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INTERVAL HAUSDORFFNESS AND INITIALITY

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Abstract. The geometric realization functor, $|?|_T : S^{\triangle^{op}} \longrightarrow KTop$, is known to commute with finite limits if the reflection of $T[1]$ into the category of T_0-spaces is a Hausdorff space, where T is a cosimplicial k-space. We have previously shown that for the categories Fco, $ConsFco$, Con, Lim, PsT, $Born$, and $PreOrd$, the initiality of the inclusion of the boundary $\dot{Y}_{0n}$ of Y_{0n} into Y_{0n} guarantees the commutation of finite limits by the geometric realization functor, $|?|_Y : S^{\triangle^{op}} \longrightarrow A$. Here we show that in the above mentioned categories the initiality condition may be viewed as a generalized Hausdorff condition on the interval Y_{01}, as is the case in the classical situation where A is the category $KTop$.

1 Introduction We have shown, in [2], that for certain categories A the geometric realization functor, $|?|_Y : S^{\triangle^{op}} \longrightarrow A$, commutes with finite limits if and only if the collection $\{i_n : \dot{Y}_{0n} \rightarrowtail Y_{0n}\}$ is strongly initial, where Y is the domain of a simplex structure, i.e. a discrete fibration, $g : Y \longrightarrow D\triangle^{op}$, in A. We have then applied this result to the categories Fco, $ConsFco$, Con, Lim, PsT, $Born$, and $PreOrd$, and have shown that the initiality of the inclusion of the boundary $\dot{Y}_{0n}$ of Y_{0n} into Y_{0n} guarantees the commutation of finite limits by the geometric realization functor, $|?|_Y : S^{\triangle^{op}} \longrightarrow A$. On the other hand, the geometric realization functor, $|?|_T : S^{\triangle^{op}} \longrightarrow KTop$, is known to commute with finite limits if the reflection of $T[1]$ into the category of T_0-spaces is a Hausdorff space, where T is a cosimplicial k-space, see [5].

In section 2 of the present article we give the preliminary resullts and then in section 3 we show that in the categories, Fco, $ConsFco$, Con, Lim, PsT, $Born$, and $PreOrd$, the initiality condition may be viewed as a generalized Hausdorff condition on the interval Y_{01}, as is the case in the classical situation where A is the category $KTop$.

2 Preliminaries. Let A be a category with finite limits and coequalizers of reflexive pairs that is geometric over the category S of sets via the morphism f. Assume that the functor $a \times - : A \longrightarrow A$ preserves extremal epis, and the direct image $f_* : A \longrightarrow S$ of f preserves reflexive coequalizers, and reflects monos and terminals. Let $g : Y \longrightarrow D\triangle^{op}$ be a simplex structure, and Let m and n be natural numbers, and $\triangle[n]$ the standard n-simplex, see [2] and [3].

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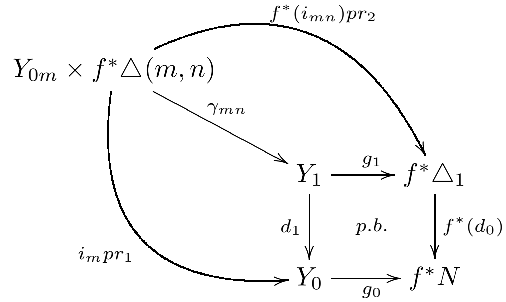
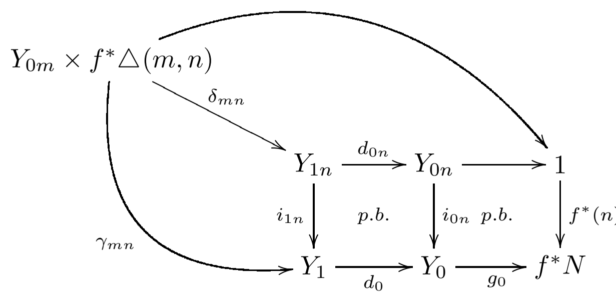
2.1. **Definition:** Define $O_{mn} : \triangle[m] \times \triangle(m,n) \longrightarrow \triangle[n]$ to be the morphism induced by composition.

