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A COMMON IMPROVEMENT OF FUZZY TOPOLOGICAL SPACES AND FUZZY (QUASI) UNIFORM SPACES

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Dedicated to my friend and colleague Horst Herrlich
on the occasion of his 70th birthday

Abstract. Using fuzzy filters in the sense of P. Eklund and W. Gähler [2], it turns out that fuzzy preuniform convergence spaces introduced in [11] form a strong topological universe in which fuzzy topological spaces as well as fuzzy (quasi) uniform spaces can be studied. Thus, better tools such as the existence of natural function spaces, the existence of one-point extensions (and consequently, the hereditariness of quotient maps), and the productivity of quotient maps are available.

0 Introduction. In 1968 fuzzy topological spaces have been introduced by C.L. Chang [1]. Together with the fuzzy continuous maps between them they form a topological construct provided that in the definition of a fuzzy topological space X the requirement is incorporated that all fuzzy subsets of X given by constant maps are fuzzy open. This has been pointed out by R. Lowen [8]. Concerning fuzzy filters, in this paper a definition due to P. Eklund and W. Gähler [2] is used which fuzzificates additionally the membership of filter elements. This leads to an alternative definition of fuzzy uniform spaces introduced by W. Gähler et al. [6] in 1998. Omitting a certain symmetry condition in this definition one obtains fuzzy quasiuniform spaces analogously to the non-fuzzy case.
In 2005 the author [10] studied preuniform convergence spaces which form a strong topological universe, i.e. a topological construct which is 1^o cartesian closed (i.e. natural function spaces exist), 2^o extensional (i.e. one-point extensions exist), and in which 3^o (arbitrary) products of quotients are quotients. Furthermore, they are suitable for generalizing topological spaces as well as quasiuniform spaces. This is very remarkable since neither topological spaces nor (quasi) uniform spaces fulfill the above mentioned convenient properties 1^o, 2^o, and 3^o with the following exception: 3^o is true for uniform spaces, and it is unknown whether 3^o is true for quasiuniform spaces (cf. [10]).

The aim of this paper is to realize that the fuzzyfication of preuniform convergence spaces which has been started in [11] leads to a strong topological universe too, and thus improves fuzzy topological spaces and fuzzy (quasi) uniform spaces. Since in non-symmetric convenient topology (cf. [10]) mainly preuniform convergence spaces are investigated we are now in the position to have a suitable framework for *non-symmetric fuzzy convenient topology.*
Finally, adding a certain symmetry condition to the definition of a fuzzy preuniform convergence space we obtain a fuzzy semiuniform convergence space, whose non-fuzzy analogue is mainly studied in convenient topology (cf. [9]). Since the construct **FSUConv** of fuzzy semiuniform convergence spaces is closely related to the construct **FPUConv** of fuzzy preuniform convergence spaces, it results that is a strong topological universe too. Therefore, the foundations of *fuzzy convenient*

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topology are also established.

The terminology of this paper corresponds to [9] and [11].

1 Preliminaries. In the following let L be a frame with different least element 0 and greatest element 1, e.g. $L = \{0, 1\}$ or $L = [0, 1]$ (= closed unit interval).

1.1. Remark. For each set X, L^X can be endowed with a partial order $\leq$ defined as follows:

$$f \leq g \quad \text{iff } f(x) \leq g(x) \quad \text{for each } x \in L.$$

As in L, for the infima and suprema in L^X the symbols $\wedge$ and $\bigwedge$ as well as $\vee$ and $\bigvee$ will be used respectively, e.g. for each pair $(f, g) \in L^X \times L^X$ and each $x \in X$, $(f \wedge g)(x) = f(x) \wedge g(x)$ and $(f \vee g)(x) = f(x) \vee g(x)$.

1.2. Definition An L-fuzzy filter (short: a *fuzzy filter*) on a non-empty set X is a map $\mathcal{F} : L^X \to L$ such that the following are satisfied:
$FFil_1$) $\mathcal{F}(\bar{l}) = l$ for each $l \in L$, where $\bar{l} : X \to L$ is defined by $\bar{l}(x) = l$ for each $x \in X$.
$FFil_2$) $\mathcal{F}(f \wedge g) = \mathcal{F}(f) \wedge \mathcal{F}(g)$ for all $f, g \in L^X$.
The set of all fuzzy filters on X is denoted by $F_L(X)$, where $F_L(\emptyset) = \emptyset$.

1.3. Remarks.

(1) If $\mathcal{F}$ is a fuzzy filter on X, then $\mathcal{F}(f) \leq \mathcal{F}(g)$ for all $f, g \in L^X$ such that $f \leq g$. Furthermore, for each $f \in L^X$, $\mathcal{F}(f) \leq \sup f = \sup \{f(x) : x \in X\}$.

(2) For each $x \in X$, there is a fuzzy filter $\dot{x} : L^X \to L$ defined by $\dot{x}(f) = f(x)$ for each $f \in L^X$.

(3) If $\mathcal{F}$ and $\mathcal{G}$ are fuzzy filters on X, then $\mathcal{F}$ is called *coarser* than $\mathcal{G}$ (or $\mathcal{G}$ is called *finer* than $\mathcal{F}$), denoted by $\mathcal{F} \subset \mathcal{G}$, iff $\mathcal{F}(f) \leq \mathcal{G}(f)$ for each $f \in L^X$.

1.4. Definition. A *fuzzy filter base* on a non-empty set X is a non-empty subset $\mathcal{B}$ of L^X such that the following are satisfied:
FB_1) $\bar{l} \in \mathcal{B}$ for each $l \in L$.
FB_2) For each $(f, g) \in \mathcal{B} \times \mathcal{B}$ there is some $h \in \mathcal{B}$ such that $h \leq f \wedge g$ and $\sup h = \sup f \wedge \sup g$.

1.5. Remark. Each fuzzy filter base $\mathcal{B}$ on X generates a fuzzy filter $\mathcal{F}$ on X defined by

$$\mathcal{F}(f) = \bigvee_{g \leq f,\ g \in \mathcal{B}} \sup g \quad \text{for each } f \in L^X.$$

Conversely, each fuzzy filter $\mathcal{F}$ on X can be generated by a fuzzy filter base on X, even a greatest one, denoted by base $\mathcal{F}$, where base $\mathcal{F} = \{f \in L^X : \mathcal{F}(f) = \sup f\}$.

1.6. Proposition. Let $f : X \to Y$ *be a map,* $\mathcal{F}$ *a fuzzy filter* on X, *and* $\mathcal{B}$ *a base of* $\mathcal{F}$. *Define* for each $g \in L^X$, $g^+ \in L^Y$ *by*

$$g^+(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} g(x) & \text{for } y \in f[X] \\ 0 & \text{otherwise} \end{cases},$$

Then $\{g^+ : g \in \mathcal{B}\} \cup \{\bar{l} : l \in L\}$ *is a base of the fuzzy filter* $f(\mathcal{F})$, *defined by* $f(\mathcal{F})(h) = \mathcal{F}(h \circ f)$ *for each* $h \in L^Y$, *where* $f(\mathcal{F})$ *is called the image of* $\mathcal{F}$ *under* f. *If* f *is surjective, then* $\{g^+ : g \in \mathcal{B}\}$ *is a base of* $f(\mathcal{F})$.

1.7. Definition. Let $f : X \to Y$ be a map, and $\mathcal{F}$ a fuzzy filter on Y. Then the *inverse image of* $\mathcal{F}$ *under* f is the coarsest fuzzy filter $\mathcal{G}$ on X such that $\mathcal{F} \subset f(\mathcal{G})$ provided that it exists. Usually, we write $f^{-1}(\mathcal{F})$ instead of $\mathcal{G}$. If $X \subset Y$ and $i : X \to Y$ denotes the inclusion map, then $i^{-1}(\mathcal{F})$ is also called the *trace* of $\mathcal{F}$.

1.8. Proposition. (cf. [4; proposition 9]). *Let $f : X \to Y$ be a map, $\mathcal{F}$ a fuzzy filter on Y, and $\mathcal{B}$ a base of $\mathcal{F}$. Then $f^{-1}(\mathcal{F})$ exists iff $\sup g = \sup(g \circ f)$ for each $g \in \mathcal{B}$.* If *$f^{-1}(\mathcal{F})$ exists, then $\{g \circ f : g \in \mathcal{B}\}$ is a base of $f^{-1}(\mathcal{F})$.*

1.9. Definition. Let $\mathcal{M}$ be a non-empty set of fuzzy filters on X. Then a fuzzy filter $\bigcap\limits_{\mathcal{F}\in\mathcal{M}} \mathcal{F}$, called the *intersection* of all $\mathcal{F} \in \mathcal{M}$, is defined by $(\bigcap\limits_{\mathcal{F}\in\mathcal{M}} \mathcal{F})(f) = \bigwedge\limits_{\mathcal{F}\in\mathcal{M}} \mathcal{F}(f)$ for each $f \in L^X$.

1.10. Proposition. *Let $\mathcal{M}$ be a non-empty set of fuzzy filters on a set X, and $f : X \to Y$ a map. Then*

$$f(\bigcap_{\mathcal{F}\in\mathcal{M}} \mathcal{F}) = \bigcap_{\mathcal{F}\in\mathcal{M}} f(\mathcal{F}).$$

Proof. For each $u \in L^X$, $f(\bigcap\limits_{\mathcal{F}\in\mathcal{M}} \mathcal{F})(u) = (\bigcap\limits_{\mathcal{F}\in\mathcal{M}} \mathcal{F})(u \circ f) = \bigwedge\limits_{\mathcal{F}\in\mathcal{M}} \mathcal{F}(u \circ f) = \bigwedge\limits_{\mathcal{F}\in\mathcal{M}} f(\mathcal{F})(u) = \Big(\bigcap\limits_{\mathcal{F}\in\mathcal{M}} f(\mathcal{F})\Big)(u)$.

1.11 Proposition. *Let $f : X \to Y$ be a map, $\mathcal{H}$ a fuzzy filter on X, and $\mathcal{K}$ a fuzzy filter on Y such that $\mathcal{K} \subset f(\mathcal{H})$. Then $f^{-1}(\mathcal{K})$ exists.*

Proof. Let $\mathcal{M} = \{\mathcal{F} \in F_L(X) : \mathcal{K} \subset f(\mathcal{F})\}$. By assumption, $\mathcal{M} \neq \emptyset$. By 1.10, $f(\bigcap\limits_{\mathcal{F}\in\mathcal{M}} \mathcal{F}) = \bigcap\limits_{\mathcal{F}\in\mathcal{M}} f(\mathcal{F}) \supset \mathcal{K}$, i.e. $\bigcap\limits_{\mathcal{F}\in\mathcal{M}} \mathcal{F}$ is the coarsest fuzzy filter on X whose image under f contains $\mathcal{K}$. By definition, $\bigcap\limits_{\mathcal{F}\in\mathcal{M}} \mathcal{F} = f^{-1}(\mathcal{K})$.

