R}^n$ be an open set, let $v : U \to \mathbb{R}^n$ and $g : U \to \mathbb{R}$ be locally bounded in U and pointwise Lipschitz in $E \subset U$. Assume that for $k = 1, 2, \ldots$, there are disjoint sets E_k and numbers $0 \leq s_k < 1$ such that $U - E = \bigcup_{k=1}^{\infty} E_k$ and one of the following conditions holds:*

(i) *$\mathcal{H}^{n-1+s_k} \llcorner E_k$ is σ-finite, and $H_{s_k}(gv)(x) = 0$ for each $x \in E_k$;*
(ii) *$\mathcal{H}^{n-1+s_k} \llcorner E_k = 0$, and $H_{s_k}(gv)(x) < \infty$ for each $x \in E_k$.*

If the $\operatorname{div} v$ belongs to $L^1_{\mathrm{loc}}(U, \mathcal{L}^n)$ and Dg belongs to $L^1_{\mathrm{loc}}(U, \mathcal{L}^n; \mathbb{R}^n)$, then

$$\int_U g \operatorname{div} v \, d\mathcal{L}^n = - \int_U Dg \cdot v \, d\mathcal{L}^n$$

whenever gv has compact support.

Proof. Find a dyadic figure $A \subset U$ whose interior contains the support of $w := gv$. Since both v and g are locally bounded, w satisfies the conditions of the Main Theorem. As $w \restriction \partial A = 0$ and $\operatorname{div} w = g \operatorname{div} v + Dg \cdot v$ almost everywhere in U, an application of the Main Theorem to w and A completes the argument. $\square$

As Theorem 3.1 is based on the Main Theorem for dyadic figures, its proof is elementary; see Remark 1.4, (5). It can be applied to studying removable sets of partial differential equations in divergence form [2, Section 4], such as the Cauchy-Riemann, Laplace, and minimal surface equations. For illustration, we generalize a classical result of Besicovitch [1].

Theorem 3.2. *Let U be an open subset of the complex plane $\mathbb{C}$, let $f : U \to \mathbb{C}$, and let $E := \{x \in U : \operatorname{Lip} f(x) < \infty\}$. Assume that for $k = 1, 2, \ldots$, there are disjoint sets E_k and numbers $0 \leq s_k < 1$ such that $U - E = \bigcup_{k=1}^{\infty} E_k$ and one of the following conditions hold:*

(i) *$\mathcal{H}^{1+s_k} \llcorner E_k$ is σ-finite, and $H_{s_k} f(z) = 0$ for each $z \in E_k$;*
(ii) *$\mathcal{H}^{1+s_k} \llcorner E_k = 0$, and $H_{s_k} f(z) < \infty$ for each $z \in E_k$.*

If the complex derivative $f'(z)$ exists for almost all $z \in U$, then f can be redefined on the set $\{z \in U : H_0 f(z) > 0\}$ so that it is holomorphic in U.

Proof. Let $\bar{\partial} := \partial/\partial x + i \, \partial/\partial y$ where $i := \sqrt{-1}$, and choose $\varphi \in C^1_c(U; C)$. Since f is locally bounded and $\bar{\partial} f(z) = 0$ for almost all $z \in U$, we obtain

$$\int_U f \, \bar{\partial} \varphi \, d\mathcal{L}^2 = - \int_U \varphi \, \bar{\partial} f \, d\mathcal{L}^2 = 0$$

by applying Theorem 3.1 to the vector fields $u := (\Re f, -\Im f)$, $v := (\Im f, \Re f)$ and functions $g := \Re \varphi$, $h := \Im \varphi$. Thus f is a distributional solution of the Cauchy-Riemann equation $\bar{\partial} f = 0$. As $\bar{\partial}$ is an elliptic operator, f is equal almost everywhere to a holomorphic function in U [12, Example 8.14]. The theorem follows, since f is continuous outside the $\mathcal{H}^1$ negligible set $\{z \in U : H_0 f(z) > 0\}$. $\square$

Let $g \in L^1(\mathbb{R}^n, \mathcal{L}^n)$, and let $\{g > t\} := \{x \in \mathbb{R}^n : g(x) > t\}$ for each $t \in \mathbb{R}$. Since the extended real-valued function $t \mapsto \|\{f > t\}\|$ is nonnegative and measurable [4, Section 5.5, Lemma 1], we can define the extended real number

$$(3.1) \qquad \|g\| := \int_{\mathbb{R}} \|\{f > t\}\| \, d\mathcal{L}^1(t),$$

called the *variation* of g. If $\|g\| < \infty$, then g is called a *BV function* in $\mathbb{R}^n$, in which case the *distributional gradient* of g is a Radon measure in $\mathbb{R}^n$, denoted by Dg. Clearly, $E \subset \mathbb{R}^n$ is a BV set if and only if the indicator χ_E of E is a BV function; in which case $\|E\| = \|\chi_E\|$.

Remark 3.3. Our geometrically intuitive definitions of BV sets and BV functions are nonstandard. However, in view of [4, Sections 5.11 and 5.5], they are equivalent to the usual definitions given in [4, Section 5.1].

