NORM ONE LINEAR PROJECTIONS AND GENERALIZED CONDITIONAL EXPECTATIONS IN BANACH SPACES

TAKASHI HONDA AND WATARU TAKAHASHI

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Abstract. Let E be a smooth, strictly convex and reflexive Banach space, let Y^* be a closed linear subspace of the dual space E^* of E and let Π_{Y^*} be the generalized projection of E^* onto Y^*. Then, the mapping E_{Y^*} of E into E defined by $E_{Y^*} = J^{-1}\Pi_{Y^*}J$ is called the generalized conditional expectation with respect to Y^*, where J is the normalized duality mapping from E into E^*. In this paper, we prove two results which are related to norm one linear projections and generalized conditional expectations in Banach spaces.

1. Introduction

Let E be a smooth Banach space and let E^* be the dual space of E. The function $\phi : E \times E \to \mathbf{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for each $x, y \in E$, where J is the normalized duality mapping from E into E^*. Let C be a nonempty closed convex subset of E and let T be a mapping from C into itself. Then, T is called generalized nonexpansive if the set $F(T)$ of fixed points of T is nonempty and

$$\phi(Tx, y) \leq \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$; see Ibaraki and Takahashi [23]. Such nonlinear operators are connected with the resolvents of maximal monotone operators in Banach spaces. When E is a smooth, strictly convex and reflexive Banach space and C is a nonempty closed convex subset of E, Alber [1] also defined a nonlinear projection Π_C of E onto C called the generalized projection. Motivated by Alber [1] and Ibaraki and Takahashi [23], Kohsaka and Takahashi [34] proved the following result: Let E be a smooth, strictly convex and reflexive Banach space, let C^* be a nonempty closed convex subset of E^* and let Π_{C^*} be the generalized projection of E^* onto C^*. Then the mapping R defined by $R = J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C^*$. When Y^* is a closed linear subspace of E^*, the authors [20] also defined the mapping $E_{Y^*} = J^{-1}\Pi_{Y^*}J$ and called E_{Y^*} the generalized conditional expectation with respect to Y^*. Then, they obtained some results for generalized conditional expectations in the Banach space.

In this paper, we study the relationship between norm one linear projections and generalized conditional expectations in a smooth, strictly convex and reflexive Banach space.

2. Preliminaries

Throughout this paper, we assume that a Banach space E with the dual space E^* is real. We denote by $\mathbf{N}$ and $\mathbf{R}$ the sets of all positive integers and all real numbers, respectively. We also denote by $\langle x, x^*\rangle$ the dual pair of $x \in E$ and $x^* \in E^*$. A Banach space E is said to be strictly convex if $\|x + y\| < 2$ for $x, y \in E$ with $\|x\| \le 1$, $\|y\| \le 1$ and $x \ne y$. A Banach space E is said to be smooth provided

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. Let E be a Banach space. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

The multivalued operator $J : E \to E^*$ is called the normalized duality mapping of E. From the Hahn-Banach theorem, $Jx \ne \emptyset$ for each $x \in E$. We know that E is smooth if and only if J is single-valued. If E is strictly convex, then J is one-to-one, i.e., $x \ne y \Rightarrow J(x) \cap J(y) = \emptyset$. If E is reflexive, then J is a mapping of E onto E^*. So, if E is reflexive, strictly convex and smooth, then J is single-valued, one-to-one and onto. In this case, the normalized duality mapping J_* from E^* into E is the inverse of J, that is, $J_* = J^{-1}$; see [43] for more details. Let E be a smooth Banach space and let J be the normalized duality mapping of E. We define the function $\phi : E \times E \to \mathbf{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. We also define the function $\phi_* : E^* \times E^* \to \mathbf{R}$ by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle x^*, J^{-1}y^* \rangle + \|y^*\|^2$$

for all $x^*, y^* \in E^*$. It is easy to see that $(\|x\| - \|y\|)^2 \le \phi(x, y)$ for all $x, y \in E$. Thus, in particular, $\phi(x, y) \ge 0$ for all $x, y \in E$. We also know the following:

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{2.1}$$

for all $x, y, z \in E$. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx) \tag{2.2}$$

for all $x, y \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Leftrightarrow x = y. \tag{2.3}$$

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z \in C : \phi(z, x) = \min_{y \in C} \phi(y, x)\}$$

is always nonempty and a singleton. Let us define the mapping Π_C of E onto C by $z = \Pi_C x$ for every $x \in E$, i.e.,

$$\phi(\Pi_C x, x) = \min_{y \in C} \phi(y, x)$$

for every $x \in E$. Such Π_C is called the generalized projection of E onto C; see Alber [1]. The following lemma is due to Alber [1] and Kamimura and Takahashi [31].