1.12 Proposition. (cf. [5; Proposition 3.8]). *Let $f : X \to Y$ be a map, $\mathcal{F}$ a fuzzy filter on X, and $\mathcal{G}$ a fuzzy filter on Y. The inverse image of $f(\mathcal{F})$ under f always exists and*

$$f^{-1}\Big(f(\mathcal{F})\Big) \subset \mathcal{F}.$$

If the inverse image of $\mathcal{G}$ under f exists, then

$$f\Big(f^{-1}(\mathcal{G})\Big) \supset \mathcal{G}.$$

1.13. Definition. Let $(X_i)_{i\in I}$ be a non-empty family of non-empty sets, and $\mathcal{F}_i$ a fuzzy filter on X_i for each $i \in I$. If $p_i : \prod\limits_{i\in I} X_i \to X_i$ denotes the i-th projection, then the coarsest fuzzy filter $\mathcal{F}$ on $\prod\limits_{i\in I} X_i$ such that $p_i(\mathcal{F}) = \mathcal{F}_i$ for each $i \in I$ is called the *product* of $(\mathcal{F}_i)_{i\in I}$, where $\prod\limits_{i\in I} \mathcal{F}_i$ is written instead of $\mathcal{F}$, or $\mathcal{F}_1 \times \mathcal{F}_2$ in case $I = \{1, 2\}$.

1.14. Proposition. (cf. [4; proposition 19]). *If I is a non-empty set and for each $i \in I, \mathcal{F}_i$ is a fuzzy filter on X_i, and $\mathcal{B}_i$ is a base of $\mathcal{F}_i$, then*

$$\mathcal{B} = \{\bigwedge_{j\in J} f_j \circ p_j : J \subset I \text{ finite and } f_j \in \mathcal{B}_j \text{ for all } j \in J\}$$

is a fuzzy filter base on $\prod\limits_{i\in I} X_i$ generating the product $\prod\limits_{i\in I} \mathcal{F}_i$ of $(\mathcal{F}_i)_{i\in I}$.

1.15. Corollary. *Let $(X_i)_{i\in I}$ be a non-empty family of non-empty sets and let $\mathcal{F}_i$ and $\mathcal{G}_i$ be fuzzy filters on X_i such that $\mathcal{F}_i \subset \mathcal{G}_i$ for each $i \in I$. Then $\prod\limits_{i\in I} \mathcal{F}_i \subset \prod \mathcal{G}_i$.*

Proof. Use 1.14 and note that $\mathcal{F}_i \subset \mathcal{G}_i$ implies base $\mathcal{F}_i \subset$ base $\mathcal{G}_i$.

1.16. Proposition. Let $f : X \to Y$ be a map and $\mathcal{F}, \mathcal{G} \in F_L(Y)$. *Then* $(f \times f)^{-1}(\mathcal{F} \times \mathcal{G})$ *exists iff* $f^{-1}(\mathcal{F})$ *and* $f^{-1}(\mathcal{G})$ *exist. If* $(f \times f)^{-1}(\mathcal{F} \times \mathcal{G})$ *exists, then*

$$(f \times f)^{-1}(\mathcal{F} \times \mathcal{G}) = f^{-1}(\mathcal{F}) \times f^{-1}(\mathcal{G}).$$

Proof.

(1) If $(f \times f)^{-1}(\mathcal{F} \times \mathcal{G})$ exists, then

$$(f \times f)\Big((f \times f)^{-1}(\mathcal{F} \times \mathcal{G})\Big) \supset \mathcal{F} \times \mathcal{G}.$$

Let p_Y^1 (resp. p_Y^2) be the first (resp. second) projection from $Y \times Y$ to Y, and p_X^1 (resp. p_X^2) the first (resp. the second) projection from $X \times X$ to X. Hence,

$$\begin{aligned} &p_Y^1\Big((f \times f)\big((f \times f)^{-1}(\mathcal{F} \times \mathcal{G})\big)\Big) \\ &= f\Big(p_X^1\big((f \times f)^{-1}(\mathcal{F} \times \mathcal{G})\big)\Big) \supset p_Y^1(\mathcal{F} \times \mathcal{G}) = \mathcal{F}, \end{aligned}$$

and analogously,

$$f\Big(p_X^2\big((f \times f)^{-1}(\mathcal{F} \times \mathcal{G})\big)\Big) \supset \mathcal{G}.$$

By 1.11, $f^{-1}(\mathcal{F})$ and $f^{-1}(\mathcal{G})$ exist.

(2) If $f^{-1}(\mathcal{F})$ and $f^{-1}(\mathcal{G})$ exist,

$$f\big(f^{-1}(\mathcal{F})\big) \supset \mathcal{F} \text{ and } f\big(f^{-1}(\mathcal{G})\big) \supset \mathcal{G}$$

(cf. 1.12). Thus, $f\big(f^{-1}(\mathcal{F})\big) \times f\big(f^{-1}(\mathcal{G})\big) = f \times f\big(f^{-1}(\mathcal{F}) \times f^{-1}(\mathcal{G})\big) \supset \mathcal{F} \times \mathcal{G}$ (cf. [11; 1.13 and 1.14]). Consequently, by 1.11, $(f \times f)^{-1}(\mathcal{F} \times \mathcal{G})$ exists.

(3) If $(f \times f)^{-1}(\mathcal{F} \times \mathcal{G})$ exists, it follows from 1) that $f^{-1}(\mathcal{F})$ and $f^{-1}(\mathcal{G})$ exist too. By [3; proposition 7], $\mathcal{B} = \{h \circ p_1 \wedge k \circ p_2 : h \in \text{ base } \mathcal{F}, k \in \text{ base } \mathcal{G}\}$ is a base of $\mathcal{F} \times \mathcal{G}$ where p_1 (resp. p_2) denotes the first (resp. second) projection from $Y \times Y$ to Y. Thus, by 1.8., $\mathcal{B}' = \{u \circ (f \times f) : u \in \mathcal{B}\}$ is a base of $(f \times f)^{-1}(\mathcal{F} \times \mathcal{G})$. Using [3; proposition 7] and 1.8 again, $\mathcal{B}'' = \{(h \circ f) \circ p_1' \wedge (k \circ f) \circ p_2' : h \in \text{ base } \mathcal{F}, k \in \text{ base } \mathcal{G}\}$ is a base of $f^{-1}(\mathcal{F}) \times f^{-1}(\mathcal{G})$, where p_1' (resp. p_2') denotes the first (resp. second) projection from $X \times X$ to X. Obviously, $(h \circ f) \circ p_1' \wedge (k \circ f) \circ p_2' = \big((h \circ p_1) \wedge (k \circ p_2)\big) \circ (f \times f)$ for all $h \in$ base $\mathcal{F}$ and $k \in$ base $\mathcal{G}$. Therefore, $\mathcal{B}' = \mathcal{B}''$, which implies

$$(f \times f)^{-1}(\mathcal{F} \times \mathcal{G}) = f^{-1}(\mathcal{F}) \times f^{-1}(\mathcal{G}).$$

1.17. Remark. In [11] the construct **FPUConv** of fuzzy preuniform convergence spaces (and fuzzy uniformly continuous maps) has been introduced where a *fuzzy preuniform convergence space* is a pair $(X, F\mathcal{J}_X)$ such that X is a set and $F\mathcal{J}_X$ is a set of fuzzy filters on $X \times X$ satisfying the following conditions:

UC_1) $(\dot{x,x}) \in F\mathcal{J}_X$ for each $x \in X$, and

UC_2) $\mathcal{F} \in F\mathcal{J}_X$ whenever $\mathcal{G} \in F\mathcal{J}_X$ and $\mathcal{G} \subset \mathcal{F}$;

and a map $f : (X, F\mathcal{J}_X) \to (Y, F\mathcal{J}_Y)$ between fuzzy preuniform convergence spaces is *fuzzy uniformly continuous* iff $(f \times f)(\mathcal{F}) \in F\mathcal{J}_Y$ for each $\mathcal{F} \in F\mathcal{J}_X$. It has been proved that **FPUConv** is a cartesian closed topological construct. In the following, further 'convenient properties' will be proved, and besides many other kinds of spaces fuzzy topological and fuzzy (quasi) uniform spaces are regarded as fuzzy preuniform convergence spaces.

2 The strong topological universe FPUConv of fuzzy preuniform convergence spaces

2.1. Theorem. **FPUConv** *is extensional. In particular, for each* $(X, F\mathcal{J}_X) \in |\mathbf{FPUConv}|$, $(X^*, F\mathcal{J}_{X^*})$ *is the one-point extension, where* $X^* = X \cup \{\infty_X\}$ *with* $\infty_X \notin X$, and $F\mathcal{J}_{X^*} = \{\mathcal{H} \in F_L(X^* \times X^*)$: *the trace of* $\mathcal{H}$ on $X \times X$ *exists and belongs to* $F\mathcal{J}_X$ *or the trace of* $\mathcal{H}$ on $X \times X$ *does not exist* $\}$.

Proof. Let $(X, F\mathcal{J}_X)$ be a fuzzy preuniform convergence space. Put $X^* = X \cup \{\infty_X\}$ with $\infty_X \notin X$, and let $i : X \to X^*$ be the inclusion map. For each $\mathcal{F} \in F_L(X \times X)$ define $\mathcal{F}^* : L^{X^* \times X^*} \to L$ by

$$\mathcal{F}^*(h) = \mathcal{F}(h|X \times X).$$

Then $\mathcal{F}^* \in F_L(X^* \times X^*)$. Furthermore,
$F\mathcal{J}_{X^*} = \{\mathcal{H} \in F_L(X^* \times X^*) : (i \times i)^{-1}(\mathcal{H})$ exists and belongs to $F\mathcal{J}_X$ or $(i \times i)^{-1}(\mathcal{H})$ does not exist $\}$
is an **FPUConv**-structure on X^*:

1. a) For each $x \in X, \dot{x} \times \dot{x} = (x, x)^{\cdot} \in F\mathcal{J}_{X^*}$, since $(i \times i)^{-1}\big((x, x)^{\cdot}\big)$ exists by 1.8 because $\sup h = \sup h|X \times X$ for each $h \in$ base $(x, x)^{\cdot}\big($ i.e. $h(x, x) = \sup h\big)$ and $(i \times i)^{-1}(\dot{x} \times \dot{x}) = i^{-1}(\dot{x}) \times i^{-1}(\dot{x})$ (cf. 1.16), where $i^{-1}(\dot{x})$ exists too and is equal to $\dot{x}$ on X, i.e. $(i \times i)^{-1}(\dot{x} \times \dot{x}) = \dot{x} \times \dot{x} \in F\mathcal{J}_X$.

   b) $\dot{\infty}_X \times \dot{\infty}_X \in F\mathcal{J}_{X^*}$ since $(i \times i)^{-1}(\dot{\infty}_X \times \dot{\infty}_X)$ does not exist because $h \in L^{X^* \times X^*}$ defined by

$$h(y, y') = \begin{cases} 1 \text{ if } (y, y') \in (X^* \times \{\infty_X\}) \cup (\{\infty_X\} \times X^*) \\ 0 \text{ if } (y, y') \in X \times X \end{cases}$$

   fulfills $(\dot{\infty}_X \times \dot{\infty}_X)(h) = \sup h$, but $\sup h \neq \sup(h|X \times X)$(cf .1.8).