2.2. **Definition:** Define the morphisms γ_{mn} and δ_{mn} by the following pullback diagrams.





where pr_1, and pr_2 are the projections of the product, and i_{mn}, and i_m are the obvious inclusions. Define λ_{mn} to be the composition $d_{0n}\delta_{mn}$.

With O_{mn} and λ_{mn} as above, we have:

2.3. **Lemma:** $|O_{mn}|_Y = \lambda_{mn}$.

Proof: By Lemma 2.2 of [2], we have $|\triangle[n]|_Y = Y_{0n}$. It can be easily shown that $|\triangle[m] \times \triangle(m,n)|_Y = Y_{0m} \times f^*\triangle(m,n)$. So we have $|O_{mn}|_Y : Y_{om} \times f^*\triangle(m,n) \longrightarrow Y_{0n}$. The diagram in the proof of [2], Lemma 2.2, together with the definitions of the maps O_{mn} and λ_{mn} show that $|O_{mn}|_Y = \lambda_{mn}$. □

The boundary $\dot{\triangle}[n]$ of $\triangle[n]$ and the boundary $\dot{Y}_{0n}$ of Y_{0n} are defined in [2], Definition 2.3, and we have:

2.4. **Lemma:** The image of O_{mn} is $Sk^m\triangle[n]$. In particular, the image of $O_{n-1,n}$ is $\dot{\triangle}[n]$.

Proof: The definition of n-skeleton together with that of O_{mn} imply that the epi-mono factorization of O_{mn} is $\triangle[m] \times \triangle(m,n) \xrightarrow{\partial_{mn}} Sk^m\triangle[n] \xrightarrow{i_{mn}} \triangle[n]$

The second assertion of the lemma follows from definition of $\dot{\triangle}[n]$. □

2.5. **Theorem:** If f_* preserves reflexive coequalizers, reflects monos, and f_*Y is filtered, then $\dot{Y}_{0n}$ in A is the image of the morphism $\lambda_{n-1,n} : Y_{0,n-1} \times f^*\triangle(n-1,n) \longrightarrow Y_{0n}$.

Proof: By Lemma 2.4, $O_{n-1,n}$ is the composition:

$$\triangle[n-1] \times \triangle(n-1,n) \xrightarrow{\partial_n} \dot{\triangle}[n] \xrightarrow{i_n} \triangle[n]$$

Applying the functor $|?|_Y$ to the above diagram, Lemma 1.3, and Lemma 2.2 of [2], give $\lambda_{n-1,n}$ as the composition $Y_{0,n-1} \times f^*\triangle(n-1,n) \xrightarrow{|\partial_n|} \dot{Y}_{0n} \xrightarrow{|i_n|} Y_{0n}$. ∂_n is an epi in S, and therefore a coequalizer. Since $|?|_Y$ preserves colimits, $|\partial_n|$ is a coequalizer in A. Since f_* preserves reflexive coequalizers, and reflects monos, $|\partial_n|$ is an e.e..

On the other hand by Lemma 2.5 of [2], and the assumption that f_* reflects monos, it follows that $i_n = |i_n|$ is a mono. Hence $\dot{Y}_{0n}$ is the image of $\lambda_{n-1,n}$. □

3 Hausdorffness. Now let A be one of the categories Fco, $ConsFco$, Con, Lim, PsT, $Born$, and $PreOrd$, see [1], and [6]. If $g : Y \longrightarrow D\triangle^{op}$ is a simplex structure, then products are preserved by the functor $|?|_Y : S^{\triangle^{op}} \longrightarrow A$, see Lemma 1.3. of [2]. So by Theorem 2.4 and Proposition 2.2 of [4], we can view Y_{01} as a linearly ordered set with minimum 0 and maximum 1. Furthermore, Y_{0n} can be identified with the set $\{(y_1, y_2, ..., y_n) : 0 \leq y_1 \leq y_2 \leq ... \leq y_n \leq 1\}$. It is not hard to see under this identification that the image of the morphism $\lambda_{n-1,n} : Y_{0,n-1} \times f^*\triangle(n-1, n) \longrightarrow Y_{0n}$, is the set, $\{(y_1, y_2, ..., y_n) \in Y_{0n} : y_1 = 0, \text{ or } y_{i-1} = y_i \text{ some } 1 < i \leq n, \text{ or } y_n = 1\}$ with the coinduced structure, see Definition 2.2. It then follows from Theorem 2.5 that $\dot{Y}_{0n} \equiv \{(y_1, y_2, ..., y_n) \in Y_{0n} : y_1 = 0, \text{ or } y_{i-1} = y_i \text{ some } 1 < i \leq n, \text{ or } y_n = 1\}$, and we have the commutative diagram:

$$\begin{array}{ccc} Y_{0,n-1} \times f^*\triangle(n-1,n) & \xrightarrow{\lambda_{n-1,n}} & Y_{0n} \\ & \underset{\partial_n}{\searrow} \quad /// \quad \underset{i_n}{\nearrow} & \\ & \dot{Y}_{0n} & \end{array}$$

where $\partial_n = |\partial_n|$. Given a map $\mu : n-1 \longrightarrow n$ in $\triangle$, the morphism $\lambda_{n-1,n}$ induces a morphism $\hat{\mu} : Y_{0,n-1} \longrightarrow Y_{0n}$ in A, which factors through $\dot{Y}_{0n}$. If μ is a mono, (epi), then $\hat{\mu}$ is the restriction of one of the maps δ_i (respectively σ_i) described in [4] p 53. Since the obvious mono$\triangle[n] \rightarrowtail \triangle[1]^n$ is a retract, if $|?|_Y$ preserves products, then$Y_{0n} \rightarrowtail Y_{01}^n$ is a retract. Hence the structure on Y_{0n} is the induced structure from Y_{01}^n. In what follows a convergence space (X, C) is denoted by X^*, and the structure $C(x)$ by $X(x)$.

3.1. **Definition:** A convergence space X^* is said to be Hausdorff if $X(x) \cap X(y) = \{[\{\phi\}]\}$ for all $x \neq y$ in X.

3.2. **Theorem:** (i) In Fco,$\dot{Y}_{0n} \overset{i_n}{\rightarrowtail} Y_{0n}$ is initial for all n if and only if Y_{01}^* is Hausdorff and $F \in Y_{01}(0)$, $F \neq [0] \rightarrow F \not\subseteq [0]$, and $F \in Y_{01}(1), F \neq [1] \rightarrow F \not\subseteq [1]$.

(ii) In $ConsFco$,$\dot{Y}_{0n} \overset{i_n}{\rightarrowtail} Y_{0n}$ is initial for all n if and only if Y_{01}^* is discrete.

(iii) In Con,$\dot{Y}_{0n} \overset{i_n}{\rightarrowtail} Y_{0n}$ is initial for all n if and only if Y_{01}^* is Hausdorff and $Y_{01}(0)$ and $Y_{01}(1)$ are discrete.

(iv) In Lim,$\dot{Y}_{0n} \overset{i_n}{\rightarrowtail} Y_{0n}$ is initial for all n if and only if Y_{01}^* is Hausdorff.

(v) In PsT,$\dot{Y}_{0n} \overset{i_n}{\rightarrowtail} Y_{0n}$ is initial for all n if and only if the following condition (I_n) holds for all n:

(I_n) If U is an ultrafilter on Y_{01}^n with U_k, the kth projection of U, in $Y_{01}(y_k)$, and $y = (y_1, y_2, ..., y_n)$ is in $\dot{Y}_{0n}$, then U contains $\partial_y \cup \dot{Y}_{0n}^c$, where ∂_y is the union of all the faces containing y and $\dot{Y}_{0n}^c$ is the complement of $\dot{Y}_{0n}$ in Y_{01}^n.

Furthermore for i_n to be initial for all n it is sufficient to have Y_{01}^* Hausdorff.

(vi) In $Born$,$\dot{Y}_{0n} \overset{i_n}{\rightarrowtail} Y_{0n}$ is initial for all n with no further conditions needed.

(vii) In $PreOrd, \dot{Y}_{0n} \xrightarrow{i_n} Y_{0n}$ is initial for all n if and only if the preorder on Y_{01} is symmetric.