Lemma 2.1 ([1, 31])**.** *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then, the following hold:*

(a) $z = \Pi_C x$ if and only if $\langle y - z, Jx - Jz \rangle \le 0$ for all $y \in C$;
(b) $\phi(z, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(z, x)$.

From this lemma, we can prove the following lemma.

Lemma 2.2. *Let M be a closed linear subspace of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times M$ Then, $z = \Pi_M x$ if and only if*

$$\langle J(x) - J(z), m \rangle = 0 \text{ for any } m \in M.$$

Let D be a nonempty closed convex subset of a smooth Banach space E, let T be a mapping from D into itself and let $F(T)$ be the set of fixed points of T. Then, T is said to be generalized nonexpansive [23] if $F(T)$ is nonempty and $\phi(Tx,u) \leq \phi(x,u)$ for all $x \in D$ and $u \in F(T)$. Let C be a nonempty subset of E and let R be a mapping from E onto C. Then R is said to be a retraction, or a projection if $Rx = x$ for all $x \in C$. It is known that if a mapping P of E into E satisfies $P^2 = P$, then P is a projection of E onto $\{Px : x \in E\}$. The mapping R is also said to be sunny if $R(Rx + t(x - Rx)) = Rx$ whenever $x \in E$ and $t \geq 0$. A nonempty subset C of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto C. The following lemmas were proved by Ibaraki and Takahashi [23].

Lemma 2.3 ([23])**.** *Let C be a nonempty closed subset of of a smooth and strictly convex Banach space E and let R be a retraction from E onto C. Then, the following are equivalent:*

(a) R is sunny and generalized nonexpansive;
(b) $\langle x - Rx, Jy - JRx\rangle \leq 0$ for all $(x, y) \in E \times C$.

Lemma 2.4 ([23])**.** *Let C be a nonempty closed sunny and generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then, the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.5 ([23])**.** *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then, the following hold:*

(a) $z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$;
(b) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z \in C : \|z - x\| = \min_{y \in C} \|y - x\|\}$$

is always nonempty and a singleton. Let us define the mapping P_C of E onto C by $z = P_C x$ for every $x \in E$, i.e.,

$$\|P_C x - x\| = \min_{y \in C} \|y - x\|$$

for every $x \in E$. Such P_C is called the metric projection of E onto C; see [43]. The following lemma is in [43].

Lemma 2.6 ([43])**.** *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then, $z = P_C x$ if and only if $\langle y - z, J(x - z)\rangle \leq 0$ for all $y \in C$.*

An operator $A \subset E \times E^*$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $R(A) = \cup\{Ax : x \in D(A)\}$ is said to be monotone if $\langle x - y, x^* - y^*\rangle \geq 0$ for any $(x, x^*), (y, y^*) \in A$. An operator A is said to be strictly monotone if $\langle x - y, x^* - y^*\rangle > 0$ for any $(x, x^*), (y, y^*) \in A$ $(x \neq y)$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, x^*) : x^* \in Ax\}$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then the set $A^{-1}0 = \{u \in E : 0 \in Au\}$ is closed and convex (see [44] for more details). Let J be the normalized duality mapping from E into E^*. Then, J is monotone. If E is strictly convex, then J is one to one and strictly monotone. The following theorem is well-known; for instance, see [43].

Theorem 2.1. *Let E be a reflexive, strictly convex and smooth Banach space and let $A\colon E \to 2^{E^*}$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$. Further, if $R(J + A) = E^*$, then $R(J + rA) = E^*$ for all $r > 0$.*

3. Generalized conditional expectations

In this section, we discuss sunny generalized nonexpansive retractions which are connected with conditional expectations in the probability theory. We start with two theorems proved by Kohsaka and Takahashi [34].