2. If $\mathcal{F} \in F\mathcal{J}_{X^*}$, and $\mathcal{G} \in F_L(X^* \times X^*)$ such that $\mathcal{F} \subset \mathcal{G}$, then a) $(i \times i)^{-1}(\mathcal{G})$ does not exist, which implies $\mathcal{G} \in F\mathcal{J}_{X^*}$, or b) $(i \times i)^{-1}(\mathcal{G})$ exists which implies $\mathcal{G} \in F\mathcal{J}_{X^*}$ too: It follows from the existence of $(i \times i)^{-1}(\mathcal{G})$ that for each $h \in$ base $\mathcal{G}, \sup h = \sup h|X \times X$, and consequently, since base $\mathcal{F} \subset$ base $\mathcal{G}, \sup h = \sup h|X \times X$ for each $h \in$ base $\mathcal{F}$, i.e. $(i \times i)^{-1}(\mathcal{F})$ exists. Furthermore, $(i \times i)^{-1}(\mathcal{F}) \subset (i \times i)^{-1}(\mathcal{G})$ (cf. 1.7 and 1.12). Since $\mathcal{F} \in F\mathcal{J}_{X^*}, (i \times i)^{-1}(\mathcal{F}) \in F\mathcal{J}_X$, and thus $(i \times i)^{-1}(\mathcal{G}) \in F\mathcal{J}_X$, i.e. $\mathcal{G} \in F\mathcal{J}_{X^*}$.

Next, $(i \times i)^{-1}(\mathcal{F}^*)$ exists and
$(*)$ $(i \times i)^{-1}(\mathcal{F}^*) = \mathcal{F}$:
The existence of $(i \times i)^{-1}(\mathcal{F}^*)$ follows, for each $h \in$ base $\mathcal{F}^*$, from

$$\sup(h|X \times X) \leq \sup h = \mathcal{F}^*(h) = \mathcal{F}(h|X \times X) \leq \sup(h|X \times X)$$

(use 1.8.).
In order to prove $(*)$, consider base $\mathcal{F} = \{k \in L^{X \times X} : \mathcal{F}(k) = \sup k\}$ and the following base $\mathcal{B}^*$ of $(i \times i)^{-1}(\mathcal{F}^*)$ according to 1.8:

$$\mathcal{B}^* = \{h|X \times X : \mathcal{F}^*(h) = \sup h\}.$$

It suffices to verify that $\mathcal{B}^* =$ base $\mathcal{F}$:

1. Let $k = h|X \times X \in \mathcal{B}^*$, i.e. $\mathcal{F}(k) = \mathcal{F}^*(h) = \sup h$. Since $\sup h = \sup k$, we obtain $k \in$ base $\mathcal{F}$.
2. Let $k \in$ base $\mathcal{F}$, i.e. $\mathcal{F}(k) = \sup k$. Define $h \in L^{X^* \times X^*}$ by

$$h(y, y') = \begin{cases} 0 \text{ if } (y, y') \in (X^* \times \{\infty_X\}) \cup (\{\infty_X\} \times X^*) \\ k(y, y') \text{ if } (y, y') \in X \times X \end{cases}$$

Then $h|X \times X = k$, and $\mathcal{F}^*(h) = \mathcal{F}(h|X \times X) = \mathcal{F}(k) = \sup k = \sup h$, i.e. $k \in \mathcal{B}^*$.
$(X^*, F\mathcal{J}_{X^*})$ is the one-point extension of $(X, F\mathcal{J}_X)$ provided that the following can be proved:

1) $(X, F\mathcal{J}_X)$ is a subspace of $(X^*, F\mathcal{J}_{X^*})$, and
2) if $(Y, F\mathcal{J}_Y)$ is a fuzzy preuniform convergence space, $(Z, F\mathcal{J}_Z)$ a subspace of $(Y, F\mathcal{J}_Y)$, and $f : (Z, F\mathcal{J}_Z) \to (X, F\mathcal{J}_X)$ a fuzzy uniformly continuous map, then

$$f^* : (Y, F\mathcal{J}_Y) \to (X^*, F\mathcal{J}_{X^*}) \text{ defined by } f^*(y) = \begin{cases} f(y) \text{ if } y \in Z \\ \infty_X \text{ if } y \in Y \setminus Z \end{cases}$$

is fuzzy uniformly continuous.

1) means that $F\mathcal{J}_X$ is equal to

$$\overline{F\mathcal{J}_X} = \{\mathcal{F} \in F_L(X \times X) : (i \times i)(\mathcal{F}) \in F\mathcal{J}_{X^*}\} :$$

Let $\mathcal{F} \in \overline{F\mathcal{J}_X}$, i.e. $(i \times i)(\mathcal{F}) \in F\mathcal{J}_{X^*}$. Then $(i \times i)^{-1}\big((i \times i)(\mathcal{F})\big) \subset \mathcal{F}$ (cf. [1.12]) which implies $\mathcal{F} \in F\mathcal{J}_X$, because $(i \times i)^{-1}\big((i \times i)(\mathcal{F})\big) \in F\mathcal{J}_X$ by assumption. Conversely, let $\mathcal{F} \in F\mathcal{J}_X$. By $(*)$, $(i \times i)^{-1}(\mathcal{F}^*) = \mathcal{F}$, and thus, $\mathcal{F}^* \in F\mathcal{J}_X*$. Since $(i \times i)(\mathcal{F}) \supset \mathcal{F}^*, (i \times i)(\mathcal{F}) \in F\mathcal{J}_X^*$, i.e. $\mathcal{F} \in \overline{F\mathcal{F}_X}$.

2) Without loss of generality, let $Z \subset Y$ and $Z \neq Y$. Now let $\mathcal{G} \in F\mathcal{J}_Y$, and let $j : Z \to Y$ be the inclusion map. By 1.8, $(j \times j)^{-1}(\mathcal{G})$ exists iff for each $k \in$ base $\mathcal{G}, \sup k = \sup(k|Z \times Z)$. Let us distinguish the following cases:
Case 1. $(j \times j)^{-1}(\mathcal{G})$ does not exist.
Case 2. $(j \times j)^{-1}(\mathcal{G})$ exists.
In each case we have to prove that $(f^* \times f^*)(\mathcal{G}) \in F\mathcal{J}_{X^*}$.

Concerning 'case 1' it suffices to prove that $(i \times i)^{-1}\big((f^* \times f^*)(\mathcal{G})\big)$ does not exist:
By assumption, there is some $k \in$ base $\mathcal{G}$ $\big($i.e. $\mathcal{G}(k) = \sup k\big)$ such that $\sup(k|Z \times Z) \neq \sup k$, i.e. $\sup(k|Z \times Z) < \sup k$. Define $h \in L^{X^* \times X^*}$ by

$$h(x, x') = \begin{cases} \sup(k|Z \times Z) \text{ if } (x, x') \in X \times X \\ \sup k \text{ if } (x, x') \in (\{\infty_X\} \times X^*) \cup (X^* \times \{\infty_X\}) \end{cases}.$$

Obviously, sup $(h|X \times X) =$ sup $(k|Z \times Z) \neq \sup k = \sup h$. Furthermore, $h \in$ base $(f^* \times f^*)(\mathcal{G})$ because $(f^* \times f^*)(\mathcal{G})(h) = \mathcal{G}\big(h \circ (f^* \times f^*)\big) = \sup k = \sup h$ (note: $k \leq h \circ (f^* \times f^*)$ by definition of h and f^*; then $\sup k \leq \bigvee_{l \leq h \circ (f^* \times f^*), l \in \text{ base } \mathcal{G}} \sup l = \mathcal{G}\big(h \circ (f^* \times f^*)\big)$, and $(f^* \times f^*)(\mathcal{G})(h) \leq \sup h (= \sup k)$ is always valid). Hence, $(j \times j)^{-1}\big((f^* \times f^*)(\mathcal{G})\big)$ does not exist (cf. 1.8).

Since in 'case 2' $(j \times j)^{-1}(\mathcal{G})$ exists and $(Z, F\mathcal{J}_Z)$ is a subspace of $(Y, F\mathcal{J}_Y)$, $(j \times j)^{-1}(\mathcal{G})$ belongs to $F\mathcal{J}_Z$.
It follows from the fuzzy uniform continuity of f that $(f \times f)\big((j \times j)^{-1}(\mathcal{G})\big) \in F\mathcal{J}_X$. In order to prove that $f^* \times f^*(\mathcal{G})$ belongs to $F\mathcal{J}_{X^*}$ it suffices therefore to check that
$\alpha)(i \times i)^{-1}\big((f^* \times f^*)(\mathcal{G})\big)$ exists, and
$\beta)$ $(i \times i)^{-1}\big((f^* \times f^*)(\mathcal{G})\big) = (f \times f)\big((j \times j)^{-1}(\mathcal{G})\big)$.
$\alpha)$ is proved indirectly: If $(i \times i)^{-1}\big((f^* \times f^*)(\mathcal{G})\big)$ would not exist there were some $h \in$ base $(f^* \times f^*)(\mathcal{G})$, i.e. $\sup h = \big((f^* \times f^*)(\mathcal{G})\big)(h) = \mathcal{G}\big(h \circ (f^* \times f^*)\big)$, such that $\sup h|X \times X \neq \sup h$, i.e. $\sup h|X \times X < \sup h$. Then, $k \in L^{Y \times Y}$ could be defined by

$$k(y, y') = \begin{cases} \sup\ (h|X \times X) \text{ if } (y, y') \in Z \times Z \\ \sup h \text{ if } (y, y') \in (Y \times Y) \backslash (Z \times Z) \end{cases}.$$

Thus, $\sup k = \sup h$, and $\sup k|Z \times Z = \sup h|X \times X$, i.e. $\sup k \neq \sup k|Z \times Z$. Furthermore, $k \in$ base $\mathcal{G}$, i.e. $\sup k \leq \mathcal{G}(k)$, since obviously $h \circ (f^* \times f^*) \leq k$, and consequently $\mathcal{G}\big(h \circ (f^* \times f^*)\big) = \sup h = \sup k \leq \mathcal{G}(k)$. Hence, $(j \times j)^{-1}(\mathcal{G})$ would not exist in contrast to our assumption in case 2.

