ON THE OPERATOR EQUATION $AB = zBA$

Muneo Chō, Jun Ik Lee and Takeaki Yamazaki

Received October 9, 2008; Revised December 10, 2008

Abstract. In this paper, we study the operator equation $AB = zBA$ for bounded operators A, B on a complex Hilbert space. In [10], J. Yang and H.-K. Du proved that if A and B are normal operators, then $|z| = 1$ by using the Fuglede-Putnam Theorem. In this paper, we give an elementary proof of this result without using the Fuglede-Putnam Theorem and some examples. Then we shall relax normality in the result by Yang and Du. A quasinormality of an operator is given by using Aluthge transformation and the operator equality. [1]

1 Introduction Commutation relations between operators on a complex Hilbert space are important for the interpretation of quantum mechanical observables and the analysis of their spectra. Accordingly, such relations have been extensively studied in the mathematical literature (see, for example, the classic study of Putnam [6]). An interesting, related aspect concerns the commutativity up to a factor for pairs of operators. Certain forms of non-commutativity can be conveniently phrased in this way. This is the case with the famous canonical (or Heisenberg) commutation relations for position Q and momentum P,

$$QP - PQ \subset iI,$$

which can be recast in the form of the Weyl relations,

$$\exp(i\alpha Q)\exp(i\beta P) = \exp(-i\alpha\beta)\exp(i\beta P)\exp(i\alpha Q), \quad \alpha, \beta \in \mathbb{R}.$$

Another example, well known in the physical context, is an anticommutation relation between Pauli spin matrices in $\mathbb{C}^2$, e.g., $\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z$, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We consider pairs of operators A and B on a complex Hilbert space $\mathcal{H}$ and explore the conditions under which they can commute up to a factor, i.e.,

$$AB = \lambda BA, \quad \lambda \in \mathbb{C}\backslash\{0\}.$$

If A and B are unitary and $\lambda = e^{2\pi i\theta}$, then the C^*-algebra which is generated from A and B is called noncommutative torus, and it is important in the area of noncommutative geometry [4].

We are interested in the following two results. First, in [3] J. B. Brooke, P. Busch and D. B. Pearson proved the following.

2000 *Mathematics Subject Classification.* 47A10.

Key words and phrases. Hilbert space; normal operator; hyponormal operator; paranormal operator.

[1]This research is partially supported by Grant-in-Aid Research No. 20540192.

Theorem A ([3]) *Let $A, B \in B(\mathcal{H})$ satisfy $AB = zBA \neq 0,\ z \in \mathbb{C}$. Then*

(1) if A or B is selfadjoint, then $z \in \mathbb{R}$;

(2) if both A and B are selfadjoint, then $z \in \{-1, 1\}$; and

(3) if both A and B are selfadjoint and one of them is positive, then $z = 1$.

Next, in [10] J. Yang and H.-K. Du showed the following.

Theorem B ([10]) *Let $A, B \in B(\mathcal{H})$ satisfy $AB = zBA \neq 0,\ z \in \mathbb{C}$. Then*

(1) if either A or B is selfadjoint and the other is normal, then $z \in \{-1, 1\}$; and

(2) if both A and B are normal, then $|z| = 1$.

We remark that for the proof of (2), Yang and Du used the Fuglede-Putnam Theorem; i.e., if N, M are normal and $NT = TM$ for some $T \in B(\mathcal{H})$, then $N^*T = TM^*$.

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is called normal and hyponormal if $T^*T = TT^*$ and $T^*T \geq TT^*$, respectively. An operator $T \in B(\mathcal{H})$ is called quasinormal and paranormal if T commutes with T^*T and $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$, respectively. It holds that

$$\text{normal} \implies \text{quasinormal} \implies \text{hyponormal} \implies \text{paranormal}.$$

For an operator $T \in B(\mathcal{H})$, the spectrum, the spectral radius, the kernel and the range of T are denoted by $\sigma(T)$, $r(T)$, $\ker(T)$ and $R(T)$, respectively.