Theorem 3.1 ([34])**.** *Let E be a smooth, strictly convex and reflexive Banach space, let C^* be a nonempty closed convex subset of E^* and let Π_{C^*} be the generalized projection of E^* onto C^*. Then the mapping R defined by $R = J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C^*$.*

Theorem 3.2 ([34])**.** *Let E be a smooth, reflexive and strictly convex Banach space and let D be a nonempty subset of E. Then, the following conditions are equivalent.*

(1) D is a sunny generalized nonexpansive retract of E;
(2) D is a generalized nonexpansive retract of E;
(3) JD is closed and convex.

In this case, D is closed.

Motivated by these theorems, the authors defined the following nonlinear operator: Let E be a reflexive, strictly convex and smooth Banach space and let J be the normalized duality mapping from E onto E^*. Let Y^* be a closed linear subspace of the dual space E^* of E. Then, the generalized conditional expectation E_{Y^*} with respect to Y^* is defined as follows:

$$E_{Y^*} := J^{-1}\Pi_{Y^*}J,$$

where Π_{Y^*} is the generalized projection from E^* onto Y^*. Such generalized conditional expectations are deeply connected with conditional expectations in the probability theory; see [21].

Let Y be a nonempty subset of a Banach space E and let Y^* be a nonempty subset of the dual space E^*. Then, we define the annihilator $Y^*_\perp$ of Y^* and the annihilator $Y^\perp$ of Y as follows:

$$Y^*_\perp = \{x \in E : f(x) = 0 \text{ for all } f \in Y^*\}$$

and

$$Y^\perp = \{f \in E^* : f(x) = 0 \text{ for all } x \in Y\}.$$

Theorem 3.3 ([2, 20])**.** *Let E be a reflexive, strictly convex and smooth Banach space and let I be the identity operator of E into itself. Let Y^* be a closed linear subspace of the dual space E^* and let E_{Y^*} be the generalized conditional expectation with respect to Y^*. Then, the mapping $I - E_{Y^*}$ is the metric projection of E onto $Y^*_\perp$. Conversely, let Y be a closed linear subspace of E and let P_Y be the metric projection of E onto Y. Then, the mapping $I - P_Y$ is the generalized conditional expectation $E_{Y^\perp}$ with respect to $Y^\perp$, i.e., $I - P_Y = E_{Y^\perp}$.*

In general, we know from Deutsch [13, 14] that the metric projection is not linear.

Let E be a normed linear space and let $x, y \in E$. We say that x is orthogonal to y in the sense of Birkhoff-James (or simply, x is BJ-orthogonal to y), denoted by $x \perp y$ if

$$\|x\| \le \|x + \lambda y\|$$

for all $\lambda \in \mathbf{R}$; see [6, 28, 29, 30]. We know that for $x, y \in E$, $x \perp y$ if and only if there exists $f \in J(x)$ with $\langle y, f\rangle = 0$; see [43]. In general, $x \perp y$ does not imply $y \perp x$. An operator T of E into itself is called left-orthogonal (resp. right-orthogonal) if for each $x \in E$, $Tx \perp (x - Tx)$ (resp. $(x - Tx) \perp Tx$).

Let E be a normed linear space and let $Y_1, Y_2 \subset E$ be closed linear subspaces. If $Y_1 \cap Y_2 = \{0\}$ and for any $x \in E$ there exists a unique pair $y_1 \in Y_1, y_2 \in Y_2$ such that

$$x = y_1 + y_2,$$

and any element of Y_1 is BJ-orthogonal to any element of Y_2, i.e., $y_1 \perp y_2$ for any $y_1 \in Y_1, y_2 \in Y_2$, then we represent the space E as

$$E = Y_1 \oplus Y_2 \text{ and } Y_1 \perp Y_2.$$

For an operator T of E into itself, the kernel of T is denoted by $\ker(T)$, i.e.,

$$\ker(T) = \{x \in E : Tx = 0\}.$$

We also know the following theorem for generalized conditional expectations in a Banach space.

