OPERATOR INEQUALITIES RELATED TO ANDO-HIAI INEQUALITY

MASATOSHI FUJII, MASATOSHI ITO, EIZABURO KAMEI AND AKEMI MATSUMOTO

Received February 6, 2009

Abstract. In this paper, firstly we shall show the equivalence relation between Ando-Hiai inequality "For $A, B > 0$, $A \sharp_\alpha B \le I$ ensures $A^r \sharp_\alpha B^r \le I$ for $r \ge 1$" and the inequalty "$A \ge B \ge 0$ ensures $A^{-r} \sharp_{\frac{r}{p+r}} B^p \le I$ for $p \ge 0$ and $r \ge 0$." Next we shall show a complementary result of Ando-Hiai inequality: If $A \sharp_\alpha B \le I$, then $A^r \sharp_\alpha B^r \le A \sharp_\alpha B$ for $0 \le r \le 1$.

1. Introduction

In this paper, a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. An operator T is said to be positive (denoted by $T \ge 0$) if $(Tx, x) \ge 0$ for all $x \in \mathcal{H}$, and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. The operator order $A \ge B$ among selfadjoint operators is naturally defined by $A - B \ge 0$. Now the most important order preserving inequality is the Löwner-Heinz inequality:

(LH) $\quad A \ge B \ge 0$ ensures $A^\alpha \ge B^\alpha$ for any $\alpha \in [0, 1]$.

Furuta inequality established in 1987 is an epoch-making extension of (LH):

Furuta inequality [7]: *If $A \ge B \ge 0$, then for each $r \ge 0$,*

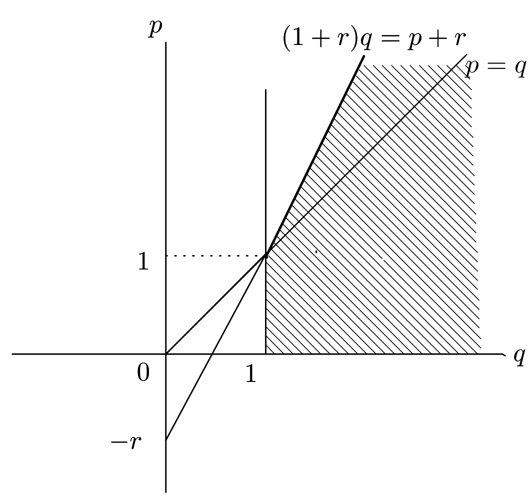
(i) $\quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \ge (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$

and

(ii) $\quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \ge (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$

hold for p and q such that $p \ge 0$ and $q \ge 1$ with

(*) $\quad (1 + r)q \ge p + r.$



Furuta inequality formally includes (LH) by putting $r = 0$ in (i) or (ii) in above. We remark that alternative proofs of it were given in [2] and [10] and also an elementary one page proof in [8]. Tanahashi [11] showed that the domain determined by (*) for p, q and r is the best possible one for Furuta inequality.

2000 *Mathematics Subject Classification.* Primary 47A63, 47A64.
Key words and phrases. Positive operators, operator mean, Ando-Hiai inequality and Furuta inequality.

As stated in [10], Furuta inequality can be arranged by using notion of α-power mean $\sharp_\alpha$ for $\alpha \in [0,1]$ introduced by Kubo-Ando as follows:

$$A \sharp_\alpha B = A^{\frac{1}{2}}(A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha A^{\frac{1}{2}}$$

for $A > 0$ and $B \geq 0$.

Theorem F. *If $A \geq B > 0$, then*

(F) $$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \leq B \leq A$$

holds for $p \geq 1$ and $r \geq 0$.

In our previous papers [6] and [5], the following result is the essence of (F), and also (C) is known as a characterization of chaotic order, that is, $\log A \geq \log B$ (see [3][4][9][12]).

Theorem A ([6])**.** *If $A \geq B > 0$, then*

(C) $$A^{-r} \sharp_{\frac{r}{p+r}} B^p \leq I$$

holds for $p \geq 0$ and $r \geq 0$.

On the other hand, Ando and Hiai [1] have showed the following inequality.

Theorem B ([1])**.** *If $A \sharp_\alpha B \leq I$ for $\alpha \in (0,1)$, then*

(AH) $$A^r \sharp_\alpha B^r \leq I$$

holds for $r \geq 1$.

By Theorem B, they obtained that for $A, B > 0$,

(AH') $A^{-1} \sharp_{\frac{1}{p}} A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}} \leq I$ implies $A^{-r} \sharp_{\frac{1}{p}} (A^{\frac{-1}{2}} B^p A^{\frac{-1}{2}})^r \leq I$ for $p \geq 1$ and $r \geq 1$.