In this paper, first, we give a simple proof for (2) of Theorem B without using the Fuglede-Putnam Theorem. Then we shall attempt to relax normality of operators in Theorem B. We will give some examples related to the results. As an application, we give a quasinormality of an operator via Aluthge transformation.

2 Results If $A \in B(\mathcal{H})$ is normal, then we have

$$\|AB\|^2 = \|B^*A^*AB\| = \|B^*AA^*B\| = \|A^*B\|^2. \tag{2.1}$$

If $B \in B(\mathcal{H})$ is normal, then we have

$$\|BA\|^2 = \|A^*B^*BA\| = \|A^*BB^*A\| = \|B^*A\|^2 = \|(B^*A)^*\|^2 = \|A^*B\|^2. \tag{2.2}$$

Therefore, we have an elementary proof for (2) of Theorem B.

Elementary proof for (2) of Theorem B. Since by (2.1) and (2.2) $\|AB\| = \|BA\|$ and $\|AB\| \neq 0$, we have $|z| = 1$. □

We shall attempt to relax normality in (2) of Theorem B. First, we have the following result:

Theorem 1 *Let $A, B \in B(\mathcal{H})$ satisfy $AB = zBA \neq 0,\ z \in \mathbb{C}$. Then*

(1) if both A^* and B are hyponormal, then $|z| \leq 1$; and

(2) if both A and B^* are hyponormal, then $|z| \geq 1$.

Proof. (1) Since both A^* and B are hyponormal, by (2.1) and (2.2), we have

$$\begin{aligned} |z|\|BA\| = \|AB\| = \|B^*A^*AB\|^{\frac{1}{2}} &\le \|B^*AA^*B\|^{\frac{1}{2}} \\ &= \|A^*B\| = \|A^*BB^*A\|^{\frac{1}{2}} \le \|A^*B^*BA\|^{\frac{1}{2}} = \|BA\|. \end{aligned}$$

Hence we have $|z| \le 1$.

(2) Since $AB = zBA$ and $z \ne 0$, we have $BA = z^{-1}AB$ for hyponormal operators A and B^*. Hence we have $|z^{-1}| \le 1$ by (1), that is, $|z| \ge 1$. □

Especially, we have the following corollary:

Corollary 1 *Let $A, B \in B(\mathcal{H})$ satisfy $AB = zBA \ne 0,\ z \in \mathbb{C}$. Then*

(1) if A is normal and B is hyponormal, then $|z| \le 1$; and

(2) if A is hyponormal and B is normal, then $|z| \ge 1$.

Note 1. If $AB = zBA \ne 0 \ \ (A, B \in B(\mathcal{H}),\ z \in \mathbb{C})$, then $r(AB) = |z| \cdot r(BA)$. Hence, if $r(AB) \ne 0$, then $|z| = 1$. And if $|z| \ne 1$, then $\sigma(AB) = \{0\} = \sigma(BA)$. Moreover, Brooke, Busch and Pearson ([3]) showed the following:
(1) $\sigma(AB) = \sigma(BA)$; and
(2) if $0 \notin \sigma(AB)$, then both A and B are invertible.

Note 2. We shall attempt to relax the normalities of A and B in (2) of Theorem B, but it is difficult by Example 1 which is stated in the next section. If the condition $|z| = 1$ is relaxed into $|z| \le 1$, we can extend (2) of Theorem B to non-normal operators case as in Theorem 1.

Theorem 2 *Let $A, B \in B(\mathcal{H})$ satisfy $AB = zBA \ne 0,\ z \in \mathbb{C}$. Let A be normal and B satisfy* $\ker(B) \subset \ker(B^*)$. *Then $|z| \le 1$, and moreover if $|z| < 1$, then $0 \in \sigma(A)$.*

To prove Theorem 2, we prepare the following lemma:

Lemma 1 *Let A, C be normal and $AB = BC$ for $B \in B(\mathcal{H})$. If $B = U|B|$ is the polar decomposition of B, then $U^*|A|^2U = U^*U|C|^2U^*U$.*