Theorem 3.4 ([20])**.** *Let E be a strictly convex, reflexive and smooth Banach space and let Y^* be a closed linear subspace of the dual space E^* of E such that for any $y_1, y_2 \in J^{-1}Y^*, y_1 + y_2 \in J^{-1}Y^*$. Then, $J^{-1}Y^*$ is a closed linear subspace of E and the generalized conditional expectation E_{Y^*} with respect to Y^* is a norm one linear projection from E to $J^{-1}Y^*$. Further, the following hold:*

(1) $E = J^{-1}Y^* \oplus \ker(E_{Y^*})$ and $J^{-1}Y^* \perp \ker(E_{Y^*})$;
(2) the operator $I - E_{Y^*}$ is the metric projection onto $\ker(E_{Y^*})$.

In general, a nonzero bounded linear projection on a Banach space has a norm which is more than or equal to 1. So, a norm one linear projection plays an important role in functional analysis; see [36, 37, 41]. Now using nonlinear functional analytic methods, we derive the following two representation theorems for norm one linear projections; see also [5, 9].

Theorem 3.5. *Let E be a strictly convex, reflexive and smooth Banach space. Any norm one linear projection P of E into itself with $Y = \{Px : x \in E\}$ can be represented as the generalized conditional expectation E_{JY} with respect to JY, where J is the normalized duality mapping of E.*

Proof. Let P be a linear projection of E into itself with $\|P\| = 1$. Then, the subsets $X = \{x \in E : Px = 0\}$ and $Y = \{Px \in E : x \in E\}$ are closed linear subspaces of E. In fact, since the operators P and $Q = I - P$ are bounded linear projections, we have that X and $Y = \{x \in E : Qx = 0\}$ are closed. Let P^* be the adjoint operator of P, i.e., $P^* : E^* \to E^*$ is a bounded linear operator defined by $\langle Px, x^* \rangle = \langle x, P^*x^* \rangle$ for any $x \in E, x^* \in E^*$. P^* is a linear projection on E^* and $\|P\| = \|P^*\| = 1$. In fact, since $\langle x, (P^*)^2x^* \rangle = \langle Px, P^*x^* \rangle = \langle P^2x, x^* \rangle = \langle Px, x^* \rangle = \langle x, P^*x^* \rangle$, we have that P^* is a linear projection on E^*.

From this, for any $x \in Y$ we have

$$\|P^*Jx\| \le \|Jx\| = \|x\|.$$

Further, for any $x \in Y$ we have

$$\begin{aligned} &\langle x, P^*Jx \rangle = \langle Px, Jx \rangle = \langle x, Jx \rangle = \|x\|^2 \\ &\Rightarrow \langle x, P^*Jx \rangle = \|x\|^2 \le \|x\|\|P^*Jx\| \\ &\Rightarrow \|x\| \le \|P^*Jx\|. \end{aligned}$$

Then, we obtain $\|P^*Jx\| = \|x\|$ and $\langle x, P^*Jx \rangle = \|x\|^2$. From the uniqueness of the normalized duality mapping, we have $P^*Jx = Jx$. The set $Y^* = \{P^*x^* \in E^* : x^* \in E^*\}$ satisfies that $JY \subset Y^*$. Since E is reflexive, we have $P^{**} = P$. So, $J_* Y^* \subset Y$. Then, we obtain that

$$JY = Y^*.$$

JY is a closed linear subspace of E^*.

From Theorem 3.4, the generalized conditional expectation E_{JY} with respect to JY is a norm one linear projection of E onto $Y \subset E$. Further, for any $x \in Y = P(E) = E_{JY}(E)$ we have $Px = E_{JY}x = x$. Setting $Y_1^* = \{E_{JY}^* x^* \in E^* : x \in E^*\}$, we have from above arguments that for any $x^* \in JY$ we have $P^* x^* = x^*$ and $E_{JY}^* x^* = x^*$. For any $x \in E$ we have

$$\begin{aligned}
\|E_{JY}x - Px\|^2 &= \langle J(E_{JY}x - Px), E_{JY}x - Px\rangle \\
&= \langle J(E_{JY}x - Px), E_{JY}x\rangle - \langle J(E_{JY}x - Px), Px\rangle \\
&= \langle E_{JY}^* J(E_{JY}x - Px), x\rangle - \langle P^* J(E_{JY}x - Px), x\rangle \\
&= \langle J(E_{JY}x - Px), x\rangle - \langle J(E_{JY}x - Px), x\rangle = 0
\end{aligned}$$

So, we obtain $P = E_{JY}$. □

Let E be a Banach space and let C be a nonempty closed convex subset of E. Then, a mapping T of C into itself with $F(T) \neq \emptyset$ is said to be quasi-nonexpansive if $\|Tx - m\| \leq \|x - m\|$ for all $m \in F(T)$ and $x \in C$.