We remark that (AH') is equivalent to the main result of log majorization. In [5], an extension of Theorem B is obtained as follows:

Theorem C ([5])**.** *If $A \sharp_\alpha B \leq I$ for $\alpha \in (0,1)$, then*

(GAH) $$A^r \sharp_{\frac{\alpha r}{(1-\alpha)s+\alpha r}} B^s \leq I$$

holds for $s \geq 1$ and $r \geq 1$.

In this paper, based on the idea of Theorem C, we shall show the equivalence relation between Theorem A and Theorem B. Next we shall show a complementary inequality related to Theorem A and Theorem B.

2. Main results

Firstly we shall show the equivalence relation between Theorem A and Theorem B.

Theorem 1. *Theorem A is equivalent to Theorem B.*

Proof of Theorem 1. Suppose that Theorem A holds and that $A \sharp_\alpha B \le I$. We put $p = \frac{1}{\alpha} > 1$. Then the assumption $A \sharp_\alpha B \le I$ says that

$$B_1 = (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha \le A^{-1} = A_1.$$

Applying Theorem A to $A_1 \ge B_1$, we have

$$A_1^{-r} \sharp_{\frac{r}{p+r}} B_1^p \le I \quad \text{for } r \ge 0.$$

Moreover it follows that for $p \ge 1$ and $r \ge 0$,

$$\begin{aligned} & A_1^{-r} \sharp_{\frac{1+r}{p+r}} B_1^p = B_1^p \sharp_{\frac{p-1}{p+r}} A_1^{-r} = B_1^p \sharp_{\frac{p-1}{p}} (B_1^p \sharp_{\frac{p}{p+r}} A_1^{-r}) \\ =& B_1^p \sharp_{\frac{p-1}{p}} (A_1^{-r} \sharp_{\frac{r}{p+r}} B_1^p) \le B_1^p \sharp_{\frac{p-1}{p}} I = B_1 \le A_1. \end{aligned}$$

Summing up the above discussion, for each $p > 1$,

$$A \sharp_{\frac{1}{p}} B \le I \;\Rightarrow\; A^r \sharp_{\frac{1+r}{p+r}} A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \le A^{-1}, \text{ or } A^{r+1} \sharp_{\frac{1+r}{p+r}} B \le I \text{ for } r \ge 0.$$

Noting that

$$B \sharp_{\frac{p-1}{p+r}} A^{r+1} = A^{r+1} \sharp_{\frac{1+r}{p+r}} B \le I,$$

we apply it for $p_1 = \frac{p+r}{p-1}$ in the following way;

$$I \ge B^{r+1} \sharp_{\frac{1+r}{p_1+r}} A^{r+1} = A^{r+1} \sharp_{\frac{1}{p}} B^{r+1}$$

by $1 - \frac{1+r}{p_1+r} = \frac{1}{p}$. Namely we obtain Theorem B.

The reverse implication: B $\Rightarrow$ A has been already shown in [5]. But we cite it for the sake of convenience: It suffices to show that (C) holds for $p, r > 1$ under the assumption $A \ge B > 0$ because it holds for $0 \le p, r \le 1$ by (LH). So we take arbitrary $p, r > 1$, and put $\alpha = \frac{r}{p+r}$ and $q = \max\{p, r\}$. Then, as noted in above, if $A \ge B > 0$, then (C) holds for $p_1 = \frac{p}{q}$ and $r_1 = \frac{r}{q}$, i.e.,

$$A^{-r_1} \sharp_{\frac{r_1}{p_1+r_1}} B^{p_1} \le I.$$

We here apply (AH) to this, that is, we have

$$I \ge A^{-r_1 q} \sharp_{\frac{r_1 q}{p_1 q + r_1 q}} B^{p_1 q} = A^{-r} \sharp_{\frac{r}{p+r}} B^p,$$

as desired. □

Next we shall show a complement related to Theorem C.

Theorem 2. *For $A, B > 0$ and $\alpha \in [0,1]$, if $A \sharp_\alpha B \le I$, then*

$$A \sharp_\alpha B \le A^\mu \sharp_{\frac{\alpha\mu}{(1-\alpha)\lambda + \alpha\mu}} B^\lambda$$

for $\mu \in [0,1]$ and $\lambda \in [0,1]$.