To prove it, we note that for an operator X with the polar decomposition $X = U|X|$, the following formula always holds:

$$\lim_{\varepsilon \to +0} |X|(|X| + \varepsilon I)^{-1} = U^*U. \tag{2.3}$$

Proof of Lemma 1. Since $AB = BC$, by the Fuglede-Putnam Theorem we have $A^*B = BC^*$ and $B^*A = CB^*$. Therefore, it holds

$$B^*BC = B^*AB = CB^*B$$

and hence $|B|C = C|B|$. Since

$$B^*A^*AB = C^*B^*BC = C^*|B|^2C = |B|C^*C|B|,$$

we have

$$\begin{aligned} B^*|A|^2B = |B|C^*C|B| &\iff |B|U^*|A|^2U|B| = |B||C|^2|B| \\ &\iff U^*UU^*|A|^2UU^*U = U^*U|C|^2U^*U \quad \text{by (2.3)} \\ &\iff U^*|A|^2U = U^*U|C|^2U^*U. \quad \square \end{aligned}$$

Proof of Theorem 2. Let $B = U|B|$ be the polar decomposition of B. Put $C = zA$ in Lemma 1. Since zA is normal, it holds

$$U^*|A|^2U = |z|^2U^*U|A|^2U^*U. \tag{2.4}$$

By $\ker(B) \subset \ker(B^*)$, $U^*U \geq UU^*$. Hence, by $U^*U \cdot UU^* = UU^*$, $U^*UU = U$ on $R(U^*)$ and $U^*UU = U = 0$ on $\ker(U)$. Therefore, by $\mathcal{H} = \overline{R(U^*)} \oplus \ker(U)$,

$$U^*UU = U = UU^*U.$$

Then by (2.4),

$$U^{*n+1}|A|^2U^{n+1} = |z|^2U^*U \cdot U^{*n}|A|^2U^n \cdot U^*U = |z|^2U^{*n}|A|^2U^n \quad (n \geq 1)$$

and

$$\begin{aligned} |z|^{2n}U^*U|A|^2U^*U &= |z|^{2(n-1)}U^*|A|^2U \quad \text{by (2.4)} \\ &= |z|^{2(n-2)}U^{*2}|A|^2U^2 \\ &= \cdots \\ &= U^{*n}|A|^2U^n. \end{aligned}$$

Assume $|z| > 1$. Then

$$\|A\|^2 \geq \|U^{*n}|A|^2U^n\| = |z|^{2n}\|U^*U|A|^2U^*U\| \longrightarrow \infty.$$

Therefore, A is unbounded and it is a contradiction to $A \in B(\mathcal{H})$. Hence $|z| \leq 1$.

Assume $|z| < 1$. Then it holds $U^{*n}U^n \geq U^*U$ by $U^*U \geq UU^*$. Hence, if $Ux \neq 0$, then $U^nx \neq 0$. Also, it holds

$$U^{*n}|A|^2U^n = |z|^{2n}U^*U|A|^2UU^* \longrightarrow 0.$$

Let a unit vector x be $Ux \neq 0$. Then $U^nx \neq 0$ for every natural number n. Put $x_n = U^nx$. Then since U is a partial isometry, $\|x_1\| \leq \|x_n\| \leq 1$ for every n and

$$\|Ax_n\|^2 = (U^{*n}|A|^2U^nx, x) \longrightarrow 0.$$

Therefore, $0 \in \sigma(A)$. $\square$

When A has some spectral condition, we can relax just normalities of A and B in (2) of Theorem B to non-normal operator case.