Theorem 3.6. *Let E be a strictly convex, reflexive and smooth Banach space and let Y^* be a closed linear subspace of the dual space E^* of E. If a projection P of E onto $J^{-1}Y^*$ is quasi-nonexpansive, that is, $\|Px - m\| \leq \|x - m\|$ for all $m \in J^{-1}Y^*$ and $x \in E$, then P is the generalized conditional expectation E_{Y^*} with respect to Y^*. Furthermore, P is a norm one linear projection.*

Proof. Let P be a projection of E onto $J^{-1}Y^*$ satisfying $\|Px - m\| \leq \|x - m\|$ for all $m \in J^{-1}Y^*$ and $x \in E$. Since Y^* is a closed linear subspace of E^* and $J^{-1}\alpha x = \alpha J^{-1}x$ for all $x \in E$ and $\alpha \in \mathbf{R}$, we have that for all $\alpha \in \mathbf{R}$ with $\alpha \neq 0$,

$$x \in J^{-1}Y^* \Leftrightarrow \alpha x \in J^{-1}Y^*.$$

Fix $x \in E$ and $m \in J^{-1}Y^*$ such that $x \notin J^{-1}Y^*$ and $m \neq 0$. For any $k > 0$, we have that $\dfrac{x}{k} - m \neq 0$. So, we have from the Hahn-Banach theorem that there exists $\xi_k \in E^*$ such that $\left\langle \dfrac{x}{k} - m, \xi_k \right\rangle = \left\| \dfrac{x}{k} - m \right\|$ and $\|\xi_k\| = 1$. Then, we have that

$$\begin{aligned}
\left\langle \frac{Px}{k} - m, \xi_k \right\rangle &\leq \left\| \frac{Px}{k} - m \right\| = \frac{1}{k}\|Px - km\| \\
&\leq \frac{1}{k}\|x - km\| = \left\| \frac{x}{k} - m \right\| \\
&= \left\langle \frac{x}{k} - m, \xi_k \right\rangle.
\end{aligned}$$

So,

$$\langle x - Px, \xi_k \rangle \geq 0.$$

We also have from the Hahn-Banach theorem that there exists ξ_m of E^* such that $\langle m, \xi_m \rangle = \|m\|$ and $\|\xi_m\| = 1$. Since the norm of E^* is strictly convex, such ξ_m is uniquely determined. In fact, if $\eta \neq \xi_m$ satisfies above properties, then $\left\| \dfrac{\eta + \xi_m}{2} \right\| < 1$ and

$$\|m\| \left\| \frac{\eta + \xi_m}{2} \right\| \geq \left\langle m, \frac{\eta + \xi_m}{2} \right\rangle = \|m\|.$$

This is a contradiction. When k tends to infinity, $\dfrac{x}{k} - m$ converges to $-m$ strongly. Further, ξ_k converges to $-\xi_m$ in weak* topology. In fact, let $k_n > 0$ for all $n \in \mathbf{N}$ and $k_n \to \infty$. Then $x_n = \dfrac{x}{k_n} - m$ converges to $-m$. Since $\{\xi_n\} = \{\xi_{k_n}\}$ is bounded, there exists a subnet $\{\xi_{n_\alpha}\}$ of $\{\xi_n\}$ converging to some $\xi \in E^*$ in weak* topology. We may show $\xi = -\xi_m$. Since the norm of E^* is lower semicontinuous in the weak* topology, we have

$$\|\xi\| \le \liminf_{\alpha} \|\xi_{n_\alpha}\| = 1.$$

On the other hand, we have that

$$\begin{aligned} |\langle -m, \xi\rangle - \|x_{n_\alpha}\|| &= |\langle -m, \xi\rangle - \langle x_{n_\alpha}, \xi_{n_\alpha}\rangle| \\ &\le |\langle -m, \xi - \xi_{n_\alpha}\rangle| + |\langle -m - x_{n_\alpha}, \xi_{n_\alpha}\rangle|. \end{aligned}$$