Proof of Theorem 2. Put $C = A^{\frac{-1}{2}} B A^{\frac{-1}{2}}$. Then $A \sharp_\alpha B \le I$ if and only if

$$A^{-1} \ge (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^\alpha = C^\alpha. \tag{2.1}$$

By (2.1) and Löwner-Heinz theorem, we have

$$\begin{aligned} A^\mu \sharp_{\frac{\alpha\mu}{(1-\alpha)\lambda+\alpha\mu}} B^\lambda &= A^{\frac{1}{2}}(A^{-(1-\mu)} \sharp_{\frac{\alpha\mu}{(1-\alpha)\lambda+\alpha\mu}} (A^{\frac{-1}{2}} B^\lambda A^{\frac{-1}{2}}))A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}(A^{-(1-\mu)} \sharp_{\frac{\alpha\mu}{(1-\alpha)\lambda+\alpha\mu}} (A^{-1} \sharp_\lambda C))A^{\frac{1}{2}} \\ &\ge A^{\frac{1}{2}}(C^{\alpha(1-\mu)} \sharp_{\frac{\alpha\mu}{(1-\alpha)\lambda+\alpha\mu}} (C^\alpha \sharp_\lambda C))A^{\frac{1}{2}} = A^{\frac{1}{2}} C^\alpha A^{\frac{1}{2}} = A \sharp_\alpha B \end{aligned}$$

for $\mu \in [0,1]$ and $\lambda \in [0,1]$. □

Corollary 3. *For $A, B > 0$ and $p > 0$, $r > 0$, if $A^{-r} \sharp_{\frac{r}{p+r}} B^p \le I$, then*

$$A^{-r} \sharp_{\frac{r}{p+r}} B^p \le A^{-t} \sharp_{\frac{t}{s+t}} B^s$$

for $s \in [0,p]$ and $t \in [0,r]$.

Proof of Corollary 3. Put $\lambda = \frac{s}{p}$, $\mu = \frac{t}{r}$ and $\alpha = \frac{r}{p+r}$. By replacing A with A^{-r} and B with B^p in Theorem 2, we can easily obtain Corollary 3. □

References

[1] T.Ando and F.Hiai, *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl., **197, 198** (1994), 113–131.

[2] M.Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator Theory, **23** (1990), 67–72.

[3] M.Fujii, T.Furuta and E.Kamei, *Furuta's inequality and its application to Ando's theorem*, Linear Algebra Appl., **179** (1993), 161–169.

[4] M.Fujii, J.F.Jiang and E.Kamei, *Characterization of chaotic order and its application to Furuta inequality*, Proc. Amer. Math. Soc., **125** (1997), 3655–3658.

[5] M.Fujii and E.Kamei, *Ando-Hiai inequality and Furuta inequality*, Linear Algebra Appl., **416** (2006), 541–545.

[6] M.Fujii, E.Kamei and R.Nakamoto, *An analysis on the internal structure of the celebrated Furuta inequality via operator mean*, Sci. Math. Jpn., **62** (2005), 421–427.

[7] T.Furuta, *$A \ge B \ge 0$ assures $(B^r A^p B^r)^{1/q} \ge B^{(p+2r)/q}$ for $r \ge 0$, $p \ge 0$, $q \ge 1$ with $(1+2r)q \ge p+2r$*, Proc. Amer. Math. Soc., **101** (1987), 85–88.

[8] T.Furuta, *An elementary proof of an order preserving inequality*, Proc. Japan Acad. Ser. A Math. Sci., **65** (1989), 126.

[9] T.Furuta, *Applications of order preserving operator inequalities*, Oper. Theory Adv. Appl., **59** (1992), 180–190.

[10] E.Kamei, *A satellite to Furuta's inequality*, Math. Japon., **33** (1988), 883–886.

[11] K.Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 141–146.

[12] M.Uchiyama, *Some exponential operator inequalities*, Math. Inequal. Appl., **2** (1999), 469–471.

(Masatoshi Fujii) Deaprtment of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, JAPAN

E-mail address: mfujii@cc.osaka-kyoiku.ac.jp

(Masatoshi Ito) Maebashi Institute of Technology, 460-1 Kamisadorimachi, Maebashi, Gunma 371-0816, JAPAN

E-mail address: m-ito@maebashi-it.ac.jp

(Eizaburo Kamei) Maebashi Institute of Technology, 460-1 Kamisadorimachi, Maebashi, Gunma 371-0816, JAPAN

E-mail address: kamei@maebashi-it.ac.jp

(Akemi Matsumoto) Nose Senior Highschool, Nose, Osaka 653-0122, JAPAN

E-mail address: m@nose.osaka-c.ed.jp