Concerning β),
$\mathcal{B} = \{(k|Z\times Z)^+ : k \in \text{base } \mathcal{G}\} \bigcup \{\bar{l} : l \in L\}$ is a base of $(f\times f)\big((j\times j)^{-1}(\mathcal{G})\big)$, where

$$(k|Z\times Z)^+(x,x') = \begin{cases} \bigvee\limits_{(z,z')\in(f\times f)^{-1}(x,x')} (k|Z\times Z)(z,z') \text{ if } (x,x')\in(f\times f)[Z\times Z] \\ 0 \text{ otherwise} \end{cases}$$

(cf. 1.6),
and $\mathcal{B}' = \{k^+|X\times X : k\in \text{base } \mathcal{G}\} \bigcup \{\bar{l} : l\in L\}$ is a base of $(i\times i)^{-1}\big((f^*\times f^*)(\mathcal{G})\big)$, where

$$k^+(x,x') = \begin{cases} \bigvee\limits_{(y,y')\in(R^*\times R^*)^{-1}(x,x')} k(y,y') \text{ if } (x,x')\in(f^*\times f^*)[Y\times Y] \\ 0 \text{ otherwise.} \end{cases}$$

Obviously for each $k\in$ base $\mathcal{G}$, $(k|Z\times Z)^+ = k^+|X\times X$, i.e. $\mathcal{B}=\mathcal{B}'$, which implies β).

2.2. Proposition. *Let I be a non-empty set, $(f_i : X_i \to Y_i)_{i\in I}$ a family of surjective maps, and $\mathcal{F}_i \in F_L(X_i)$ for each $i\in I$. Then*

$$\prod_{i\in I} f_i(\prod_{i\in I}\mathcal{F}_i) = \prod_{i\in I} f_i(\mathcal{F}_i).$$

Proof: For each $i\in I$, let $\mathcal{B}_i =$ base $\mathcal{F}_i$, and let $p_i : \prod\limits_{i\in I} X_i \to X_i$ be the i-th projection. By 1.14, $\mathcal{B} = \{\bigwedge\limits_{j\in J} h_j\circ p_j : J\subset I$ is finite, and $h_j\in\mathcal{B}_j$ for each $j\in J\}$ is a base of $\prod\limits_{i\in I}\mathcal{F}_i$, and by 1.6, $\mathcal{B}^+ = \{(\bigwedge\limits_{j\in J} h_j\circ p_j)^+ : J\subset I$ is finite and $h_j\in\mathcal{B}_j$ for each $j\in J\}$ is a base of $\prod\limits_{i\in I} f_i(\prod\limits_{i\in I}\mathcal{F}_i)$, where $(\bigwedge\limits_{j\in J} h_j\circ p_j)^+ \in L^{\prod_{i\in I} Y_i}$ is defined by

$$\big(\bigwedge_{j\in J}(h_j\circ p_j)\big)^+\big((y_i)\big) = \bigvee_{(x_i)\in(\prod\limits_{i\in I} f_i)^{-1}\big((y_i)\big)} \big(\bigwedge_{j\in J} h_j\circ p_j\big)\big((x_i)\big)$$

since $\prod\limits_{i\in I} f_i : \prod\limits_{i\in I} X_i \to \prod\limits_{i\in I} Y_i$ is surjective.
Furthermore, for each $i\in I$, $\mathcal{B}_i^+ = \{h_i^+ : h_i\in\mathcal{B}_i\}$ is a base of $f_i(\mathcal{F}_i)$, where $h_i^+\in L^{Y_i}$ is defined by

$$h_i^+(y_i) = \bigvee_{x_i\in f_i^{-1}(y_i)} h_i(x_i), \text{ and}$$

$$\widetilde{\mathcal{B}}^+ = \{\bigwedge_{j\in J} h_j^+\circ p_j' : J\subset I \text{ is finite, and } h_j\in\mathcal{B}_j \text{ for each } j\in J\}$$

is a base of $\prod\limits_{i\in I} f_i(\mathcal{F}_i)$, where $p_j' : \prod\limits_{i\in I} Y_i \to Y_j$ denotes the j-th projection. Since L fulfills an inifinite distributive law it is easily checked that for each finite $J\subset I$,

$$\left(\bigwedge_{j\in J} h_j\circ p_j\right)^+ = \bigwedge_{j\in J} h_j^+\circ p_j'.$$

Thus, $\mathcal{B}^+ = \widetilde{\mathcal{B}}^+$, i.e. the assertion is proved.

2.3. Theorem. *In* **FPUConv** *products of quotient maps are quotient maps.*

Proof. Let I be a non-empty set, $\big(f_i : (X_i, F\mathcal{J}_{X_i}) \to (Y_i, F\mathcal{J}_{Y_i})\big)_{i\in I}$ a family of quotient maps, and

$$\begin{array}{ccc} (X, F\mathcal{J}_X) & \xrightarrow{\prod_{i\in I} f_i} & (Y, F\mathcal{J}_Y) \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p'_i} \\ (X_i, F\mathcal{J}_{X_i}) & \xrightarrow{f_i} & (Y_i, F\mathcal{J}_{Y_i}) \end{array}$$

the corresponding product diagram in **FPUConv**, where $(X, F\mathcal{J}_X) = \prod_{i\in I}(X_i, F\mathcal{J}_{X_i})$, and $(Y, F\mathcal{J}_Y) = \prod_{i\in I}(Y_i, F\mathcal{J}_{Y_i})$. Since all f_i are surjective, $\prod_{i\in I} f_i$ is surjective. For each $i \in I, F\mathcal{J}_{Y_i} = \{\mathcal{F} \in F_L(Y_i\times Y_i) :$ there is some $\mathcal{G}_i \in F\mathcal{J}_{X_i}$ with $(f_i\times f_i)(\mathcal{G}_i) \subset \mathcal{F}\}$ because f_i is a quotient map. Let $F\mathcal{J}'_Y = \{\mathcal{K} \in F_L(Y\times Y) :$ there is some $\mathcal{G} \in F\mathcal{J}_X$ with $(\prod_{i\in I} f_i\times\prod_{i\in I} f_i)(\mathcal{G}) \subset \mathcal{K}\}$. Then $F\mathcal{J}_Y = \{\mathcal{H} \in F_L(Y \times Y) : (p'_i \times p'_i)(\mathcal{H}) \in F\mathcal{J}_{Y_i}$ for each $i \in I\}$ is equal to $F\mathcal{J}'_Y$, i.e. $\prod_{i\in I} f_i$ is a quotient map:

1) If $\mathcal{K} \in F\mathcal{J}'_Y$, then there is some $\mathcal{G} \in F\mathcal{J}_X$ with $(\prod_{i\in I} f_i \times \prod_{i\in I} f_i)(\mathcal{G}) \subset \mathcal{K}$. Since $\prod_{i\in I} f_i$ is fuzzy uniformly continuous, it follows that $\mathcal{K} \in F\mathcal{J}_Y$.
2) If $\mathcal{H} \in F\mathcal{J}_Y$, then $(p'_i \times p'_i)(\mathcal{H}) \in F\mathcal{J}_{Y_i}$ for each $i \in I$. Thus, for each $i \in I$, there is some $\mathcal{G}_i \in F\mathcal{J}_{X_i}$ such that $(f_i \times f_i)(\mathcal{G}_i) \subset (p'_i \times p'_i)(\mathcal{H})$. Identifying $\prod_{i\in I} X_i \times X_i$ with $\prod_{i\in I} X_i\times\prod_{i\in I} X_i$, and $\prod_{i\in I} Y_i\times Y_i$ with $\prod_{i\in I} Y_i\times\prod_{i\in I} Y_i$, we get $\prod_{i\in I}(f_i\times f_i) = \prod_{i\in I} f_i\times\prod_{i\in I} f_i$, and using 2.2.,$\Big(\prod_{i\in I} f_i \times \prod_{i\in I} f_i\Big)\Big(\prod_{i\in I}\mathcal{G}_i\Big) = \Big(\prod_{i\in I} f_i \times f_i\Big)\Big(\prod_{i\in I}\mathcal{G}_i\Big) = \prod_{i\in I}(f_i \times f_i)(\mathcal{G}_i) \subset \prod_{i\in I}(p'_i \times p'_i)(\mathcal{H}) \subset \mathcal{H}$ which implies $\mathcal{H} \in F\mathcal{J}'_Y$ since $\prod_{i\in I}\mathcal{G}_i \in F\mathcal{J}_X$.

2.4. Theorem. **FPUConv** *is a strong topological universe.*

Proof. 2.1, 2.3 and [11; 2.5].

3 Fuzzy preconvergence spaces and fuzzy topological spaces

3.1. Remark. In [11] it has been proved that the construct **FPConv** of fuzzy preconvergence spaces (and fuzzy uniformly continuous maps) is bicoreflective in **FPUConv**, and concretely isomorphic to the construct **FGConv** of fuzzy generalized convergence spaces (and fuzzy continuous maps). Furthermore, **FPConv** is closed under formation of finite products in **FPUConv** (cf. [11; 3.9]). Next, we will prove the following:

3.2. Proposition. **FPConv** *is closed under formation of subspaces* (in **FPUConv**).

Proof. Let $(Y, F\mathcal{J}_Y) \in |\mathbf{FPConv}|$, and $(X, F\mathcal{J}_X)$ a subspace of $(Y, F\mathcal{J}_Y)$ in **FPUConv**, where $X \subset Y$. It suffices to prove that $F\mathcal{J}_X \subset F\mathcal{J}_{q_{F\mathcal{J}_X}}$ since the inverse inclusion is always valid. Let $\mathcal{H} \in F\mathcal{J}_X$. Then $(i \times i)(\mathcal{H}) \in F\mathcal{J}_Y = F\mathcal{J}_{q_{F\mathcal{J}_Y}}$ (where $i : X \to Y$ denotes the inclusion map), i.e. $(i \times i)(\mathcal{H}) \supset \dot{y} \times \mathcal{G}$ for some $(\mathcal{G}, y) \in q_{F\mathcal{J}_Y}$. By 1.12, 1.11, and 1.16 we obtain

$$\mathcal{H} \supset (i \times i)^{-1}\big((i \times i)(\mathcal{H})\big) \supset (i \times i)^{-1}(\dot{y} \times \mathcal{G}) = i^{-1}(\dot{y}) \times i^{-1}(\mathcal{G}).$$

Since $i^{-1}(\dot{y})$ exists, $y \in X$ $\big($ By 1.12, $\dot{y} \subset i(i^{-1}(\dot{y}))$, i.e. for each $u \in L^Y, \dot{y}(u) = u(y) \leq i^{-1}(\dot{y})(u \circ i) \leq \sup(u \circ i)$; if $y \in Y\backslash X$, then for $u \in L^Y$ defined by

$$u(z) = \begin{cases} 0 \text{ if } z \in X \\ 1 \text{ if } z \in Y\backslash X \end{cases},$$