Since $\langle -m, \xi - \xi_{n_\alpha}\rangle \to 0$ and $\langle -m - x_{n_\alpha}, \xi_{n_\alpha}\rangle \to 0$, we have

$$\|x_{n_\alpha}\| \to -\langle m, \xi\rangle = \langle m, -\xi\rangle.$$

Since $\|x_{n_\alpha}\| \to \|m\|$, we have $\langle m, -\xi\rangle = \|m\|$. So we have

$$\|m\| = \langle m, -\xi\rangle \le \|m\|\|\xi\|$$

and hence $\|\xi\| \ge 1$. Therefore, we have $\|\xi\| = 1$, $\langle m, -\xi\rangle = \|m\|$ and $\xi = -\xi_m$. Any weak* convergent subnet of $\{\xi_n\}$ converges to $-\xi_m$ in weak* topology. Then, we have that ξ_k converges to $-\xi_m$ in weak* topology as $k \to \infty$. So, we obtain

$$\langle x - Px, \xi_m\rangle \le 0.$$

We know that $Jm = \|m\|\xi_m$. Then we have that $\langle x - Px, Jm\rangle \le 0$ for any $m \in J^{-1}Y^*$ with $m \ne 0$. Since Y^* is a closed linear subspace of E^*, we have that

$$\langle x - Px, y^*\rangle = 0$$

for any $y^* \in Y^*$. We also know that for any x, $Px \in J^{-1}Y^*$. That $Px \in J^{-1}Y^*$ and $\langle x - Px, y^*\rangle = 0$ for any $y^* \in Y^*$ imply that $JPx \in Y^*$ and $\langle J^{-1}Jx - J^{-1}JPx, y^*\rangle = 0$ for any $y^* \in Y^*$. From the definition of Π_{Y^*}, we have $JPx = \Pi_{Y^*}Jx$. So, we obtain $Px = J^{-1}\Pi_{Y^*}Jx$. This implies $Px = E_{Y^*}x$. So, we have $P = E_{Y^*}$. We also have that the range of P is convex. In fact, let $x, y \in P(E) = J^{-1}Y^*$ and $0 \le \alpha \le 1$. Putting $z = \alpha x + (1-\alpha)y$, we have $\|Pz - x\| \le \|z - x\|$ and $\|Pz - y\| \le \|z - y\|$. Hence, we have that

$$\|x - y\| \le \|x - Pz\| + \|Pz - y\| \le \|x - z\| + \|z - y\| = \|x - y\|.$$

This implies that $\|x - z\| = \|x - Pz\|$ and $\|y - z\| = \|y - Pz\|$. Since E is strictly convex, we have $z = Pz$. Therefore, $P(E) = J^{-1}Y^*$ is convex. So, for any $y_1, y_2 \in J^{-1}Y^*$, we have $\dfrac{y_1 + y_2}{2} \in J^{-1}Y^*$. From $\dfrac{y_1 + y_2}{2} \in J^{-1}Y^*$, we have $y_1 + y_2 \in J^{-1}\frac{1}{2}Y^* = J^{-1}Y^*$. Then, for any $y_1, y_2 \in J^{-1}Y^*$, we have $y_1 + y_2 \in J^{-1}Y^*$. So, we have from Theorem 3.4 that $J^{-1}Y^*$ is a closed linear subspace of E. Further, from Theorem 3.4, the mapping $P = E_{Y^*}$ is a norm one linear projection. □

Using Theorems 3.5 and 3.6, we obtain the following corollary.

Corollary 3.1. *Let E be a strictly convex, reflexive and smooth Banach space and let Y^* be a closed linear subspace of the dual space E^* of E. If the generalized conditional expectation E_{Y^*} is a quasi-nonexpansive projection of E onto $J^{-1}Y^*$, then it is a norm one linear projection and $J^{-1}Y^*$ is a closed linear subspace in E. Conversely, any norm one linear projection is a quasi-nonexpansive generalized conditional expectation.*

Proof. If the generalized conditional expectation E_{Y^*} is also a quasi-nonexpansive projection of E onto $J^{-1}Y^*$, from Theorem 3.6 it is a norm one linear projection.