Proof: (i) Since $\dot{Y}_{0n}$ has the coinduced structure, by 3.2.3. of [6], which holds for Fco as well, we have $\dot{Y}_{0n}(y) = \{F \in F(\dot{Y}_{0n}) : \exists x \in Y_{0,n-1}, \mu \in \triangle(n-1,n), G \in Y_{0,n-1}(x)$ such that $\hat{\mu}(x) = y$, and $\hat{\mu}(G) \subseteq F\}$. Some computation shows that i_n is initial if and only if the following condition holds:

(I_n) For all $y = (y_1, y_2, ..., y_n) \in \dot{Y}_{0n}$ and all $F_k \in Y_{01}(y_k)$, $k = 1, 2, ..., n$, there is $x \in Y_{0,n-1}$, $\mu \in \triangle(n-1, n)$, and $G \in Y_{0,n-1}(x)$ such that $\hat{\mu}(x) = y$ and $\hat{\mu}(G) \subseteq [(F_1 \otimes F_2 \otimes ... \otimes F_n) \wedge \dot{Y}_{0n}]$. Now suppose i_n is initial for all n and so (I_n) holds for all n. Therefore (I_3) holds. Given $x < y$, $F \in Y_{01}(x)$, $F' \in Y_{01}(y)$, let $y_1 = y_2 = x$, $y_3 = y$, $F_1 = F_2 = F$, $F_3 = F'$ and note that $(y_1, y_2, y_3) \in \dot{Y}_{03}$, $F_k \in Y_{01}(y_k)$, and $\hat{\delta}_2(y_1, y_2) = (y_1, y_2, y_3)$. (I_3) implies the existence of a G in $Y_{02}(y_1, y_3)$ such that $\hat{\delta}_2(G) \subseteq [(F_1 \otimes F_2 \otimes F_3) \wedge \dot{Y}_{03}]$. $M \in G$ implies there are A_k in F_k such that $\hat{\delta}_2(M)$ contains $(A_1 \times A_2 \times A_3) \cap \dot{Y}_{03}$ which contains $\{(a, b, b) : a \in A_1, b \in A_2 \cap A_3, a \leq b\}$. It follows that $a \in A_1, b \in A_2 \cap A_3, a \leq b$ implies $a = b$, or equivalently (*) $a \in A_1, b \in A_2 \cap A_3 \to a \geq b$. Let $A = A_1 \cap A_2 \in F_1 \wedge F_2 = F \wedge F = F$. By (*) we have (**) $a \in A, b \in A \cap A_3 \to a \geq b$. From (**) it follows that if b_1 and b_2 are in $A \cap A_3$ then $b_1 = b_2$, hence $A \cap A_3 \subseteq \{z_0\}$ for some z_0. This implies $F \wedge F' = [\{\phi\}]$ or otherwise $F \wedge F' = [z_0]$, in which case by (*) it follows that $[z_0, 1] \in F$, where $[z_0, 1] = \{x \in Y_{01} : z_0 \leq x \leq 1\}$. Now let $y_1 = x$, $y_2 = y_3 = y$, $F_1 = F$, $F_2 = F_3 = F'$. A similar argument shows that either $F \wedge F' = [\{\phi\}]$ or $F \wedge F' = [z_1]$ and $[0, z_1] \in F'$. Combining the above results we conclude that for $x < y$, $F \in Y_{01}(x)$, $F' \in Y_{01}(y)$ we have either $F \wedge F' = [\{\phi\}]$ or $(F \wedge F' = [z_0]$, $[z_0, 1] \in F$ and $[0, z_0] \in F')$. Assume $F \wedge F' \neq [\{\phi\}]$. Let $y_1 = y_2 = x$, $y_3 = y$, $F_1 = [x]$, $F_2 = F$, and $F_3 = F'$, then $(y_1, y_2, y_3) \in \dot{Y}_{03}$, $F_k \in Y_{01}(y_k)$ for $k = 1, 2, 3$. Apply (I_3) to get a filter $G \in Y_{02}(y_1, y_3)$ such that $\hat{\delta}_2(G) \subseteq [(F_1 \otimes F_2 \otimes F_3) \wedge \dot{Y}_{03}]$. $M \in G$ implies $\hat{\delta}_2(M)$ contains $(A_1 \times A_2 \times A_3) \cap \dot{Y}_{03}$ for some $A_k \in F_k$. It follows that $\{(a, b, b) : a \in A_1, b \in A_2 \cap A_3, a \leq b\} \subseteq \hat{\delta}_2(M)$. Therefore $a \in A_1$, $b \in A_2 \cap A_3 \to a \geq b$. $A_1 \in F_1 = [x]$, so $x \in A_1$, hence we have: (***) $b \in A_2 \cap A_3 \to x \geq b$. Let $A'_2 = A_2 \cap [z_0, 1] \in F$ and $A'_3 = A_3 \cap [0, z_0] \in F'$. So $A'_2 \cap A'_3 \in F \wedge F' \neq [\{\phi\}]$. This implies $A'_2 \cap A'_3 \neq \phi$. On the other hand $A'_2 \cap A'_3 \subseteq [0, z_0] \cap [z_0, 1] = \{z_0\}$, therefore $\{z_0\} = A'_2 \cap A'_3 \subseteq A_2 \cap A_3$, that is $z_0 \in A_2 \cap A_3$ and so by (***) we have $x \geq z_0$. Finally by letting $y_1 = x$, $y_2 = y_3 = y$, $F_1 = F$, $F_2 = F'$, and $F_3 = [y]$ and applying (I_3) we conclude $y \leq z_0$. It follows that $y \leq z_0 \leq x$, that is $y \leq x$ a contradiction. Hence $F \wedge F' = [\{\phi\}]$ and we have proved for $x \neq y$, $F \in Y_{01}(x)$, $G \in Y_{01}(y)$ implies $F \wedge G = [\{\phi\}]$. It is easy to see that this is just the Hausdorffness defined in Definition 1.6. This proves that Y^*_{01} is Hausdorff. To show $F \in Y_{01}(0)$, $F \neq [0]$ implies $F \not\subseteq [0]$, we use the fact that (I_2) holds. Let $y_1 = y_2 = 0$ and take $F \in Y_{01}(0)$ and assume that $F \subseteq [0]$. Then $x = 0$ and $\mu = \delta_1$ or δ_2 and there is a filter G in $Y_{01}(0)$ such that $\hat{\delta}_1(G) \subseteq [(F \otimes F) \wedge \dot{Y}_{02}]$ or $\hat{\delta}_2(G) \subseteq [(F \otimes F) \wedge \dot{Y}_{02}]$. If $\hat{\delta}_1(G) \subseteq [(F \otimes F) \wedge \dot{Y}_{02}]$, then $M \in G$ implies $\hat{\delta}_1(M)$ contains $(A \times B) \cap \dot{Y}_{02}$ for some A and B in F. $0 \in A$ since $F \subseteq [0]$. Therefore $0 \times A \subseteq \hat{\delta}_1(M) = \Delta M$. Thus $A \subseteq \{0\}$ and so F contains $[0]$. It follows that $F = [0]$. If $\hat{\delta}_2(G) \subseteq [(F \otimes F) \wedge \dot{Y}_{02}]$, then $M \in G$ implies $\hat{\delta}_2(M)$ contains $(A \times B) \cap \dot{Y}_{02}$ for some A and B in F. $\Delta(A \cap B) \subseteq (A \times B) \cap \dot{Y}_{02}$, therefore $\Delta(A \cap B) \subseteq 0 \times M$. It follows that $A \cap B \subseteq \{0\}$. $A \cap B \in F$ so F contains $[0]$ and thus $F = [0]$. A similar argument shows that $F \in Y_{01}(1)$ and $F \neq [1]$ implies $F \not\subseteq [1]$. To prove the sufficiency, we need to show (I_n) holds for all n. We show (I_3) holds, the rest is similar.