Let $\Omega \subset \mathbb{R}^n$ be a bounded *Lipschitz domain*. A function $g \in L^1(\Omega, \mathcal{L}^n)$ is called a BV function in Ω if its extension to $\mathbb{R}^n$ by zero is a BV function in $\mathbb{R}^n$. If g is a BV function in Ω, then a finite limit

$$\operatorname{Tr} g(x) := \lim_{r \to 0+} \frac{1}{|\Omega \cap B(x,r)|} \int_{\Omega \cap B(x,r)} g \, d\mathcal{L}^n$$

exists for $\mathcal{H}^{n-1}$ almost all $x \in \partial\Omega$, and the function $x \mapsto \operatorname{Tr} g(x)$ belongs to $L^1(\partial\Omega, \mathcal{H}^{n-1})$; see [4, Section 5.3].

Theorem 3.4. *Let Ω be a bounded Lipschitz domain, let $v \in C(\operatorname{cl}\Omega; \mathbb{R}^n)$, and let*

$$E := \{ x \in \Omega : \operatorname{Lip} v(x) < \infty \}.$$

Assume that for $k = 1, 2, \ldots$, there are disjoint sets E_k and numbers $0 < s_k < 1$ such that $\operatorname{cl}\Omega - E = \bigcup_{k=1}^{\infty} E_k$ and one of the following conditions holds:

(i) *$\mathcal{H}^{n-1+s_k} \llcorner E_k$ is σ-finite, and $H_{s_k} v(x) = 0$ for each $x \in E_k$;*
(ii) *$\mathcal{H}^{n-1+s_k} \llcorner E_k = 0$, and $H_{s_k} v(x) < \infty$ for each $x \in E_k$.*

If the divergence $\operatorname{div} v$ belongs to $L^1(\Omega, \mathcal{L}^n)$, then

$$(3.2) \qquad \int_\Omega g \operatorname{div} v \, d\mathcal{L}^n = \int_{\partial\Omega} (\operatorname{Tr} g) v \cdot \nu_\Omega \, d\mathcal{H}^{m-1} - \int_\Omega v \cdot d(Dg)$$

for each bounded BV function g in Ω.

Proof. There is a sequence $\{v_i\}$ in $C^1(\mathbb{R}^n, \mathbb{R}^n)$ that converges uniformly to v on $\operatorname{cl}\Omega$. According to [4, Section 5.3], the identity (3.2) holds for each v_i, and

$$\lim \int_{\partial\Omega} (\operatorname{Tr} g) v_i \cdot \nu_\Omega \, d\mathcal{H}^{m-1} = \int_{\partial\Omega} (\operatorname{Tr} g) v \cdot \nu_\Omega \, d\mathcal{H}^{m-1},$$

$$\lim \int_\Omega v_i \cdot d(Dg) = \int_\Omega v \cdot d(Dg).$$

by the dominated convergence theorem. Let $w_i = v_i - v$ and $c_i := \sup_{x \in \operatorname{cl}\Omega} |w_i(x)|$. The Fubini and Main theorems, together with equality (3.1), yield

$$\left| \int_{\operatorname{cl}\Omega} g \operatorname{div} w_i \, d\mathcal{L}^n \right| = \left| \int_{\mathbb{R}} \left(\int_{\{g>t\}} \operatorname{div} w_i \right) d\mathcal{L}^1(t) \right|$$

$$= \left| \int_{\mathbb{R}} \left(\int_{\partial_*\{g>t\}} w_i \cdot \nu_{\{g>t\}} \, d\mathcal{H}^{m-1} \right) d\mathcal{L}^1(t) \right|$$

$$\leq c_i \int_{\mathbb{R}} \|\{g > t\}\| \, d\mathcal{L}^1(t) = c_i \|g\|.$$

The theorem follows, since $\lim c_i = 0$ implies

$$\lim \int_\Omega g \operatorname{div} v_i \, d\mathcal{L}^n = \int_\Omega g \operatorname{div} v \, d\mathcal{L}^n. \quad \square$$

References

[1] A.S. Besicovitch, *On sufficient conditions for a function to be analytic, and behaviour of analytic functions in the neighbourhood of non-isolated singular points*, Proc. London Math. Soc. **32** (1931), 1–9.

[2] T. De Pauw and W.F. Pfeffer, *The Gauss-Green theorem and removable sets for PDEs in divergence form*, Adv. Math. **183** (2004), 155–182.

[3] ———, *The divergence theorem for unbounded vector fields*, Trans. Amer. Math. Soc. **359** (2007), 5915–5929.

[4] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, 1992.

[5] K.J. Falconer, *The Geometry of Fractal Sets*, Cambridge Univ. Press, Cambridge, 1985.

[6] E.J. Howard, *Analycity of almost everywhere differentiable functions*, Proc. Amer. Math. Soc. **110** (1990), 745–753.

[7] W.B. Jurkat, *The divergence theorem and Perron integration with exceptional sets*, Czechoslovak Math. J. **43** (1993), 27–45.

[8] W.F. Pfeffer, *Derivation and Integration*, Cambridge Univ. Press, New York, 2001.

[9] ———, *The Gauss-Green theorem in the context of Lebesgue integration*, Bull. London Math. Soc. **37** (2005), 81–94.

[10] ———, *A devil's platform*, Amer. Math. Monthly, to appear.

[11] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1987.

[12] ———, *Functional Analysis*, McGraw-Hill, New York, 1991.

[13] S. Saks, *Theory of the Integral*, Dover, New York, 1964.

[14] I. Tamanini and C. Giacomelli, *Un tipo di approssimazione "dall'interno" degli insiemi di perimetro finito*, Rend. Mat. Acc. Lincei **(9) 1** (1990), 181–187.

Department of Mathematics, University of California, Davis, CA 95616

E-mail address: `wfpfeffer@ucdavis.edu, washek@mcn.org`