$u(y) = 1 \leq \sup u|X = 0$, i.e. $0 = 1$ - a contradiction. $\big)$. Let $(\dot{y})_X \in F_L(X)$ and $(\dot{y})_Y \in F_L(Y)$ be defined by

$$(\dot{y})_X(u) = u(y) \text{for each } u \in L^X \text{ , and}$$

$$(\dot{y})_Y(v) = v(y) \text{ for each } v \in L^Y, \text{ respectively .}$$

Then

$$(*) \qquad (\dot{y})_X \subset i^{-1}\big((\dot{y})_Y\big)$$

because base $(\dot{y})_X = \{u \in L^X : u(y) = \sup u\}$ is contained in the base $\mathcal{B} = \{v|X : v \in \text{base}(\dot{y})_Y\}$ of $i^{-1}\big((\dot{y})_Y\big)$ $\big[u \in \text{base}(\dot{y})_X$, i.e. $u(y) = \sup u$, implies $u = v|X$ with $v(y) = \sup v$, i.e. $u \in \mathcal{B}$, where $v \in L^Y$ is defined by

$$v(z) = \begin{cases} u(z) \text{ if } z \in X \\ 0 \text{ if } z \in Y \backslash X \end{cases} \Big].$$

Since $(\dot{y})_X$ is a fuzzy ultrafilter (cf. [2]), it follows from $(*)$

$$i^{-1}\big((\dot{y})_Y\big) = (\dot{y})_X.$$

Thus, $\mathcal{H} \supset (\dot{y})_X \times i^{-1}(\mathcal{G})$, where $(\dot{y})_X \times i^{-1}(\mathcal{G}) \in F\mathcal{J}_X$ (note that $(\dot{y})_Y \times \mathcal{G} \in F\mathcal{J}_Y$ by assumption, and $i \times i\big((\dot{y})_X \times i^{-1}(\mathcal{G})\big) = (\dot{y})_Y \times i\big(i^{-1}(\mathcal{G})\big) \supset (\dot{y})_Y \times \mathcal{G}\big)$. Consequently, $\mathcal{H} \in F\mathcal{J}_{q_{F\mathcal{J}_X}}$.

3.3. Theorem. **FGConv** *is extensional. In particular, for each* $(X, q) \in |\mathbf{FGConv}|$ *the one-point extension* (X^*, q^*) *is formed as follows:*

$$X^* = X \cup \{\infty_X\} \text{ with } \infty_X \notin X, \textit{ and}$$

$$q^* = \{(\mathcal{F}^*, x^*) \in F_L(X^*) \times X^* : i^{-1}(\dot{x}^*) \textit{ does not exist or}$$
$$i^{-1}(\mathcal{F}^*) \textit{ does not exist or } (i^{-1}(\mathcal{F}^*), x^*) \in q\}$$

where $i : X \to X^*$ *denotes the inclusion map.*

Proof. Since **FGConv** is bicoreflectively embedded in **FPUConv** (cf. 3.1) and closed under formation of subspaces (cf. 3.2), it follows from [9; 3.25] that **FGConv** is extensional, and the one-point extensions in **FGConv** are formed as in **FPUConv** and then the bicoreflector is applied, i.e. the underlying fuzzy generalized convergence space has to be formed (see [11; 3.2] for the definition):
Let $(X, q) \in |\mathbf{FGConv}|$, and let $(X^*, F\mathcal{J}_{X^*})$ be the one-point extension of $(X, F\mathcal{J}_q) \in |\mathbf{FPUConv}|$ where $F\mathcal{J}_q = \{\mathcal{H} \in F_L(X \times X) : \text{there is some } (\mathcal{F}, x) \in q \text{ with } \dot{x} \times \mathcal{F} \subset \mathcal{H}\}$.
If (X^*, q^*) is the one-point extension of (X, q), then $(\mathcal{F}^*, x^*) \in q^*$ iff $\dot{x} \times \mathcal{F}^* \in F\mathcal{J}_{X^*}$, i.e.

1. $(i \times i)^{-1}(\dot{x}^* \times \mathcal{F}^*)$ exists and belongs to $F\mathcal{J}_q$ or
2. $(i \times i)^{-1}(\dot{x}^* \times \mathcal{F}^*)$ does not exist.

By 1.16, the second case is equivalent to

$$\begin{cases} a) i^{-1}(\dot{x}^*) \text{ does not exist} \\ \text{or} \\ b) i^{-1}(\mathcal{F}^*) \text{ does not exist.} \end{cases}$$

The first case is equivalent to $\big(i^{-1}(\mathcal{F}^*), x^*\big) \in q$, which is easily checked (use 1.16 again).

3.4. Theorem. In **FGConv** *products of quotient maps are quotient maps.*

Proof. Since **PGConv** is bicoreflectively embedded in **FPUConv** it is topological, and the initial and final structures in **FGConv** are formed as follows:

1. If X is a set, $\big((X_i, q_i)\big)_{i\in I}$ a family of fuzzy generalized convergence spaces, and $(f_i : X \to X_i)_{i\in I}$ a family of maps, then $q = \{(\mathcal{F}, x) \in F_L(X) \times X : \big(f_i(\mathcal{F}), f_i(x)\big) \in q_i$ for each $i \in I\}$ is the initial **FGConv**-structure on X w.r.t. the given data.
2. If X is a set,$\big((X_i, q_i)\big)_{i\in I}$ a family of fuzzy generalized convergence spaces, and $(f_i : X_i \to X)_{i\in I}$ a family of maps such that $X = \bigcup\limits_{i\in I} f_i[X_i]$, then
   $q = \{(\mathcal{F}, x) \in F_L(X) \times X :$ there are some $i \in I$ and some $(\mathcal{F}_i, x_i) \in q_i$ with $f_i(\mathcal{F}_i) \subset \mathcal{F}$ and $f_i(x_i) = x\}$
   is the final **FGConv**-structure on X w.r.t. the given data.

Let I be a non-empty set, $\big(f_i : (X_i, q_i) \to (Y_i, r_i)\big)_{i\in I}$ a family of quotient maps, and

$$\begin{array}{ccc} (\prod_{i\in I} X_i, q) & \xrightarrow{\prod\limits_{i\in I} f_i} & (\prod_{i\in I} Y_i, r) \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p'_i} \\ (X_i, q_i) & \xrightarrow{f_i} & (Y_i, r_i) \end{array}$$

the corresponding product diagram in **FGConv**, where $\prod\limits_{i\in I} f_i$ is surjective since all f_i are surjective. It remains to prove the following equivalence:
$(\mathcal{H}, y) \in r\Big($ i.e. $\big(p'_i)(\mathcal{H}), p'_i(y)\big) \in r_i$ for each $i \in I\Big)$
$\Leftrightarrow$ There is some $\big(\mathcal{G}, (x_i)\big) \in q$ with $(\prod\limits_{i\in I} f_i)(\mathcal{G}) \subset \mathcal{H}$ and $(\prod\limits_{i\in I} f_i)\big((x_i)\big) = \big(f_i(x_i)\big)_{i\in I} = y$.

" $\Leftarrow$ ". Since the above diagram commutes, $\prod\limits_{i\in I} f_i$ is fuzzy continuous, and the assertion is obvious.
" $\Rightarrow$ ".$\big(p'_i(\mathcal{H}), p'_i(y)\big) \in r_i$ for each $i \in I$ implies the existence of some $(\mathcal{G}_i, x_i) \in q_i$ with $f_i(\mathcal{G}_i) \subset p'_i(\mathcal{H})$, and $f_i(x_i) = p'_i(y)$ for each $i \in I$. Thus, using 2.2., $\prod\limits_{i\in I} f_i(\prod\limits_{i\in I} \mathcal{G}_i) = \prod\limits_{i\in I} f_i(\mathcal{G}_i) \subset \prod\limits_{i\in I} p'_i(\mathcal{H}) \subset \mathcal{H}$. Since $\big(\prod\limits_{i\in I} \mathcal{G}_i, (x_i)\big) \in q$, $\Big(\prod\limits_{i\in I} f_i(\prod\limits_{i\in I} \mathcal{G}_i), \prod\limits_{i\in I} f_i\big((x_i)\big)\Big) \in r$, where $\prod\limits_{i\in I} f_i\big((x_i)\big) = \big(f_i(x_i)\big) = y$. Consequently, $(\mathcal{H}, y) \in r$.

3.5. Theorem. FGConv ($\simeq$ FPConv) *is a strong topological universe.*

Proof. 3.3., 3.4., and [11; 3.11].

3.6. Definitions

1) A *fuzzy topological space* is a pair (X, t) where X is a set and $t \subset L^X$ a *fuzzy topology* on X, i.e the following are satisfied:

$FTop_1$) All constant maps from X to L belong to t (this includes the empty map $\emptyset : \emptyset \to L$ in case $X = \emptyset$).

$FTop_2$) $f, g \in t$ implies $f \wedge g \in t$.

$FTop_3$) $s \subset t$ implies $\bigvee s \in t$.

If t is a fuzzy topology on X, then the elements of t are called *fuzzy open subsets* of X.

2) A map $f : (X, t) \to (X', t')$ between fuzzy topological spaces is called *fuzzy continuous* provided that $f^{-1}(g') = g' \circ f \in t$ for each $g' \in t'$.
3) Let (X, t) be a fuzzy topological space, and $f \in L^X$. Then the *interior* of f with respect to t, denoted by $\text{int}_t f$, is defined as follows:

$$\text{int}_t f = \bigvee_{g\le f,\ g\in t} g$$

3.7. Remarks.

1) Concerning the interior of a fuzzy subset w.r.t. a fuzzy topology t, the following are satisfied:

   α) $\mathrm{int}_t c = c$ for each constant map $c : X \to L$.

   β) $\mathrm{int}_t f \leq f$ for each $f \in L^X$.

   γ) $\mathrm{int}_t f \wedge \mathrm{int}_t g = \mathrm{int}_t(f \wedge g)$ for all $f, g \in L^X$.

   δ) $\mathrm{int}_t\ (\mathrm{int}_t f) = \mathrm{int}_t f$ for each $f \in L^X$.

   If (X, t) is a fuzzy topological space, then for each $x \in X$, a fuzzy filter $\mathcal{U}_t(x) : L^X \to L$ is defined by
   $$\mathcal{U}_t(x)(f) = (\mathrm{int}_t f)(x) \text{ for each } f \in L^X,$$
   called the *fuzzy neighborhood filter of x with respect to t*.

2) If (X, q) is a fuzzy generalized convergence space, then for each $x \in X$, the *fuzzy neighborhood filter $\mathcal{U}_q(x)$ of x with respect to q* is defined by
   $\big(\mathcal{U}_q(x)\big)(g) = (\bigcap_{\mathcal{H} \xrightarrow{q} x} \mathcal{H})(g)$ for each $g \in L^X$, where $(\bigcap_{\mathcal{H} \xrightarrow{q} x} \mathcal{H})(g) = \bigwedge_{\mathcal{H} \xrightarrow{q} x} \mathcal{H}(g)$. Furthermore, for each $g \in L^X$ the *interior* $\mathrm{int}_q g$ *of* g *w.r.t.* q can be defined as follows:
   $(\mathrm{int}_q g)(x) = \big(\mathcal{U}_q(x)\big)(g)$ for each $x \in X$, and $\mathrm{int}_q \emptyset = \emptyset$ in case $X = \emptyset$.
   Hence, the interior of a fuzzy subset with respect to q fulfills the three conditions corresponding to $\alpha), \beta), \gamma)$ in 3.7.1).