Let $0 \neq x < y \neq 1$, F_1, $F_2 \in Y_{01}(x)$ and $F_3 \in Y_{01}(y)$. Define $G = [((F_1 \wedge F_2) \otimes F_3) \wedge Y02] \in Y_{02}(x, y)$. Note that $\hat{\delta}_2(x, y) = (x, x, y)$. We need to show that $\hat{\delta}_2(G) \subseteq [(F_1 \otimes F_2 \otimes F_3) \wedge \dot{Y}_{03}]$. $M \in G$ implies M contains $((A \cap B) \times C) \cap Y_{02}$ for some $A \in F_1$, $B \in F_2$, and $C \in F_2$. So

$\hat{\delta}_2(M)$ contains $(\Delta(A \cap B) \times C) \cap \dot{Y}_{03}$. Since F_1 and F_2 are in $Y_{01}(x)$, $x \neq 0, 1$, Hausdorffness implies $F_1 \wedge [0] = [\{\phi\}]$, $F_1 \wedge [1] = [\{\phi\}]$, etc. So we can assume without loss of generality that $0 \notin A$, $1 \notin B$. It follows that $\Delta(A \cap B) = (A \times B) \cap \dot{Y}_{02}$, and therefore we have $\hat{\delta}_2(M)$ contains $(A \times B \times C) \cap \dot{Y}_{03} \in [(F_1 \otimes F_2 \otimes F_3) \wedge \dot{Y}_{03}]$, and so $\hat{\delta}_2(M) \in [(F_1 \otimes F_2 \otimes F_3) \wedge \dot{Y}_{03}]$ as desired. The cases for (y_1, y_2, y_3) on the remaining faces of $\dot{Y}_{03}$ follow similarly.

(ii) Let C denote the constant structure on Y_{01}. It is not hard to check that i_n is initial for all n, if and only if given F_k in C, $k = 1, 2, ..., n$, there is G in $Y^*_{0,n-1}$ such that $\hat{\delta}_k(G) \subseteq [(F_1 \otimes F_2 \otimes ... \otimes F_n) \wedge \dot{Y}_{0n}]$ for some k. Suppose i_n is initial for all n. Let $F \in C$, it follows that there is $G \in Y^*_{02}$ such that $\hat{\delta}_k(G) \subseteq [F \otimes F \otimes F \wedge \dot{Y}_{03}]$. $M \in G$ implies $\hat{\delta}_k(M)$ contains $(A_1 \times A_2 \times A_3) \cap \dot{Y}_{03}$, for some A_1, A_2, A_3 in F. Let $A = A_1 \cap A_2 \cap A_3$, then we have (*) $(A \times A \times A) \cap \dot{Y}_{03} \subseteq \hat{\delta}_k(M)$. It follows that (**) if x and y are in A and $x \leq y$, then both (x, x, y) and (x, y, y) are in $\hat{\delta}_k(M)$. If $k = 0$, then $\hat{\delta}_0(M) = M \times 1$ and (*) implies $A \subseteq \{1\}$, thus F contains [1]. If $k = 1$, then $\hat{\delta}_1(M) = \{(a, b, b) : (a, b) \in M\}$ and (**) implies $x, y \in A$ and $x \leq y \to x = y$ or equivalently $x, y \in A \to x \geq y$. It follows that $A \subseteq \{z_0\}$ for some z_0 and therefore $[z_0] \subseteq F$. If $k = 2$, it follows similarly that $[z_0] \subseteq F$ for some z_0. If $k = 3$, it follows that $[0] \subseteq F$. Hence in either case F in C implies $[z] \subseteq F$ for some z, that is, C is discrete. The sufficiency is trivial.

(iii) The proof is similar to that of Fco. In this case since $[x] \cap F$ is in $Y_{01}(x)$ for all F in $Y_{01}(x)$, it follows that for $x = 0$ or 1, $Y_{01}(x)$ is discrete.

(iv) In Lim, i_n is initial for all n if and only if $y = (y_1, y_2, ..., y_n) \in \dot{Y}_{0n}$ and $F_k \in Y_{01}(y_k)$, $k = 1, 2, ..., n$, implies there is a finite number of (x_i, μ_i) in $Y_{0,n-1} \times \triangle(n-1, n)$, and G_i in $Y_{0,n-1}(x_i)$ such that $\hat{\mu}_i(x_i) = y$ and $\bigcap_i [\hat{\mu}_i(G_i)] \subseteq [(F_1 \otimes F_2 \otimes ... \otimes F_n) \wedge \dot{Y}_{0n}]$. The rest is similar to the proof for Fco.