Theorem 3 *Let $A, B \in B(\mathcal{H})$ satisfy $AB = zBA \neq 0$, $z \in \mathbb{C}$. Let A be hyponormal and B be paranormal. If A is invertible or 0 is an isolated point of $\sigma(A)$, then $|z| = 1$.*

To prove Theorem 3, we prepare the following lemma:

Lemma 2 ([7]) *Let T be paranormal. If $\mathcal{X}$ is an invariant subspace for T, then the restriction $T_{|\mathcal{X}}$ is also paranormal.*

Proof of Theorem 3. We only prove as 0 is an isolated point of $\sigma(A)$. Since $\mathcal{H} = \overline{R(A^*)} \oplus \ker(A)$, we decompose $A = \begin{pmatrix} A_1 & 0 \\ A_2 & 0 \end{pmatrix}$ on $\overline{R(A^*)} \oplus \ker(A)$. Since A is hyponormal,

$$A^*A = \begin{pmatrix} A_1^*A_1 + A_2^*A_2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad AA^* = \begin{pmatrix} A_1A_1^* & A_1A_2^* \\ A_2A_1^* & A_2A_2^* \end{pmatrix},$$

it holds $0 \geq A_2A_2^*$. Hence $A_2 = 0$ and $A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$. Since $\sigma(A) = \sigma(A_1) \bigcup \{0\}$, 0 is an isolated point of $\sigma(A_1)$ if $0 \in \sigma(A_1)$. Since A_1 is hyponormal, 0 is an eigenvalue of A_1. It's a contradiction and hence A_1 is invertible. Let $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$ on $\overline{R(A^*)} \oplus \ker(A)$. By $z \neq 0$ and $AB = zBA$, it holds $A_1B_2 = B_3A_1 = 0$. Since A_1 is invertible, we have $B_2 = B_3 = 0$. Hence, $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_4 \end{pmatrix}$. Therefore, $\overline{R(A^*)}$ is invariant for B and hence B_1 is paranormal by Lemma 2. Since $B_1 = zA_1^{-1}B_1A_1$ and $r(B_1) \neq 0$, we have $|z| = 1$. □

Especially, we have the corollary.

Corollary 2 *Let $A, B \in B(\mathcal{H})$ satisfy $AB = zBA \neq 0,\ z \in \mathbb{C}$. Let A be normal and B be paranormal. If A is invertible or 0 is an isolated point of $\sigma(A)$, then $|z| = 1$.*

Let $T = U|T|$ be the polar decomposition of an operator T. The Aluthge transform $\Delta(T)$ is defined by $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ in [1]. The Aluthge transform has good properties, and many authors study it (see [2, 5, 8, 9]). We are interested in the condition of operators A, B such that $AB = BA$ holds under $AB = zBA$ for some $z \in \mathbb{C}$. But it is difficult to give a solution. We can obtain a partial solution for the problem.

Theorem 4 *Let $T \in B(\mathcal{H})$ satisfy $T = z\Delta(T)$ for some $z \in \mathbb{C}$. If $\sigma(T) \neq \{0\}$, then $T = \Delta(T)$; i.e., T is quasinormal.*

Proof. By $r(\Delta(T)) = r(T) = |z|r(\Delta(T))$ and $r(T) \neq 0$, we have $|z| = 1$. Let $T = U|T|$ be the polar decomposition with the kernel condition $\ker(T) = \ker(U)$. Then we have

$$\begin{aligned} T = z\Delta(T) &\Longleftrightarrow U|T| = z|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} \\ &\Longleftrightarrow U|T|^{\frac{1}{2}}U^*U = z|T|^{\frac{1}{2}}UU^*U \quad \text{by (2.3)} \\ &\Longleftrightarrow U|T|^{\frac{1}{2}} = z|T|^{\frac{1}{2}}U. \qquad (2.5) \end{aligned}$$

Then by $|T|^{\frac{1}{2}} \geq 0$ and (1) of Theorem A, $z \in \mathbb{R}$, that is, $z = \pm 1$. If $z = -1$, then by (2.5),

$$U|T|^{\frac{1}{2}} + |T|^{\frac{1}{2}}U = 0 \iff |T|^{\frac{1}{2}} + U^*|T|^{\frac{1}{2}}U = 0.$$

Since both $|T|^{\frac{1}{2}}$ and $U^*|T|^{\frac{1}{2}}U$ are positive, we have $|T| = 0$ and $T = 0$. It is a contradiction to $\sigma(T) \neq \{0\}$. Hence $z = 1$. □