3) By means of the interior operator $\mathrm{int}_q : L^X \to L^X$ defined as under 2) for each fuzzy generalized convergence space (X, q) a fuzzy topology t_q is defined by
   $$f \in t_q \text{ iff } \mathrm{int}_q f = f.$$
   $\big($ note: $f \in L^X$ is fuzzy open in (X, t_q) iff $f(x) \leq \mathcal{H}(f)$ for each $(\mathcal{H}, x) \in q\big)$.

3.8. Definitions.

1) A fuzzy generalized convergence space (X, q) is called *topological* iff there is a fuzzy topology t on X such that
   $$(\mathcal{F}, x) \in q \text{ iff } \mathcal{F} \supset \mathcal{U}_t(x).$$

2) A fuzzy preuniform convergence space $(X, F\mathcal{J}_X)$ is called *topological* provided that it is a fuzzy preconvergence space whose underlying fuzzy generalized convergence space $(X, q_{F\mathcal{J}_X})$ (cf. [11; 3.1 and 3.2])is topological.

3.9. Remark. Obviously, *a fuzzy preuniform convergence space $(X, F\mathcal{J}_X)$ is topological iff there is a fuzzy topology t on X such that $F\mathcal{J}_X = \{\mathcal{F} \in F_L(X \times X) :$ there is some $x \in X$ with $\mathcal{F} \supset \dot{x} \times \mathcal{U}_t(x)\}$.*

3.10. Proposition. *The construct* **T-FGConv** *of topological fuzzy generalized convergence spaces (and fuzzy continuous maps) is concretely isomorphic to the construct* **FTop** *of fuzzy topological spaces (and continuous maps), and bireflective in* **FGConv**.

Proof.

1) For each fuzzy topology t on a set X define a topological fuzzy generalized convergence space structure q_t by $(\mathcal{F}, x) \in q_t$ iff $\mathcal{F} \supset \mathcal{U}_t(x)$.
   Then the following are satisfied:

   1. $t_{q_t} = t$ for each fuzzy topology t on a set X, and
   2. $q_{t_q} = q$ for each topological fuzzy generalized convergence space structure q on a set X.
   Concerning 1., $f \in t$ iff $\mathrm{int}_t f = f$, and $f \in t_{q_t}$ iff $\mathrm{int}_{q_t} f = f$. Thus, the proof is finished if $\mathrm{int}_t f = \mathrm{int}_{q_t} f$ for each $f \in L^X$: For each $x \in X$, $(\mathrm{int}_{q_t} f)(x) = (\bigcap_{\mathcal{H} \xrightarrow{q_t} x} \mathcal{H})(f) = \mathcal{U}_t(x)(f) = (\mathrm{int}_t f)(x)$.
   2. follows from 1., since $q = q_t$ for some fuzzy topology t and $t_{q_t} = t$, i.e. $q = q_t = q_{t_{q_t}} = q_{t_q}$.
   Furthermore, we have:

3. If $f : (X, q) \to (X', q')$ is a fuzzy continuous map between fuzzy generalized convergence spaces, then $f : (X, t_q) \to (X', t_{q'})$ is fuzzy continuous, i.e. $h' \circ f \in t_q$ for each $h' \in t_{q'}$:

   Since $f : (X, q) \to (X', q')$ is fuzzy continuous, $\big(f(\mathcal{H}), f(x)\big) \in q'$ for each $(\mathcal{H}, x) \in q$, and if $h' \in t_{q'}, h'\big(f(x)\big) \leq f(\mathcal{H})(h') = \mathcal{H}(h' \circ f)$, i.e. $h' \circ f \in t_q$.

4. If $f : (X, t) \to (X', t')$ is a fuzzy continuous map between fuzzy topological spaces, then $f : (X, q_t) \to (X', q_{t'})$ is fuzzy continuous too:

   Let $(\mathcal{F}, x) \in q_t$, i.e. $\mathcal{F} \supset \mathcal{U}_t(x)$. In order to prove that $\big(f(\mathcal{F}), f(x)\big) \in q_{t'}$, i.e. $f(\mathcal{F}) \supset \mathcal{U}'_t\big(f(x)\big)$, it must be checked that for each $h \in L^{X'}, \mathcal{U}_{t'}\big(f(x)\big)(h) = (\mathrm{int}_{t'}h)\big(f(x)\big) \leq f(\mathcal{F})(h) = \mathcal{F}(h \circ f)$. Since $f : (X, t) \to (X', t')$ is fuzzy continuous, and $\mathrm{int}_{t'}h \in t', (\mathrm{int}_{t'}h) \circ f \in t$. Then it follows from $\mathcal{F} \supset \mathcal{U}_t(x)$ that $\mathcal{U}_t(x)\big((\mathrm{int}_{t'}h) \circ f\big) = \Big(\mathrm{int}_t\big((\mathrm{int}_{t'}h) \circ f\big)\Big)(x) = \big((\mathrm{int}_{t'}h) \circ f\big)(x) \leq \mathcal{F}\big((\mathrm{int}_{t'}h) \circ f\big) \leq \mathcal{F}(h \circ f)$.

It follows from 1., 2., 3., and 4., that **T-FGConv** is concretely isomorphic to **FTop**.

2) If $(X, q) \in |\mathbf{FGConv}|$, then $1_X : (X, q) \to (X, q_{t_q})$ is the bireflection of (X, q) w.r.t. **T-FGConv**: Obviously, $q \leq q_{t_q}$. Let $(X', q') \in |\mathbf{T\text{-}FGConv}|$, i.e. $q' = q_{t'}$ for some fuzzy topology t' on X', and $f : (X, q) \to (X', q')$ a fuzzy continuous map. Then we have to prove that $f : (X, q_{t_q}) \to (X', q')$ is fuzzy continuous too. In other words: It must be checked that $f : (X, t_q) \to (X', t')$ is fuzzy continuous: Let $g' \in t'$, i.e. $g' = \mathrm{int}_{t'}g'$. In order to prove $g' \circ f \in t_q$, let $(\mathcal{H}, x) \in q$. Then $f(\mathcal{H}) \supset \mathcal{U}_{t'}\big(f(x)\big)$ by assumption. Thus,

$$g'\big(f(x)\big) = (\mathrm{int}_{t'}g')\big(f(x)\big) = \Big(\mathcal{U}_{t'}\big(f(x)\big)\Big)(g') \leq f(\mathcal{H})(g') = \mathcal{H}(g' \circ f).$$

Consequently, $g' \circ f \in t_q$.

3.11. Remark. By means of 3.10 *we need not distinguish between topological fuzzy generalized convergence spaces and fuzzy topological spaces.*

3.12. Definition. Let (X, q) be a fuzzy generalized convergence space and $1_X : (X, q) \to (X, q_{t_q})$ the bireflection of (X, q) w.r.t. **T-FGConv**. Then $(X, q_{t_q})\big(\text{or } (X, t_q)\big)$ is called *the underlying fuzzy topological space of the fuzzy generalized convergence space* (X, q).

3.13. Corollary *The construct* **T-FPUConv** *of topological fuzzy preuniform convergence spaces (and fuzzy uniformly continuous maps) is concretely isomorphic to* **T-FGConv**.

Proof. The concrete isomorphism from **FPConv** to **FGConv** (cf. [11; 3.4]) leads to a concrete isomorphism from **T-PUConv** to **T-FGConv**.

3.14. Remark. It follows from 3.10 and 3.13 that **FTop** is concretely isomorphic to **T-FPUConv**, i.e. *we need not distinguish between fuzzy topological spaces and topological fuzzy preuniform convergence spaces, in other words:* **FTop** *can be embedded into* **FPUConv**.

4 Fuzzy semiuniform convergence spaces and fuzzy filter spaces

4.1. Remark. In [11] we have proved that the construct **FSUConv** of fuzzy semiuniform convergence spaces (and fuzzy uniformly continuous maps) is cartesian closed, and bireflective and bicoreflective in **FPUConv**. In the following further convenient properties are verified.

4.2. Proposition. **FSUConv** *is extensional and one-point extensions in* **FSUConv** *are formed as in* **FPUConv**.

Proof. By [9; 3.2.6] it suffices to prove that **FSUConv** is closed under formation of one-point extensions in **FPUConv**: Let $(X, F\mathcal{J}_X) \in |\mathbf{FSUConv}|$, and let $(X^*, F\mathcal{J}_{X^*})$ be the one-point extension of $(X, F\mathcal{J}_X)$ in **FPUConv**, i.e. $X^* = X \cup \{\infty_X\}$ with $\infty_X \notin X$, and $F\mathcal{J}_{X^*} = \{\mathcal{F}^* \in F_L(X^* \times X^*) : (i \times i)^{-1}(\mathcal{F}^*)$ exists and belongs to $F\mathcal{J}_X$ or $(i \times i)^{-1}(\mathcal{F}^*)$ does not exist$\}$, where $i : X \to X^*$ denotes the inclusion map. In order to prove that $(X^*, F\mathcal{J}_{X^*}) \in |\mathbf{FSUConv}|$, let $\mathcal{F}^* \in F\mathcal{J}_{X^*}$.
Then
1. $(i \times i)^{-1}(\mathcal{F}^*)$ exists and belongs to $F\mathcal{J}_X$
or
2. $(i \times i)^{-1}(\mathcal{F}^*)$ does not exist.
In the first case $(i \times i)^{-1}(\mathcal{F}^{*-1})$ exists, and $(i \times i)^{-1}(\mathcal{F}^{*-1}) = \big((i \times i)^{-1}(\mathcal{F}^*)\big)^{-1}$ which implies that $(i \times i)^{-1}(\mathcal{F}^{*-1})$ belongs to $F\mathcal{J}_X$, i.e. $\mathcal{F}^{*-1} \in F\mathcal{J}_{X^*}$ (The existence of $(i \times i)^{-1}(\mathcal{F}^{*-1})$ follows from the fact that $\{h^{-1} : h \in$ base $\mathcal{F}^*\}$ is a base of $\mathcal{F}^{*-1}$, and for each $h \in$ base $\mathcal{F}^*, \sup h^{-1} = \sup h = \sup h|X \times X = \sup h^{-1}|X \times X$ since $(i \times i)^{-1}(\mathcal{F}^*)$ exists. Furthermore, $\mathcal{B} = \{h^{-1}|X \times X : h \in$ base $\mathcal{F}^*\}$ is a base of $(i \times i)^{-1}(\mathcal{F}^{*-1})$ as well as a base of $\big((i \times i)^{-1}(\mathcal{F}^*)\big)^{-1}$ because $(h|X \times X)^{-1} = h^{-1}|X \times X$ for each $h \in$ base $\mathcal{F}^*$.).
Concerning the second case, by assumption, there is some $h \in \mathcal{B} =$ base $\mathcal{F}^*$ such that sup $h = \sup h^{-1} \neq \sup h|X \times X = \sup h^{-1}|X \times X$, i.e. there is some $h^{-1} \in \mathcal{B}^{-1}$ such that sup $h^{-1} \neq \sup h^{-1}|X \times X$, which implies that $(i \times i)^{-1}(\mathcal{F}^{*-1})$ does not exist, i.e. $\mathcal{F}^{*-1} \in F\mathcal{J}_{X^*}$.