(v) Using 3.2.9. of [6], one can show $\dot{Y}_{0n}(y) = \{F \in F(\dot{Y}_{0n}) : U$ an ultrafilter $\supseteq F \to \exists x \in Y_{0,n-1}, \mu \in \triangle(n-1, n)$, and ultrafilter $E \in Y_{0,n-1}(x)$ such that $\hat{\mu}(x) = y$ and $[\hat{\mu}(E) = U]\}$ for each n. It is easy to show that i_n is initial if and only if given $y \in \dot{Y}_{0n}$, $F \in Y^n_{01}(y)$, we have $[F \wedge \dot{Y}_{0n}]$ is in $\dot{Y}_{0n}(y)$. Being in PsT, it then follows that i_n is initial if and only if for any $y \in \dot{Y}_{0n}$ and U an ultrafilter in $Y^n_{01}(y)$, $[U \wedge \dot{Y}_{0n}] \in \dot{Y}_{0n}(y)$. On the other hand $[U \wedge \dot{Y}_{0n}] \in \dot{Y}_{0n}(y)$, if and only if either $[U \wedge \dot{Y}_{0n}] = [\{\phi\}]$ or there is $x \in Y_{0,n-1}$, $\mu \in \triangle(n-1, n)$, and an ultrafilter $E \in Y_{0,n-1}(x)$ such that $[U \wedge \dot{Y}_{0n}] = [\hat{\mu}(E)]$. Using the fact that U is an ultrafilter, and so for a given set either the set or its complement is in U, it is easy to show that $[U \wedge \dot{Y}_{0n}] = [\{\phi\}]$ exactly when U contains $\dot{Y}^c_{0n}$ and that $[U \wedge \dot{Y}_{0n}] = [\hat{\mu}(E)]$ exactly when U contains ∂_y.

Combining the above facts yields the statement (I_n) as required. To prove the other assertion, suppose Y^*_{01} is Hausdorff. To show (I_1) holds, let $y = 0$ in $\dot{Y}_{01}$, and U an ultrafilter in $Y_{01}(0)$. Y^*_{01} Hausdorff implies $U \neq [1]$, therefore $\{1\} \notin U$. Since U is an ultrafilter, it follows that $\{1\}^c = [0, 1) \in U$. If $\{0\} \in U$, then $U = [0]$, otherwise $(0, 1] \in U$, hence $(0, 1) = [0, 1) \cap (0, 1] \in U$, that is $U = [0]$ or $(0, 1) = \dot{Y}^c_{01} \in U$. For $y = 1$, a similar argument shows if U is an ultrafilter in $Y_{01}(1)$, then $U = [1]$ or $\dot{Y}^c_{01} \in U$. Hence (I_1) holds. To show (I_2) holds, let $y = (x, x)$, $x \neq 0, 1$, U an ultrafilter in $Y^2_{01}(x, x)$. It follows that U_1 and U_2 are in $Y_{01}(x)$. $x \neq 0$ and Y^*_{01} Hausdorff imply $U_1 \neq [0]$ and so there is $A \in U_1$ such that $0 \notin A$. Similarly there is $B \in U_2$ such that $1 \notin B$. If U does not contain $\dot{Y}^c_{02}$, then $\dot{Y}_{02} \in U$. Also $A \times B \in U_1 \otimes U_2 \subseteq U$, therefore $(A \times B) \cap \dot{Y}_{02} \in U$. $(A \times B) \cap \dot{Y}_{02} = \Delta(A \cap B) \subseteq \partial_y$. It follows that $\partial_y \in U$. For $y = (0, x)$, $x \neq 0, 1$, a similar argument shows that $\dot{Y}^c_{02}$ is in U, etc. To show (I_3) holds, let $y = (y_1, y_2, y_3)$, $0 \neq y_1 = y_2 < y_3 \neq 1$, and U an ultrafilter in $Y^3_{01}(y)$. Since $y_1 \neq 0$, $y_2 \neq y_3$, $y_3 \neq 1$, and Y^*_{01} is Hausdorff, it follows that there are sets $A \in U_1$, $B \in U_2$, $C \in U_3$ such that $0 \notin A$, $1 \notin C$, and $B \cap C = \phi$. If $\dot{Y}^c_{03} \notin U$, then $\dot{Y}_{03} \in U$ and so $(A \times B \times C) \cap \dot{Y}_{03} \in U$. But $(A \times B \times C) \cap \dot{Y}_{03} \subseteq \partial_y$, hence $\partial_y \in U$. Other cases follow similarly. Same argument holds for (I_n), $n > 3$.