Conversely, from Theorem 3.5, any norm one linear projection is both a generalized conditional expectation and a quasi-nonexpansive projection. □

4. Ando's theorem

Let $(\Omega, \mathfrak{A}, \mu)$ be a probability space. For any p with $1 \leq p < \infty$, let $L^p(\mathfrak{A})$ be the space of real valued measurable functions such that $\int_\Omega |x(\omega)|^p d\mu < \infty$. For $p = \infty$ we denote by $L^\infty(\mathfrak{A})$ the space of real valued measurable functions such that $\operatorname{ess\,sup}|x(\omega)| < \infty$. For any p with $1 \leq p < \infty$, $L^\infty(\mathfrak{A})$ is a subspace of $L^p(\mathfrak{A})$. For any p with $1 \leq p < \infty$, the space $E = L^p(\mathfrak{A})$ with $\|x\| = (\int_\Omega |x(\omega)|^p d\mu)^{1/p}$ is a Banach space. Further, we know that the space $L^p(\mathfrak{A})$ with $1 < p < \infty$ is uniformly convex and uniformly smooth. The dual pair of $E = L^p(\mathfrak{A})$ with $1 < p < \infty$ is described as follows: $L^p(\mathfrak{A})^* = L^q(\mathfrak{A})$ in which q is the conjugate exponent of p, i.e., q satisfies $\frac{1}{p} + \frac{1}{q} = 1$. The duality is given as $\langle x, x^* \rangle = \int_\Omega x^*(\omega)x(\omega)d\mu$ for $x \in L^p(\mathfrak{A})$ and $x^* \in L^q(\mathfrak{A})$. For any p, q with $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we denote by J_p the normalized duality mapping of $L^p(\mathfrak{A})$. Then, we know that for any $x \in L^p(\mathfrak{A})$ with $x \neq 0$,

$$J_p x = \|x\|_p^{2-p} |x|^{p-2} x$$

and J_p: $L^p(\mathfrak{A}) \to L^q(\mathfrak{A})$ is one to one and onto. Further, we know that $J_p^{-1} = J_q : L^q(\mathfrak{A}) \to L^p(\mathfrak{A})$.

For p with $1 \leq p \leq \infty$ and $A \in \mathfrak{A}$, we define a linear projection $1_A : L^p(\mathfrak{A}) \to L^p(\mathfrak{A})$ as follows: For any $x = x(\omega) \in L^p(\mathfrak{A})$,

$$1_A x = 1_A x(\omega) = \begin{cases} x(\omega), & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A. \end{cases}$$

And a family of subsets $\mathfrak{R}$ is called a ring if and only if $\emptyset \in \mathfrak{R}$ and for all A and B in $\mathfrak{R}$, we have $A \cup B \in \mathfrak{R}$ and $B \backslash A \in \mathfrak{R}$. A ring $\mathfrak{R}$ is called a σ ring if and only if any countable union of sets in $\mathfrak{R}$ is in $\mathfrak{R}$.

In 1966, Ando [3] showed that for $1 < p < \infty$ with $p \neq 2$, all norm one linear projections on L^p are similar to conditional expectations; see also [18]. Using our representation theorem (Theorem 3.5), we shall derive Ando's theorem ([35, p. 160]). Before deriving it, we need the following lemmas.

Lemma 4.1 ([35])**.** *Let P be a norm one linear projection on $E = L^p(\mathfrak{A})$ with $1 < p < \infty$ and $p \neq 2$. If $y \in P(E)$, then*

$$1_{supp\{y\}} \circ P = P \circ 1_{supp\{y\}},$$

where $supp\{y\} = \{\omega \in \Omega : y = y(\omega) \neq 0\} \in \mathfrak{A}$.