4.3. Theorem. FSUConv *is a strong topological universe.*

Proof.
1) By [11; 2.8] **FSUConv** is a cartesian closed topological construct.
2) By 4.2, **FSUConv** is extensional.
3) Since **FSUConv** is bireflective and bicoreflective in **FPUConv** (cf. [11; 2.7]), products and quotients in **FSUConv** are formed as in **FPUConv**. Thus, by 2.3, in **FSUConv** products of quotient maps are quotient maps.

4.4. Remark. In [11] it has been proved that the (topological) construct **FFil** of fuzzy filter spaces (and fuzzy Cauchy continuous maps) is concretely isomorphic to a subconstruct of **FPUConv**, denoted by **FFil-D-FPUConv** and called the construct of **FFil**-determined fuzzy preuniform convergence spaces, i.e. it can be embedded in **FPUConv**, even in **FSUConv** (use [11; 4.3]), and **FFil** $\cong$ **FFil-D-FPUConv** is a cartesian closed topological construct. Before proving futher convenient properties we need the following:

4.5. Proposition. FFil-D-FPUConv *is bireflective in* **FPUConv** (*and thus in* **FSUConv**).

Proof.
a) **FFil-D-FPUConv** is closed under the formation of subspaces in **FPUConv:**
Let $(X, F\mathcal{J}_X) \in |\mathbf{FFil\text{-}D\text{-}FPUConv}|$ and let $(U, F\mathcal{J}_U)$ be a subspace of $(X, F\mathcal{J}_X)$ in **FPUConv**, where without loss of generality $U \subset X$ and $U \neq \emptyset$. Then we have to prove that $F\mathcal{J}_U \subset F\mathcal{J}_{\gamma_{F\mathcal{J}_U}}$, where $F\mathcal{J}_{\gamma_{F\mathcal{J}_U}} = \{\mathcal{F} \in F_L(U \times U)$: there is some $\mathcal{G} \in F_L(U)$ with $\mathcal{G} \times \mathcal{G} \in F\mathcal{J}_U$ and $\mathcal{G} \times \mathcal{G} \subset \mathcal{F}\}$, since $F\mathcal{J}_{\gamma_{F\mathcal{J}_U}} \subset F\mathcal{J}_U$ is always valid. Let $\mathcal{F} \in F\mathcal{J}_U$. By assumption there is some $\mathcal{G} \in F_L(X)$ such that $\mathcal{G} \times \mathcal{G} \subset (i \times i)(\mathcal{F})$ and $\mathcal{G} \times \mathcal{G} \in F\mathcal{J}_X$. Thus by 1.11, $(i \times i)^{-1}(\mathcal{G} \times \mathcal{G})$ exists, and by 1.16, $i^{-1}(\mathcal{G})$ exists too and we have
$(*)$ $(i \times i)^{-1}(\mathcal{G} \times \mathcal{G}) = i^{-1}(\mathcal{G}) \times i^{-1}(\mathcal{G})$.
Using 1.12,
$(**)$ $\mathcal{F} \supset (i \times i)^{-1}\big((i \times i)(\mathcal{F})\big)$.
It follows from $(*)$ and $(**)$ that $\mathcal{F} \supset i^{-1}(\mathcal{G}) \times i^{-1}(\mathcal{G})$ where $i^{-1}(\mathcal{G}) \in \gamma_{F\mathcal{J}_U} = \{\mathcal{H} \in F_L(U) : (i \times i)(\mathcal{H} \times \mathcal{H}) \in F\mathcal{J}_X\}$ because of
$(i\times i)\big((i\times i)^{-1}(\mathcal{G}\times\mathcal{G})\big) \supset \mathcal{G}\times\mathcal{G}$ by 1.12, i.e. $(i\times i)\big((i\times i)^{-1}(\mathcal{G}\times\mathcal{G})\big) \in F\mathcal{J}_X$ since $\mathcal{G}\times\mathcal{G} \in F\mathcal{J}_X$.
Consequently, $\mathcal{F} \in F\mathcal{J}_{\gamma_{F\mathcal{J}_U}}$.

b) **FFil-D-FPUConv** is closed under the formation of products in **FPUConv**: Let $(X_i, F\mathcal{J}_{X_i})_{i\in I}$ be a family of **FFil**-determined fuzzy preuniform convergence spaces where without loss of generality $X_i \neq \emptyset$ for each $i \in I$, and $(X, F\mathcal{J}_X)$ their product in **FPUConv**, i.e. $X = \prod_{i\in I} X_i$, and $F\mathcal{J}_X = \{\mathcal{F} \in F_L(\prod_{i\in I} X_i \times \prod_{i\in I} X_i) : p_i \times p_i(\mathcal{F}) \in F\mathcal{J}_{X_i}$ for each $i \in I\}$ where $p_i : \prod_{i\in I} X_i \to X_i$ denotes the i-th projection. In order to prove that $(X, F\mathcal{J}_X)$ is **FFil**-determined, let $\mathcal{F} \in F\mathcal{J}_X$. Then for each $i \in X$, there is some $\mathcal{G}_i \in F_L(X_i)$ such that $\mathcal{G}_i \times \mathcal{G}_i \in F\mathcal{J}_{X_i}$ and $p_i \times p_i(\mathcal{F}) \supset \mathcal{G}_i \times \mathcal{G}_i$ since $(X_i, F\mathcal{J}_{X_i})$ is **FFil**-determined. Consequently,

(1) $\prod_{i\in I} \mathcal{G}_i \times \mathcal{G}_i \subset \prod_{i\in I} p_i \times p_i(\mathcal{F})$ (cf. 1.15),
and

(2) $\prod_{i\in I} (p_i \times p_i)(\mathcal{F}) \subset \mathcal{F}$ (where $\prod_{i\in I} X_i \times \prod_{i\in I} X_i$ and $\prod_{i\in I} X_i \times X_i$ are identified).
Furthermore,

(3) $\prod_{i\in I} \mathcal{G}_i \times \prod_{i\in I} \mathcal{G}_i \subset \prod_{i\in I} \mathcal{G}_i \times \mathcal{G}_i$:
For each $i \in I$ let p_i^1 and p_i^2 be the first and the second projection with domain $X_i \times X_i$ respectively, and let p_1 and p_2 be the first and the second projection with domain $\prod_{i\in I} X_i \times \prod_{i\in I} X_i$ respectively. Then $p_1 = \prod_{i\in I} p_i^1$ and $p_2 = \prod_{i\in I} p_i^2$ (up to identification). Using 2.2., we obtain $p_1(\prod_{i\in I} \mathcal{G}_i \times \mathcal{G}_i) = \prod_{i\in I} p_i^1(\prod_{i\in I} \mathcal{G}_i \times \mathcal{G}_i) = \prod_{i\in I} p_i^1(\mathcal{G}_i \times \mathcal{G}_i) = \prod_{i\in I} \mathcal{G}_i$, and analogously, $p_2(\prod_{i\in I} \mathcal{G}_i \times \mathcal{G}_i) = \prod_{i\in I} \mathcal{G}_i$. Hence, $\prod_{i\in I} \mathcal{G}_i \times \prod_{i\in I} \mathcal{G}_i \subset \prod_{i\in I} \mathcal{G}_i \times \mathcal{G}_i$.

It follows from (1), (2) and (3) that $\mathcal{F} \supset \prod_{i\in I} \mathcal{G}_i \times \prod_{i\in I} \mathcal{G}_i$, where $\prod_{i\in I} \mathcal{G}_i \in \gamma_{F\mathcal{J}_X}$, i.e. $\prod_{i\in I} \mathcal{G}_i \times \prod_{i\in I} \mathcal{G}_i \in F\mathcal{J}_X$, since $p_i \times p_i(\prod_{i\in I} \mathcal{G}_i \times \prod_{i\in I} \mathcal{G}_i) = p_i(\prod_{i\in I} \mathcal{G}_i) \times p_i(\prod_{i\in I} \mathcal{G}_i) = \mathcal{G}_i \times \mathcal{G}_i \in F\mathcal{J}_{X_i}$ for each $i \in I$. Thus, $(X, F\mathcal{J}_X)$ is **FFil**-determined.

c) **FFil-D-FPUConv** contains all indiscrete **FPUConv**-objects: For each set M and each $l \in L$ define $\bar{l}^M \in L^M$ by $\bar{l}^M(x) = l$ for each $x \in M$. Let $(X, F\mathcal{J}_X) \in |\mathbf{FPUConv}|$ be indiscrete, i.e. $F\mathcal{J}_X = F_L(X \times X)$. If $\mathcal{F} \in F\mathcal{J}_X$, then $\mathcal{B}_{X\times X} = \{\bar{l}^{X\times X} : l \in L\}$ is a fuzzy filter base generating a fuzzy filter $(\mathcal{B}_{X\times X}) \subset \mathcal{F}$. Furthermore, the fuzzy filter base $\mathcal{B}_X = \{\bar{l}^X : l \in L\}$ generates a fuzzy filter $\mathcal{G} = (\mathcal{B}_X)$ such that $\mathcal{G} \times \mathcal{G} \subset \mathcal{F}$. (Note that the base $\mathcal{B}'_{X\times X} = \{v \circ p_2 \wedge u \circ p_1 : u, v \in \mathcal{B}_X\}$ of $\mathcal{G} \times \mathcal{G}$ coincides with $\mathcal{B}_{X\times X}$). Thus, $(X, F\mathcal{J}_X)$ is **FFil**-determined.

By a), b) and c) the assertion of the proposition is proved (cf. [9; 2.2.11.2) and 2.2.4]).

4.6. Theorem. FFil-D-FPUConv ($\cong$ **FFil**) *is a strong topological universe.*

Proof. Using [11; 4.6] and 4.5 **Fil-D-FPUConv** is a bicoreflective and bireflective subconstruct of **FPUConv**. Since **FPUConv** is a strong topological universe, **FFil-D-FPUConv** is a strong topological universe too where the natural function spaces and one-point extensions in **FFil-D-FPUConv** arise from the natural function spaces and one-point extensions in **FPUConv** by bicoreflective modification.