(vi) By 3.3.3. of [6], we have the structure $\dot{B}_{0n}$ on $\dot{Y}_{0n}$ is $\dot{B}_{0n} = \{B \subseteq \dot{Y}_{0n} : B \subseteq \partial_n(M)$, for some $M \in B(Y_{0,n-1} \times \triangle(n-1,n))\}$. Some computation shows that $\dot{B}_{0n} = \{B \subseteq \dot{Y}_{0n} : B \subseteq \dot{\cup}\hat{\mu}_i(N_i), \mu_i \in \triangle(n-1,n), N_i \in B_{0,n-1}\}$. To show i_n is initial, we show if $B \subseteq \dot{Y}_{0n}$, and $B \in B_{0n}$, then $B \in \dot{B}_{0n}$. Given such a set B, we have $B \subseteq B_1 \times B_2 \times ... \times B_n$, where B_i is the ith projection of B. We then have $B = B \cap \dot{Y}_{0n} \subseteq (B_1 \times B_2 \times ... \times B_n) \cap \dot{Y}_{0n} \subseteq \hat{\delta}_0(B_1 \times B_2 \times ... \times B_{n-1} \cap Y_{0,n-1}) \cup \hat{\delta}_1(B_1 \times B_2 \times ... \times B_{n-2} \times (B_{n-1} \cap B_n) \cap Y_{0,n-1}) \cup ... \cup \hat{\delta}_{n-1}((B_1 \cap B_2) \times B_3 \times ... \times B_n \cap Y_{0,n-1}) \cup \hat{\delta}_n(B_2 \times ... \times B_n \cap Y_{0,n-1})$. Hence B is contained in a finite union of sets of the form $\hat{\mu}_i(N_i)$, and each N_i is in $B_{0,n-1}$. Therefore B is in $\dot{B}_{0n}$ as required.

(vii) Suppose i_n is initial for all n. Consider $Y_{0,n-1} \times \triangle(n-1,n) \xrightarrow{\partial_n} \dot{Y}_{0n}$. By 3.1.3. of [6], $y \leq y'$ in $\dot{Y}_{0n}$ if and only if there is a ∂_n-chain from y to y', that is, a finite chain $y = \omega_0 \leq \omega_1 \leq ... \leq \omega_m = y'$ such that for each $k = 0, 1, ..., m-1$, there is a pair $(b_k, \mu_k) \leq (b'_k, \mu'_k)$ in $Y_{0,n-1} \times \triangle(n-1,n)$ such that $\hat{\mu}_k(b_k) = \omega_k$ and $\hat{\mu}'_k(b'_k) = \omega_{k+1}$. Since $\triangle(n-1,n)$ has the discrete structure, it follows that $\mu_k = \mu'_k$ for each k. $0 \not\leq 1$ in $\dot{Y}_{01}$, that is 0 and 1 are not related in $\dot{Y}_{01}$, since otherwise the existence of the chain implies 0=1. Since i_1 is initial, it follows that $0 \not\leq 1$ in Y_{01}. In fact this is sufficient for i_1 to be initial. Suppose $x \neq 0$ and $0 \leq x$ in Y_{01}. It follows that $(0,x) \leq (x,x)$ in Y_{02}. Since i_2 is initial we conclude that $(0,x) \leq (x,x)$ in $\dot{Y}_{02}$. The existence of the chain will imply $x \leq 0$ in Y_{01}. One can similarly show that if $x \leq 0$ then $0 \leq x$, and that the same holds with 0 replaced by 1. In fact i_1 and i_2 are initial if and only if $0 \not\leq 1$, $x \leq 0 \Leftrightarrow 0 \leq x$, and $x \leq 1 \Leftrightarrow 1 \leq x$. Similar computation shows i_1, i_2 and i_3 are initial if and only if $x_1 \leq x_2 \Leftrightarrow x_2 \leq x_1$. It thus follows that if i_n is initial for all n, then the preorder on Y_{01} is symmetric. The converse can easily be proved. □

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