An operator T is nilpotent if $T^n = 0$ holds for some positive integer n, and T is quasinilpotent if $\sigma(T) = \{0\}$ (i.e., $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = 0$). Obviously, if T is nilpotent, then it is quasinilpotent. But the converse does not hold. In [10], Yang and Du proved the following: *Let $A, B \in B(\mathcal{H})$ satisfy $AB = zBA \neq 0$ ($z \in \mathbb{C}$). If A is normal and $R(B)$ is dense, then AB is not nilpotent.* About this result, we give a result as follows:

Theorem 5 *Let $A, B \in B(\mathcal{H})$ satisfy $AB = zBA \neq 0$, $z \in \mathbb{C}$. If $R(A)$ is dense and $\ker(B) = \{0\}$, then AB is not nilpotent.*

Proof. Assume $(AB)^n = 0$ for a natural number n. Then $(AB)^n = z^n(BA)^n = z^n B(AB)^{n-1}A = 0$. Since $z \neq 0$, $R(A)$ is dense and $\ker(B) = \{0\}$, we have $(AB)^{n-1} = 0$. Therefore, by induction we have $AB = 0$. It is a contradiction. □

3 Examples In (2) of Theorem B, we have $|z| = 1$ if $AB = zBA$ holds for normal operators A and B. One might think that Theorem 1 can be extended to non-normal operators case. But it is difficult from the following example:

Example 1 *Let $\mathcal{H} = \ell^2$ and, for $z \in \mathbb{C}$,*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & z & 0 & 0 & \cdots \\ 0 & 0 & z^2 & 0 & \cdots \\ 0 & 0 & 0 & z^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \tag{3.1}$$

Then the operator equation $AB = zBA$ holds. Also, A is normal and B is quasinormal. We have the following three cases:

(1) If $|z| = 1$, then $\sigma(AB) \neq \{0\}$.

(2) If $|z| < 1$, then $\sigma(AB) = \{0\}$. In this case, A is compact.

(3) If $|z| > 1$, then A is unbounded.

In Theorem 3, we obtain $T = \Delta(T)$ if $T = z\Delta(T)$ and $\sigma(T) \neq \{0\}$. We can expect that the second assumption $\sigma(T) \neq \{0\}$ can be omitted. But it is false as the following example:

Example 2 *Let $\mathcal{H} = \ell^2$ and, for $0 < z < 1$, $T \in B(\mathcal{H})$ be*

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & z^2 & 0 & 0 & \cdots \\ 0 & 0 & z^4 & 0 & \cdots \\ 0 & 0 & 0 & z^6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & z^2 & 0 & 0 & \cdots \\ 0 & 0 & z^4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Then the operator equation $T = \frac{1}{z}\Delta(T)$ holds. But $T \neq \Delta(T)$.

In Theorem 4, we obtain that AB is not nilpotent, but we do not know whether AB is quasinilpotent or not. Relating to the question, we give an example.

Example 3 *Let $\mathcal{H} = \ell^2$ and, for $z \in \mathbb{C}$, $A, B \in B(\mathcal{H})$ be*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & z & 0 & 0 & \cdots \\ 0 & 0 & z^2 & 0 & \cdots \\ 0 & 0 & 0 & z^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then A is normal, $R(B) = \mathcal{H}$ and $AB = z^{-1}BA$. It holds that

(1) if $|z| < 1$, then $\sigma(AB) = \{0\}$; i.e., AB is quasinilpotent;

(2) if $|z| = 1$, then $\sigma(AB) \neq \{0\}$; i.e., AB is not quasinilpotent.

By Example 3, there exists an operator B with dense range and a normal operator A such that AB is quasinilpotent. But AB must not be nilpotent by [10].

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Muneo Chō
Department of Mathematics
Kanagawa University
Yokohama 221-8686, Japan
e-mail: chiyom01@kanagawa-u.ac.jp

Jun Ik Lee
Department of Mathematics Education
SangMyung University
Seoul 110-743, Korea
e-mail: jilee@smu.ac.kr

Takeaki Yamazaki
Department of Mathematics
Kanagawa University
Yokohama 221-8686, Japan
e-mail: yamazt26@kanagawa-u.ac.jp