4.7. Remarks.

1) Since **FFil** is concretely isomorphic to **FFil-D-FPUConv** it follows from 4.6 that **FFil** is a strong topological universe where the natural function spaces in **FFil** have been described in [11; 4.8], and the one-point extensions are formed as follows:
Let $(X, \gamma) \in |\mathbf{FFil}|, X^* = X \cup \{\infty_X\}$ with $\infty_X \notin X$, and let $i : X \to X^*$ be the inclusion map. Then (X^*, γ^*) is the one-point extension of (X, γ), where $\gamma^* = \{\mathcal{F} \in F_L(X^*) : i^{-1}(\mathcal{F})$ exists and belongs to γ or $i^{-1}(\mathcal{F})$ does not exist $\}$.

2) Corresponding to the non-fuzzy case a fuzzy filter space (X, γ) is called *complete* provided that for each $\mathcal{F} \in \gamma$ there is some $x \in X$ such that $\mathcal{F} \cap \dot{x} \in \gamma$. Furthermore, the construct **CFFil** of complete fuzzy filter spaces (and fuzzy Cauchy continuous maps) is bicoreflective in **FFil** and closed under formation of products in **FFil**. In [7] a fuzzy Kent convergence space (X, q) (cf. [11; 3.5]) for the definition) is called *symmetric* iff $(\mathcal{F}, y) \in q$ whenever $(\mathcal{F}, x) \in q$ and $\mathcal{F}(f) \leq f(y)$ for all $f \in L^X$. It is easily checked that the construct $\mathbf{FKConv_S}$ of symmetric fuzzy Kent convergence spaces (and fuzzy continuous maps) is concretely isomorphic to **CFFil**. This implies that $\mathbf{FKConv_S}$ is a strongly cartesian closed topological construct, but it is not extensional since extensionality is not even fulfilled in the case $L = \{0, 1\}$, i.e. in the non-fuzzy case (cf. [9; 3.2.7. ④]). Obviously, symmetric fuzzy Kent convergence spaces can also be described as complete **FFil**-determined fuzzy preuniform convergence spaces, where a fuzzy preuniform convergence space $(X, F\mathcal{J}_X)$ is called complete provided that its underlying fuzzy filter space $(X, \gamma_{F\mathcal{J}_X})$ (cf. [11; 4.2]) is complete.

3) Let (X, γ) be a fuzzy filter space, and $q_\gamma = \{(\mathcal{F}, x) \in F_L(X) \times X : \mathcal{F} \cap \dot{x} \in \gamma\}$. Then (X, q_γ) is a symmetric fuzzy Kent convergence space, called *the underlying symmetric fuzzy Kent convergence space of* (X, γ). In particular, a fuzzy filter space (X, γ) is complete iff each $\mathcal{F} \in \gamma$ converges in (X, q_γ). If $(X, \gamma_{F\mathcal{J}_X})$ is the underlying fuzzy filter space of a fuzzy preuniform convergence space $(X, F\mathcal{J}_X)$ (cf. 2)), then $(X, q_{\gamma_{F\mathcal{J}_X}})$ is also *called the underlying symmetric fuzzy Kent convergence space of* $(X, F\mathcal{J}_X)$.

5 Fuzzy (quasi) uniform spaces

5.1. Definitions.(cf. [6; 13]).

1) (a) Let X be a non-empty set and $\mathcal{U}$ a fuzzy filter on $X \times X$. Consider the following conditions:

FU$_1$) $\mathcal{U} \subset (\dot{x,x})$ for each $x \in X$,
FU$_2$) $\mathcal{U} = \mathcal{U}^{-1}$, where $\mathcal{U}^{-1}(u) = \mathcal{U}(u^{-1})$ for each $u \in L^{X \times X}$ and $u^{-1}(x, y) = u(y, x)$ for each $(x, y) \in X \times X$.
If FU$_1$) is fulfilled, then consider also
FU$_3$) $\mathcal{U} \subset \mathcal{U} \circ \mathcal{U}$, where $\mathcal{U} \circ \mathcal{U}$ is the fuzzy filter on $X \times X$ defined by $\mathcal{U} \circ \mathcal{U}(u) = \bigvee_{v \in \text{ base } \mathcal{U},\ v \circ v \leq u} \mathcal{U}(v)$ for each $u \in L^{X \times X}$, and for each $v \in L^{X \times X}$ the composition $v \circ v$ is defined by $v \circ v(x, y) = \bigvee_{z \in X} v(x, z) \wedge v(z, y)$ for each $(x, y) \in X \times X$.
Then $\mathcal{U}$ is called a *fuzzy quasiuniformity* on X provided that FU$_1$) and FU$_3$) are satisfied and it is called a *fuzzy uniformity* on X provided that FU$_1$), FU$_2$) and FU$_3$) are fulfilled.

(b) A fuzzy (quasi) uniformity on the empty set $\emptyset$ is a map $\mathcal{U} : L^\emptyset \to L$, where $L^\emptyset = \{\emptyset\}$, such that $\mathcal{U}(\emptyset) = 1$.

(c) A *fuzzy* (*quasi*) *uniform space* is a pair $(X,\mathcal{U})$ where X is a set and $\mathcal{U}$ a fuzzy (quasi) uniformity on X.

2) A map $f : (X,\mathcal{U}) \to (X',\mathcal{U}')$ between fuzzy quasiuniform spaces is called *fuzzy uniformly continuous* iff $\mathcal{U}' \subset (f \times f)(\mathcal{U})$.

3) The construct of fuzzy uniform spaces (resp.fuzzy quasiuniform spaces) [and fuzzy uniformly continuous maps] is denoted by **FUnif** (resp. **FQUnif**).

5.2. Remark. In case $L = \{0, 1\}$ the fuzzy (quasi) uniform spaces may be identified with the usual (quasi) uniform spaces.

5.3. Definition. A fuzzy preuniform convergence space $(X, F\mathcal{J}_X)$ is called (*quasi*) *uniform* provided that there is a fuzzy (quasi) uniformity $\mathcal{U}$ on X such that $F\mathcal{J}_X = [\mathcal{U}]$, where $[\mathcal{U}] = \{\mathcal{F} \in F_L(X \times X) : \mathcal{U} \subset \mathcal{F}\}$.

5.4. Remarks. 1) It is easily checked that the construct **QU-FPUConv** of quasiuniform fuzzy preuniform convergence spaces (and fuzzy uniformly continuous maps) is concretely isomorphic

to **FQUnif**. Obviously, a uniform fuzzy preuniform convergence space is a fuzzy semiuniform convergence space, and the construct **U-FSUConv** of uniform fuzzy semiuniform convergence spaces (and fuzzy uniformly continuous maps) is concretely isomorphic to **FUnif**. Therefore, fuzzy quasiuniform spaces (resp. fuzzy uniform spaces) can be studied in the better behaved framework of fuzzy preuniform convergence spaces (resp. fuzzy semiuniform convergence spaces). Thus, *fuzzy topological spaces as well as fuzzy (quasi) uniform spaces can be improved by regarding them as fuzzy preuniform convergence spaces.*
2) By 1) **FQUnif** and **FUnif** may be regarded as subcontructs of **FPUConv** (up to isomorphism). They are bireflective in **FPUConv**:
For each $(X, F\mathcal{J}_X) \in |\mathbf{FPUConv}|, 1_X : (X, F\mathcal{J}_X) \to (X, [\mathcal{U}])$ is the bireflection of $(X, F\mathcal{J}_X)$ with respect to **FQUnif** (resp. **FUnif**) where $\mathcal{U}$ is the finest fuzzy quasiuniformity (resp. finest fuzzy uniformity) which is contained in each $\mathcal{F} \in F\mathcal{J}_X$. Consequently, **FQUnif** and **FUnif** are topological constructs. $(X, \mathcal{U})$ is *called the underlying fuzzy (quasi) uniform space of* $(X, F\mathcal{J}_X)$.
3) The initial structures in **FUnif** (or **FQUnif**) are formed as follows:
Let X be a non-empty set, $((X_i, \mathcal{U}_i))_{i\in I}$ a family of fuzzy (quasi) uniform spaces, $\mathcal{B}_i$ a base of $\mathcal{U}_i$ for each $i \in I$, and $(f_i : X \to X_i)_{i\in I}$ a family of maps. In case $I \neq \emptyset$, $\mathcal{B} = \{ \bigwedge_{j\in J} u_j \circ (f_j \times f_j) : J \subset I$ non-empty and finite, $u_j \in \mathcal{B}_j$ for each $j \in J\}$ is a base of the *initial fuzzy (quasi) uniformity* on X with respect to the given data.
In case $I = \emptyset$,
$\mathcal{B} = \{\bar{l} : l \in L\}$ is a base of the initial fuzzy (quasi) uniformity on X with respect to the given data, i.e. a base of the *indiscrete fuzzy (quasi) uniformity* on X.
(Note: The unique fuzzy uniformity on the empty set is initial.)

References

[1] Chang, C.L.: 1968, 'Fuzzy topological spaces', *J. Math. Anal. Appl.* **24**, 182–189.

[2] Eklund, P., and W. Gähler: 1992, 'Fuzzy filter functors and convergence', in: Applications of Category Theory to Fuzzy Subsets, 11th *International Seminar on Fuzzy Set Theory*, Linz 1989, pp. 109–136, Kluwer, Dordrecht.

[3] Eklund, P., and W. Gähler: 1992, 'Contributions to fuzzy convergence', in: Recent Developments of General Topology and its Applications, International Conference in Memory of Felix Hausdorff (eds. W. Gähler, H. Herrlich, and G. Preuß), pp. 118–123, *Mathematical Research* **67**, Akademie Verlag, Berlin.

[4] Gähler, W.: 1995, 'Convergence', *Fuzzy Sets and Systems* **73**, 97–129.

[5] Gähler, W.: 1995, 'The general fuzzy filter approach to fuzzy topology I', *Fuzzy Sets and Systems* **76**, 205–224.

[6] Gähler, W., F. Bayoumi, A. Kandil and A. Nouh : 1998, 'The theory of global fuzzy neighborhood structures. (III). Fuzzy uniform structures', *Fuzzy Sets and Systems* **98**, 175–199.

[7] Jandeck, T.-U.: 2003, *Fuzzy structures for semiuniform convergence spaces*, Ph.D. thesis, Free University Berlin, Logos Verlag, Berlin

[8] Lowen, R.: 1976, 'Fuzzy topological spaces and fuzzy compactness', *J. Math. Anal. Appl.* **56**, 621–633.

[9] Preuß, G.: 2002, *Foundations of Topology - An Approach to Convenient Topology*, Kluwer, Dordrecht.

[10] Preuß, G.: 2005, 'Non-symmetric convenient topology and its relations to convenient topology', *Topology Proceedings* **29**, 595–611.

[11] Preuß, G.: 2006, 'Fuzzy natural function spaces', *The Journal of Fuzzy Mathematics* **14**, 977–993.

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