Lemma 4.2 ([35])**.** *Suppose $1 < p < \infty$ and $p \neq 2$, and let P be a norm one linear projection on $L^p(\mathfrak{A})$. Define $\mathfrak{F}_\circ$ to be the set of supports of all functions whose equivalence classes are in $P(E)$. Then, the following hold:*

(1) $\mathfrak{F}_\circ$ is a sub-σ ring of $\mathfrak{A}$;
(2) for fixed $y \in P(E)$, $y^{-1} \cdot Px$ is $\mathfrak{F}_\circ$ measurable for any $x \in E$ such that $supp\{x\} \subset supp\{y\}$.

If $y \in L^p(\mathfrak{A}, \mu)$, then the measure $|y|^p d\mu$ restricted to any sub-ring $\mathfrak{F}_\mathfrak{o}$ of $\mathfrak{A}$ is finite. By the Radon-Nikodym theorem we may define the conditional expectation $E_{\mathfrak{F}_\mathfrak{o},|y|^p}$ for the measure $|y|^p d\mu$ relative to sub-ring $\mathfrak{F}$. $E_{\mathfrak{F}_\mathfrak{o},|y|^p}$ is uniquely determined by the equation

$$\int_A z \cdot |y|^p d\mu = \int_A \left(E_{\mathfrak{F}_\mathfrak{o},|y|^p} z\right) |y|^p d\mu \qquad (A \in \mathfrak{F}_\mathfrak{o}) \tag{4.1}$$

for $z \in L^1(\Omega, \mathfrak{A}, |y|^p d\mu)$, and the condition that $E_{\mathfrak{F}_\mathfrak{o},|y|^p} z$ is $\mathfrak{F}_\mathfrak{o}$ measurable; see [35, p. 158–159]. Using these lemmas, we can obtain Ando's theorem.

Theorem 4.1 ([35])**.** *Suppose $1 < p < \infty$ with $p \neq 2$ and that P is a norm one linear projection on $E = L^p(\mathfrak{A}, \mu)$. If $y \in P(E)$ and $x \in E$ such that $supp\{x\} \subset supp\{y\}$, then*

$$Px = yE_{\mathfrak{F}_\mathfrak{o},|y|^p}(x \cdot y^{-1}).$$

Proof. We may consider that $\|y\| = 1$. From Theorem 3.3 and the same argument in the proof of Theorem 3.6, we have for any $x \in E$,

$$\langle x - Px, Jy \rangle = 0.$$

Since for any $z_1, z_2 \in E$, $\int_\Omega z_1 \cdot Jz_2 d < \infty$ and from Lemma 4.1 we have $supp\{Px\} \subset supp\{y\}$, we have that for any $x \in E$ such that $supp\{x\} \subset supp\{y\}$ and $A \in \mathfrak{F}_\mathfrak{o}$,

$$\begin{aligned}
&\langle 1_A x - P \circ 1_A x, Jy \rangle = 0 \\
\Rightarrow &\langle 1_A x - 1_A \circ Px, Jy \rangle = 0 \\
\Rightarrow &\int_\Omega 1_A x \cdot Jy d\mu = \int_\Omega 1_A \circ Px \cdot Jy d\mu \\
\Rightarrow &\int_A x \cdot Jy d\mu = \int_A Px \cdot Jy d\mu \\
\Rightarrow &\int_A x \cdot |y|^p y^{-1} d\mu = \int_A Px \cdot |y|^p y^{-1} d\mu \\
\Rightarrow &\int_A (x \cdot y^{-1}) |y|^p d\mu = \int_A (Px \cdot y^{-1}) |y|^p d\mu < \infty.
\end{aligned}$$

Since $Px \cdot y^{-1}$ is $\mathfrak{F}_\mathfrak{o}$ measurable, from the uniqueness of $E_{\mathfrak{F}_\mathfrak{o},|y|^p}$ we have

$$Px \cdot y^{-1} = E_{\mathfrak{F}_\mathfrak{o},|y|^p}(x \cdot y^{-1}).$$

Since $supp\{Px\} \subset supp\{y\}$, we obtain

$$Px = yE_{\mathfrak{F}_\mathfrak{o},|y|^p}(x \cdot y^{-1}).$$

□

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(T. Honda) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan

E-mail address: `honda4@is.titech.ac.jp`

(W. Takahashi) Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan

E-mail address: `wataru@is.titech.ac.jp`